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PRODUCT AND DUAL PRODUCT OF SUBMODULES

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Abstract

For a commutative ring (with identity) R and an R-module M, we introduce the notions of product and dual product of submodules of M and obtain some related results.

1. Introduction

Throughout this paper R denotes a commutative ring with identity and for a submodule N of an R-module M, $\mathrm{Ann}_R^k(N)$ denotes $(\mathrm{Ann}_R(N))^k$. Also $\mathbb{Z}$ denotes the ring of integers.

Now let M be an R-module. In this paper, the concepts of *product* and *coproduct* of submodules of M are introduced and they are used to define *nilpotent*, *conilpotent*, *naturally prime*, and *naturally coprime submodules*. It is shown, among other results, that (see (2.1) (b), 3.9, and 3.10) every naturally prime (resp. naturally coprime) submodule of M is prime (resp. coprime). In this case, we show that the converse is also true when M is a multiplication (resp. comultiplication) module. Also it is proved that (see 3.14) if M is a Noetherian comultiplication R-module, then M has only a finite number of minimal submodules. Finally we obtain a characterization (see (2.1) (f), 3.15, and 3.16) for faithful multiplication domains (resp. comultiplication codomains).

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2. Auxiliary Results

Definition 2.1. (a) An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$.

(b) An R-module M is said to be a *comultiplication module* (see [2]) if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$.

(c) A proper submodule N of an R-module M is said to be *prime* if for any $r \in R$ and any $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$. This implies that $(N :_R M)$ is a prime ideal of R.

(d) A non-zero submodule N of an R-module M is said to be *second* (see [5]) if for each $a \in R$ the homothety $N \xrightarrow{a} N$ is either surjective or zero. This implies that $\mathrm{Ann}_R(N)$ is a prime ideal of R.

(e) Let M be an R-module. The dual notion of $Z(M)$, the set of zero divisors of M, is denoted by $W(M)$ and defined by

$$W(M) = \{a \in R : \text{the homothety } M \xrightarrow{a} M \text{ is not surjective}\}.$$

(f) An R-module M is said to be a *domain* (resp. *codomain*) (see [4]) if $Zd(M) = 0$ (resp. $W(M) = 0$).

(g) Let N be a non-zero (resp. proper) submodule of an R-module M. Then N is said to be *large* (resp. *small*) *submodule* of M (see [1]) if for every non-zero (resp. proper) submodule L of M, $N \cap L \neq 0$ (resp. $L + N \neq M$).

Example 2.2 (see [2]). Let p be a prime number. Then $M = \mathbb{Z}(p^{\infty})$ is a comultiplication $\mathbb{Z}$-module but $\mathbb{Z}$ (as a $\mathbb{Z}$-module) is not a comultiplication module.

Example 2.3. Let p be a prime number. Then the $\mathbb{Z}$-module $\mathbb{Z}(p^{\infty})$ is a codomain.

3. Main Results

Definition 3.1. Let M be an R-module and let T be the set of all submodules of M. Consider the map $\phi : T \times T \to T$ defined by

$$(N, K) \mapsto (N :_R M)(K :_R M)M.$$

Then $\phi(N, K)$ is denoted by NK and called the *product of N and K*.

Definition 3.2. Let M be an R-module and let T be the set of all submodules of M. Consider the map $\phi : T \times T \to T$ defined by

$$(N, K) \mapsto (0 :_M \operatorname{Ann}_R(N)\operatorname{Ann}_R(K)).$$

Then $\phi : (N, K)$ is denoted by $C(NK)$ and called the *coproduct of N and K*.

Example 3.3. For submodules $2\mathbb{Z}$ and $3\mathbb{Z}$ of $\mathbb{Z}$ (as a $\mathbb{Z}$-module), we have $(2\mathbb{Z})(3\mathbb{Z}) = 6\mathbb{Z}$, and $C((2\mathbb{Z})(3\mathbb{Z})) = \mathbb{Z}$.

Definition 3.4. Let M be an R-module. Then a submodule N of M is said to be *nilpotent* (resp. *conilpotent*) if there exists a positive integer k such that $N^k = 0$ (resp. $C(N^k) = M$), where N^k (resp. $C(N^k)$) means the product (resp. coproduct) of N, k times.

