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A NOTE ON FUZZY MAPPINGS

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Abstract

We introduce the concepts of surjective fuzzy mapping, injective fuzzy mapping, bijective fuzzy mapping on a set and we obtain some of their properties.

1. Introduction

Since Zadeh [5] has introduced the notion of fuzzy set on a set, many researchers are engaged in extending the notions of relations to the broader framework of the fuzzy setting. The result that the theory of fuzzy mappings was developed in [2, 3, 4] among several others. Nemitz [2] dealt with the notion of fuzzy equivalence relations and fuzzy functions as fuzzy relations. Ounalli and Jaoua [3] introduced the notion of difunctional relation on a set and investigated its properties. Sidky [4] introduced the concepts of t-fuzzy mapping and t-fuzzy partition. In the present paper, we introduce the concepts of surjective fuzzy mapping, injective fuzzy mapping, bijective fuzzy mapping on a set and we give some more results in connection with fuzzy mappings. Henceforth, without loss of generality, we assume that all fuzzy relations are defined on a fixed universe U.

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2. Preliminaries

In this section, we review some basic definitions and results from [2, 3] for reference purposes.

Definition 1. The *scalars sets of a fuzzy relation R*, written $\Phi(R)$, is defined as follows:

$$\Phi(R) = \{\alpha \neq 0 \mid \exists(x, y) \in U \times U, R(x, y) = \alpha\}.$$

Definition 2. Let R be a fuzzy relation and $a \in \Phi(R)$. The α*-cut relative to R*, written R_α, is a crisp relation such that for all $x, y \in U$:

$$R_\alpha(x, y) = \begin{cases} 1, & \text{if } R(x, y) \geq \alpha \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. A fuzzy relation R is *fuzzy difunctional* if and only if it satisfies condition $RR^{-1}R \subseteq R$, which is equivalent to $RR^{-1}R = R$.

Definition 4. Let R be a crisp relation on U. The *image set of* $x \in U$, written xR, is defined by $xR = \{y \mid (x, y) \in R\}$. R is *difunctional* if and only if, for all $x, y \in U$, $xR \cap yR \neq \emptyset \Rightarrow xR = yR$.

Definition 5. A fuzzy function is a *fuzzy relation R* such that for all $a \in \Phi(R)$, R_a is a crisp function.

Definition 6. Let R be a fuzzy function. We say that

1. R is *injective* if for all $\alpha \in \Phi(R)$, R_α is injective.
2. R is *surjective* if for all $\alpha \in \Phi(R)$, R_α is surjective.
3. R is *bijective* if for all $\alpha \in \Phi(R)$, R_α is bijective.

3. Main Results

In this section, we study properties of fuzzy functions and related topics in details.

Theorem 1. *Let S and R be two fuzzy relations. For two fuzzy relations R and S, we have* $(S \circ R)_\alpha = S_\alpha \circ R_\alpha$ *for all* $\alpha \in \Phi(R)$.

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Proof. We must prove that $(S \circ R)_\alpha(x, y) = (S_\alpha \circ R_\alpha)(x, y)$ for all $x, y \in U$. If $(S \circ R)_\alpha(x, y) = 1$, then $(S \circ R)(x, y) \geq \alpha$. This means that there exists $z_0 \in U$ such that $R(x, z_0) \wedge S(z_0, y) \geq \alpha$. This implies that $R(x, z_0) \geq \alpha$ and $S(z_0, y) \geq \alpha$. Hence, we have $R_\alpha(x, z_0) = 1$ and $S_\alpha(z_0, y) = 1$. On the other hand, we note that

$$\begin{aligned}(S_\alpha \circ R_\alpha)(x, y) &= \bigvee_{z \in U}[R_\alpha(x, z) \wedge S_\alpha(z, y)] \\ &\geq R_\alpha(x, z_0) \wedge S_\alpha(z_0, y) \\ &= 1.\end{aligned}$$

This implies $(S_\alpha \circ R_\alpha)(x, y) = 1$. Next, let $(S_\alpha \circ R_\alpha)(x, y) = 0$. Then, $\bigvee_{z \in U}[R_\alpha(x, z) \wedge S_\alpha(z, y)] = 0$. So, for all $z \in U$, $R_\alpha(x, z) \wedge S_\alpha(z, y) = 0$. This implies, for all $z \in U$, $R_\alpha(x, z) = 0$ or $S_\alpha(z, y) = 0$. On the other hand, we note that

$$(S \circ R)_\alpha(x, y) = \bigvee_{z \in U}[R_\alpha(x, z) \wedge S_\alpha(z, y)] = 0.$$

Hence, $(S \circ R)_\alpha = S_\alpha \circ R_\alpha$. This completes the proof.

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Theorem 2. *Let R and S be two fuzzy functions. If R and S are injective, then $S \circ R$ is injective.*

Proof. Let $\alpha \in \Phi(R)$ be fixed, and let $(S \circ R)_\alpha(x_1, z) = 1$ and $(S \circ R)_\alpha(x_2, z) = 1$. Then, $(S \circ R)_\alpha(x_1, z) = 1$ implies $(S \circ R)(x_1, z) \geq \alpha$. So, by the definition of composition, $(S \circ R)(x_1, z) \geq \alpha$ means there exists $y \in U$ such that $R(x_1, y) \wedge S(y, z) \geq \alpha$.