Example 3.5. Let p be a prime number and consider $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. Then the submodule $N = \mathbb{Z}(1/p + \mathbb{Z})$ of $\mathbb{Z}(p^\infty)$ is nilpotent but it is not conilpotent. Also the submodule $2\mathbb{Z}$ of $\mathbb{Z}$ is conilpotent but it is not nilpotent.

Theorem 3.6. *Let M be an R-module and let N, H and K be submodules of M. Then we have the following*:

(a) $HN + HK \subseteq H(N + K)$ *and if M is a multiplication R-module, then the equality holds.*

(b) $C(H(N \cap K)) \subseteq C(HN) \cap C(HK)$ *and if M is a comultiplication R-module, then the equality holds.*

(c) *If M is a multiplication module and N is a nilpotent submodule of M, then N is a small submodule of M.*

(d) *If M is a comultiplication module and N is a conilpotent submodule of M, then N is a large submodule of M.*

(e) *If R is an Artinian ring, then $Rad(M)$ (resp. $Soc(M)$) is nilpotent (resp. conilpotent). (Here $Rad(M)$ denotes the intersection of all maximal submodules of M.)*

Proof. (a) It is clear that $HN \subseteq H(N + K)$. This in turn implies that $HN + HK \subseteq H(N + K)$. If M is a multiplication module, then we have

$$\begin{aligned} HN + HK &= (H :_R M)((N :_R M)M + (K :_R M)M) \\ &= (H :_R M)(N + K) \\ &= H(N + K). \end{aligned}$$

(b) It is clear that $C(H(N \cap K)) \subseteq C(HN)$. This in turn implies that

$$C(H(N \cap K)) \subseteq C(HN) \cap C(HK).$$

If M is a comultiplication module, then

$$\begin{aligned} C(HN) \cap C(HK) &= (0 :_M \operatorname{Ann}_R(H)(\operatorname{Ann}_R(N) + \operatorname{Ann}_R(K))) \\ &= (0 :_M \operatorname{Ann}_R(H)\operatorname{Ann}_R(N \cap K)) \\ &= C(H(N \cap K)). \end{aligned}$$

(c) Let N be a nilpotent submodule of multiplication R-module M and let $N + K = M$, where K is a submodule of M. Since N is nilpotent, there exists a positive integer n such that $N^n = 0$. Now we have $KN + N^2 = NM$ by part (a). Since M is a multiplication module, $KN + N^2 = N$. So $N^2 + K = M$. This in turn implies that $N^n + K = M$. Thus $M = K$ as desired.

(d) Let N be a conilpotent submodule of comultiplication R-module M and let $N \cap K = 0$, where K is a submodule of M. Since N is conilpotent there exists a positive integer n such that $C(N^n) = M$. Now we have $C(KN) \cap C(N^2) = C(N0)$ by part (b). Since M is a comultiplication module, $C(KN) \cap C(N^2) = N$. So $C(N^2) \cap K = 0$. This in turn implies that $C(N^n) \cap K = 0$. Thus $K = 0$ as desired.

(e) Since R is an Artinian ring, there exists a positive integer n such that $(J(R))^n = 0$. Further we have $\mathrm{Rad}(M) = J(R)M$ by [1, 15.18]. Thus

$$(\mathrm{Rad}(M))^n = (\mathrm{Rad}(M) :_R M)^n M = (J(R)M :_R M)^n M = (J(R))^n M = 0.$$

To see the second assertion, we have $\mathrm{Soc}_R(M) = (0 :_M J(R))$ by [1, 15.17]. Hence

$$\begin{aligned} C((\mathrm{Soc}(M))^n) &= (0 :_M \mathrm{Ann}_R^n(\mathrm{Soc}(M))) \\ &= (0 :_M \mathrm{Ann}_R^n(0 :_M J(R))) \\ &= (0 :_M (J(R))^n) = M. \end{aligned}$$

So the proof is completed.

Lemma 3.7. *Let M be an R-module. Then the following statements are equivalent*:

(a) 0 *is the only nilpotent submodule of M.*

(b) *For all submodules N, K of M with* $NK = 0$, *we have* $N \cap K = 0$.

Proof. (a) $\Rightarrow$ (b) If N and K are submodules of M with $NK = 0$, then

$$(N \cap K)^2 = (N \cap K :_R M)^2 M \subseteq (N :_R M)(K :_R M)M = NK = 0.$$

Thus $(N \cap K)^2 = 0$. Hence by (a), we have $N \cap K = 0$.

(b) $\Rightarrow$ (a) This is clear.

Lemma 3.8. *Let M be an R-module. Then the following statements are equivalent*:

(a) *M is the only conilpotent submodule of M.*

(b) *For all submodules N, K of M with* $C(NK) = M$, *we have* $N + K = M$.

Proof. (a) $\Rightarrow$ (b) If N and K are submodules of M with $C(NK) = M$, then

$$\begin{aligned} C(N+K)^2 &= (0 :_M \operatorname{Ann}_R^2(N+K)) \\ &\supseteq (0 :_M \operatorname{Ann}_R(N)\operatorname{Ann}_R(K)) \\ &= C(NK) = M. \end{aligned}$$

Thus $C(N+K)^2 = M$. Hence by (a), we have $N + K = M$.

(b) $\Rightarrow$ (a) This is clear.

Definition 3.9. Let M be an R-module. Then a proper (resp. non-zero) submodule H of M is said to be *naturally prime* (resp. *naturally coprime*), if for submodules N and K of M, the relation $NK \subseteq H$ (resp. $H \subseteq C(NK)$) implies that $N \subseteq H$ (resp. $H \subseteq N$) or $K \subseteq H$ (resp. $H \subseteq K$).

Theorem 3.10. *Let M be an R-module. Then we have the following*:

(a) *If P is a naturally prime submodule of M, then P is a prime submodule of M. Furthermore, if M is a multiplication R-module, then the converse is true.*

(b) *If S is a naturally coprime submodule of M, then S is a second submodule of M. Furthermore, if M is a comultiplication R-module, then the converse is true.*

Proof. (a) Suppose that $rm \in P$, where $r \in R$, $m \in M$. Then $rRm \subseteq P$. Hence $(Rm :_R M)rM \subseteq P$. Thus $(Rm :_R M)(rM :_R M)M \subseteq P$. This implies that $Rm \subseteq P$ or $rM \subseteq P$ because P is naturally prime. Therefore, $m \in P$ or $r \in (P :_R M)$ as desired. Now assume that M is a multiplication module and P is a prime submodule of M. Suppose that $NK \subseteq P$, where N and K are submodules of M and $K \not\subseteq P$. Hence there exists $x \in K$ such that $x \notin P$. Now $(N :_R M)Rx \subseteq P$ implies that $(N :_R M)M \subseteq (P :_R M)M$. Thus $N \subseteq P$ as desired.

(b) Suppose that S is a naturally coprime submodule of M and $r \in R$. Then

$$\begin{aligned} S &\subseteq (0 :_M r\mathrm{Ann}_R(rS)) \\ &= ((0 :_M r) :_M \mathrm{Ann}_R(rS)) \\ &= (0 :_M \mathrm{Ann}((0 :_M r))\mathrm{Ann}_R(rS)). \end{aligned}$$

This implies that $S \subseteq (0 :_M r)$ or $S \subseteq rS$. Thus $rS = 0$ or $rS = S$ as desired. Now assume that M is a comultiplication module and S is a second submodule of M. Suppose that $S \subseteq C(NK)$, where N and K are submodules of M and $S \not\subseteq K$. These follow that $(\mathrm{Ann}_R(K))S \subseteq N$ and $(\mathrm{Ann}_R(K))S \neq 0$. Now since S is second, $(\mathrm{Ann}_R(K))S = S$. Therefore, $S \subseteq N$ as desired.

Corollary 3.11. *Let M be an R-module. If P is naturally prime (resp. naturally coprime) submodule of M, then $(P :_R M)$ (resp. $Ann_R(P)$) is a prime ideal of R.*

Proof. Use 3.10 (a) and 2.1 (c) (resp. 3.10 (b) and 2.1 (d)).

Example 3.12. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_p$ as a $\mathbb{Z}$-module, where p is a prime number. Then $N := p\mathbb{Z} \oplus \mathbb{Z}_p$ is a maximal submodule of M. Hence N is a prime submodule of M. But N is not naturally prime submodule of M because

$$(\mathbb{Z} \oplus 0)^2 = p^2\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z}_p) = p^2\mathbb{Z} \oplus 0 \subseteq p\mathbb{Z} \oplus \mathbb{Z}_p,$$

whereas $\mathbb{Z} \oplus 0 \not\subseteq p\mathbb{Z} \oplus \mathbb{Z}_p$.