Similarly, we see that $(S \circ R)_\alpha(x_2, z) = 1$ means there exists $y' \in U$ such that $R(x_2, y') \wedge S(y', z) \geq \alpha$. Hence, above inequalities entail $S(y, z) \geq \alpha$ and $S(y', z) \geq \alpha$. So $S_\alpha(y, z) = 1$ and $S_\alpha(y', z) = 1$. Since S is injective, we get $y = y'$. On the other hand, from inequalities $R(x_1, y) \geq \alpha$ and $R(x_2, y') \geq \alpha$, we have $R(x_1, y) \geq \alpha$ and $R(x_2, y) \geq \alpha$. This implies $R_\alpha(x_1, y) = 1$ and $R_\alpha(x_2, y) = 1$. Hence, since R is injective, we have $x_1 = x_2$. This completes the proof.

Theorem 3. *Let R and S be two fuzzy functions. If R and S are surjective, then $S \circ R$ is surjective.*

Proof. It is sufficient to show that $(S \circ R)_\alpha$ is surjective for all $\alpha \in \Phi(R)$. Now, for fixed $\alpha \in \Phi(R)$, let $z \in U$ be any given. Since S is surjective, S_α is surjective. Hence, for this $z \in U$, there exists $y \in U$ such that $S_\alpha(y, z) = 1$. Also, for this $y \in U$, since R is surjective, R_α is surjective. This entails there exists $x \in U$ such that $R_\alpha(x, y) = 1$.

On the other hand,

$$\begin{aligned}(S_\alpha \circ R_\alpha)(x, z) &= \bigvee_{y \in U}[R_\alpha(x, y) \wedge S_\alpha(y, z)] \\ &\geq R_\alpha(x, y) \wedge S_\alpha(y, z) \\ &= 1.\end{aligned}$$

Hence, $(S_\alpha \circ R_\alpha)(x, z) = 1$. Therefore $S \circ R$ is surjective. This completes the proof.

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Theorem 4. *Let S and R be two fuzzy functions. If $S \circ R$ is injective, then R is injective.*

Proof. Let $\alpha \in \Phi(R)$ be fixed. Then, we show that R_α is injective. Now, let $R_\alpha(x_1, y) = 1 = R_\alpha(x_2, y)$. Then, for this $y \in U$, since R_α is an ordinary function, there exists $z \in U$ such that $S_\alpha(y, z) = 1$. This means that

$$\begin{aligned}(S \circ R)_\alpha(x_1, z) &= (S_\alpha \circ R_\alpha)(x_1, z) \\ &= \bigvee_{y \in U}[R_\alpha(x_1, y) \wedge S_\alpha(y, z)] \\ &\geq R_\alpha(x_1, y) \wedge S_\alpha(y, z) \\ &= 1.\end{aligned}$$

Thus $(S \circ R)_\alpha(x_1, z) = 1$. Similarly, we get easily that $(S \circ R)_\alpha(x_2, z) = 1$. Hence, we have $(S \circ R)_\alpha(x_1, z) = 1 = (S \circ R)_\alpha(x_2, z)$. Since $(S \circ R)$ is injective, this entails $x_1 = x_2$. Therefore, R is injective.

Theorem 5. *Let R and S be two fuzzy functions. If* $S \circ R$ *is surjective, then S is surjective.*

Proof. Let $z \in U$ be any given. Since $(S \circ R)_\alpha$ is surjective, there exists $x \in U$ such that $(S \circ R)_\alpha(x, z) = (S_\alpha \circ R_\alpha)(x, z) = 1$. This means that there exists $y \in U$ such that $R_\alpha(x, y) \wedge S_\alpha(y, z) = 1$. Hence, we have $R_\alpha(x, y) = 1$ and $S_\alpha(y, z) = 1$. Therefore, S_α is surjective. This completes the proof.

Theorem 6. *Let R and S be two fuzzy functions. If* $S \circ R$ *is injective and R is surjective, then S is injective.*

Proof. Let $\alpha \in \Phi(R)$ be fixed. Then, we show that S_α is injective. Now, let $S_\alpha(y_1, z) = 1 = S_\alpha(y_2, z)$. Since R_α is surjective, for these $y_1, y_2 \in U$ there exist $x_1, x_2 \in U$ such that $R_\alpha(x_1, y_1) = 1 = R_\alpha(x_2, y_2)$. So

$$\begin{aligned}(S \circ R)_\alpha(x_1, z) &= (S_\alpha \circ R_\alpha)(x_1, z)\\ &= \bigvee_{z \in U}[R_\alpha(x_1, y) \wedge S_\alpha(y, z)]\\ &\geq R_\alpha(x_1, y_1) \wedge S_\alpha(y_1, z)\\ &= 1.\end{aligned}$$

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This implies $(S \circ R)_\alpha(x_1, z) = 1$. Similarly, we see that $(S \circ R)_\alpha(x_2, z) = 1$. Since $(S \circ R)_\alpha$ is injective, we have $x_1 = x_2 = x$. Also, from $R_\alpha(x_1, y_1) = 1 = R_\alpha(x_2, y_2)$, we get $R_\alpha(x, y_1) = 1 = R_\alpha(x, y_2)$. This entails that $y_1 = y_2$. Therefore, S_α is injective. This completes the proof.