Example 3.13. Let p be a prime number and consider $M = \mathbb{Z}(p^\infty) \oplus p\mathbb{Z}$ as a $\mathbb{Z}$-module. For a submodule $N = \mathbb{Z}(1/p + \mathbb{Z})$ of $\mathbb{Z}(p^\infty)$, the submodule $N \oplus 0$ of M is a minimal. Hence $N \oplus 0$ is a second submodule of M by [5, 1.6]. But $N \oplus 0$ is not naturally coprime submodule of M because

$$N \oplus 0 \subseteq \mathbb{Z}(p^\infty) \oplus p\mathbb{Z} = (0 :_{\mathbb{Z}(p^\infty)\oplus p\mathbb{Z}} 0) = C((0 \oplus p\mathbb{Z})^2),$$

while $N \oplus 0 \not\subseteq 0 \oplus p\mathbb{Z}$.

Theorem 3.14. *Let M be a Noetherian comultiplication R-module. Then M has only a finite number of minimal submodules.*

Proof. Consider the set of all submodules $\sum_{i=1}^{r} K_i$, where each K_i is a minimal submodule of *M*. This set is non-empty by [3, 3.2], so it has a maximal element, say $\sum_{i=1}^{n} K_i$. Hence for any minimal submodule *N*, we have $N + \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} K_i$. Now by 3.10 (b), *N* is naturally coprime because it is a second module by [5, 1.6]. This in turn implies that $N \subseteq K_i$ for some *i*. Since K_i is minimal, $N = K_i$ and the proof is completed.

Theorem 3.15. *Let M be a faithful multiplication R-module. Then the following are equivalent*:

(a) *M is a domain.*

(b) $KN = 0$ *implies that* $N = 0$ *or* $K = 0$, *where* N *and* K *are submodules of M.*

Proof. (a) $\Rightarrow$ (b) Assume that $NK = 0$. If $K = 0$, then we are done. If $K \neq 0$, then there exists $0 \neq m \in K = (K :_R M)M$. Now since *M* is a domain $(N :_R M)m = 0$ implies that $(N :_R M) = 0$. Thus $N = 0$ as desired.

(b) $\Rightarrow$ (a) Suppose that $0 \neq r \in Zd(M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$. This in turn implies that $0 = Rrm = (Rm)(rM)$. Hence by part (b), $Rm = 0$ or $rM = 0$. Since M is faithful, $m = 0$. This contradiction shows that $Zd(M) = 0$ and the proof is completed.

Theorem 3.16. *Let M be a faithful comultiplication R-module. Then the following are equivalent*:

(a) *M is a codomain.*

(b) $C(NK) = M$ *implies that* $N = M$ *or* $K = M$, *where N and K are submodules of M.*

Proof. (a) $\Rightarrow$ (b) Assume that $C(NK) = M$. Hence

$$M = (0 :_M \operatorname{Ann}_R(N)\operatorname{Ann}_R(K)) = (N :_M \operatorname{Ann}_R(K)),$$

so that $(\mathrm{Ann}_R(K))M \subseteq N$. If $(\mathrm{Ann}_R(K))M = M$, then $N = M$ and we are done. If $(\mathrm{Ann}_R(K))M \neq M$, then $\mathrm{Ann}_R(K) \subseteq W(M) = 0$. Hence $K = M$ as desired.

(b) $\Rightarrow$ (a) Assume that $0 \neq r \in W(M)$. So $rM \neq M$. Then we have

$$\begin{aligned} M &= (0 :_M r\mathrm{Ann}_R(rM)) \\ &= ((0 :_M r) :_M \mathrm{Ann}_R(rM)) \\ &= (0 :_M \mathrm{Ann}_R((0 :_M r))\mathrm{Ann}_R(rM)). \end{aligned}$$

Now by part (b), $rM = M$ or $(0 :_M r) = M$. Since M is faithful, it follows that $rM = M$ which is a contradiction. Hence $W(M) = 0$ and the proof is completed.

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