Theorem 7. *Let R and S be two fuzzy functions. If* $S \circ R$ *is surjective and S is injective, then R is surjective.*

Proof. Let $\alpha \in \Phi(R)$ be fixed. Then, we show that R_α is surjective. Now, let $y \in U$ be any given. Since S_α is an ordinary function, there exists $z \in U$ such that $S_\alpha(y, z) = 1$. By hypothesis, since $S \circ R$ is surjective, $(S \circ R)_\alpha$ is surjective. Hence, for this $z \in U$, there exists $x \in U$ such that $(S \circ R)_\alpha(x, z) = 1$. This implies that there exists $y' \in U$

such that $R_\alpha(x, y') \wedge S_\alpha(y', z) = 1$. This means that $S_\alpha(y', z) = 1$. Since S_α is injective, the equalities $S_\alpha(y, z) = 1 = S_\alpha(y', z)$ imply $y = y'$. Hence, we have $R_\alpha(x, y) = R_\alpha(x, y') = 1$. Therefore, R is surjective. This completes the proof.

Theorem 8. *Let a fuzzy relation R be reflexive. If R is fuzzy difunctional and R_α is anti-symmetric for all $\alpha \in \Phi(R)$, then R is a fuzzy function.*

Proof. Let $\alpha \in \Phi(R)$ be fixed. Then, we show that R_α is an ordinary function. Now, let $x \in U$ be any given. Since R is reflexive, $R(x, x) = 1$, and so, $R(x, x) = 1 \geq \alpha$. This implies that $R_\alpha(x, x) = 1$ for all $\alpha \in \Phi(R)$. Next, let $R_\alpha(x, y_1) = 1$ and $R_\alpha(x, y_2) = 1$. Since R is reflexive, R_α is reflexive. Hence, we have $R_\alpha(y_1, y_1) = 1$ and $R_\alpha(y_2, y_2) = 1$. So $xR_\alpha \cap y_1R_\alpha \neq \varnothing$ and $xR_\alpha \cap y_2R_\alpha \neq \varnothing$. Since R is fuzzy difunctional, we get $xR_\alpha = y_1R_\alpha$ and $xR_\alpha = y_2R_\alpha$. This means that $y_1R_\alpha = y_2R_\alpha$, and so, $y_1 \in y_2R_\alpha$ and $y_2 \in y_1R_\alpha$, from which it follows that $(y_1, y_2) \in R_\alpha$ and $(y_2, y_1) \in R_\alpha$. Since R_α is anti-symmetric, we have $y_1 = y_2$. Therefore, R_α is an ordinary function. This completes the proof.

Theorem 9. *If R is a fuzzy function, then R is fuzzy difunctional.*

Proof. Let $\alpha \in \Phi(R)$ be fixed. Suppose that $xR_\alpha \cap yR_\alpha \neq \varnothing$ and $x, y \in U$. Let $z \in xR_\alpha$. Then $R_\alpha(x, z) = 1$. $xR_\alpha \cap yR_\alpha \neq \varnothing$ implies there exists $z' \in U$ such that $z' \in xR_\alpha \cap yR_\alpha$. This means that $R_\alpha(x, z') = 1$ and $R_\alpha(y, z') = 1$. Since R is a fuzzy function, R_α is an ordinary function. Thus, we have $z = z'$. This means that $R_\alpha(y, z) = 1$, and so, $z \in yR_\alpha$. Hence, we have $xR_\alpha \subseteq yR_\alpha$. Similarly, if $z \in yR_\alpha$, then $R_\alpha(y, z) = 1$. Since $xR_\alpha \cap yR_\alpha \neq \varnothing$, there exists $z' \in U$ such that $z' \in xR_\alpha \cap yR_\alpha$, which implies that $R_\alpha(x, z') = 1$ and $R_\alpha(y, z') = 1$. Combining $R_\alpha(y, z) = 1$ and $R_\alpha(y, z') = 1$, we get $z = z'$, and so, $R_\alpha(x, z) = 1$ and $R_\alpha(x, z') = 1$. This leads $z \in xR_\alpha$. Hence, we have $yR_\alpha \subseteq xR_\alpha$. Therefore, $xR_\alpha = yR_\alpha$. This complete the proof.

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Theorem 10. *Let a fuzzy relation R be reflexive and let R_α be difunctional and anti-symmetric for all $\alpha \in \Phi(R)$. Then R is fuzzy difunctional if and only if R is a fuzzy function.*

Proof. It follows from Theorems 8 and 9.

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