Far East J. Math. Sci. (FJMS) **25**(3) (2007), 417-431
This paper is available online at http://www.pphmj.com

C^r STRONG CELL DECOMPOSITIONS IN NON-VALUATIONAL WEAKLY o-MINIMAL REAL CLOSED FIELDS

HIROSHI TANAKA and TOMOHIRO KAWAKAMI

(Received February 8, 2007)

Abstract

A structure $\mathcal{R} = (R, <, \ldots)$ equipped with a dense linear ordering $<$ without endpoints is said to be *o-minimal* (*weakly o-minimal*) if every definable subset of R is a finite union of intervals (convex sets), respectively. A weakly o-minimal structure $\mathcal{R} = (R, <, +, \ldots)$ expanding an ordered group $(R, <, +)$ is said to be *non-valuational* if for every cut $\langle C, D\rangle$ definable in $\mathcal{R}$ we have that $\inf\{y - x : x \in C, y \in D\} = 0$. L. van den Dries proved that every o-minimal expansion of a real closed field admits a C^r cell decomposition for each positive integer r. In this paper, we prove the non-valuational weakly o-minimal version of it.

1. Introduction

Weak o-minimality was introduced by Dickmann (see [3]). He showed that every real closed ring is weakly o-minimal in the language $L = \{<, +, -, \cdot, 0, 1, \mathrm{Div}\}$, where the symbol "Div" is interpreted as $x \,\mathrm{Div}\, y \Leftrightarrow \exists z(y = xz)$. After that several fundamental results of weakly o-minimality were proved by Macpherson et al. in [6].

2000 Mathematics Subject Classification: 03C64, 14P10, 14P20.

Keywords and phrases: weakly o-minimal, C^r strong cells, C^r strong cell decompositions, real closed fields.

Non-valuational weakly o-minimal expansions of ordered groups and ordered fields were studied by Macpherson et al. in [6], by Wencel in [7], and by Dolich in [4]. Now, it is known that the model theory of weakly o-minimal structures does not develop as smoothly as that of o-minimal structures, see [6]. However non-valuational weakly o-minimal expansions of ordered groups are very similar to o-minimal structures. In particular, Wencel showed that every non-valuational weakly o-minimal expansion of an ordered group admits an o-minimal style cell decomposition (say *strong cell decomposition*) in [7].

On the other hand, differentiability and analyticity properties of definable functions for weakly o-minimal expansions of real closed fields are scarcely studied (see [6, Open problem 3]). In this paper, we study differentiability properties of definable functions for non-valuational weakly o-minimal expansions of real closed fields. Consequently, we prove that each definable function in one variable for non-valuational weakly o-minimal expansions of real closed fields is piecewise differentiable (Proposition 3.1). Moreover, we prove that every non-valuational weakly o-minimal expansion of a real closed field admits a C^r strong cell decomposition for each positive integer r (Theorem 2.10).

Throughout this paper, "definable" means "definable possibly with parameters" and we assume that a structure $\mathcal{R} = (R, <, \ldots)$ is a dense linear ordering $<$ without endpoints. The set of positive integers is denoted by $\mathbb{N}$. The reader is assumed to be familiar with fundamental results of o-minimality; see, for example, [5] or [2].

2. Preliminaries and the Main Theorem

In this section, we introduce some definitions and facts for weakly o-minimal structures and state our main theorem.

A subset A of R is said to be *convex* if $a, b \in A$ and $c \in R$ with $a < c < b$, then $c \in A$. Moreover if $A = \emptyset$ or $\inf A, \sup A \in R \cup \{-\infty, +\infty\}$, then A is called an *interval* in R. We say that $\mathcal{R}$ is *o-minimal* (*weakly o-minimal*) if every definable subset of R is a finite union of intervals (convex sets), respectively.

For any subsets C, D of R, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D \rangle$ of non-empty subsets of R is called a *cut* in $\mathcal{R}$ if $C < D$, $C \cup D = R$ and D has no lowest element. A cut $\langle C, D \rangle$ is said to be *definable* in $\mathcal{R}$ if the sets C, D are definable in $\mathcal{R}$. The set of all cuts definable in $\mathcal{R}$ will be denoted by $\overline{R}$. Note that we have $R = \overline{R}$ if $\mathcal{R}$ is o-minimal. We define a linear ordering on $\overline{R}$ by $\langle C_1, D_1 \rangle < \langle C_2, D_2 \rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(R, <)$ as a substructure of $(\overline{R}, <)$ by identifying an element $a \in R$ with the definable cut $\langle (-\infty, a], (a, +\infty) \rangle$. We equip R $(\overline{R})$ with the *interval topology* (the open intervals form a base), and each product R^n $((\overline{R})^n)$ with the corresponding product topology, respectively.

Recall the notion of definable functions from [7]. Let $n \in \mathbb{N}$ and $A \subset R^n$ definable. A function $f : A \to \overline{R}$ is said to be *definable* if the set $\Gamma_<(f) := \{\langle x, y \rangle \in R^{n+1} : x \in A, y < f(x)\}$ is definable. A function $f : A \to \overline{R} \cup \{-\infty, +\infty\}$ is said to be *definable* if f is a definable function from A to $\overline{R}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $A \subset R^n$ definable. Suppose that $f : A \to R$ is a function. Then the following conditions are equivalent.*

(1) *The function f is definable.*

(2) *The graph $\Gamma(f) := \{\langle x, y \rangle \in R^{n+1} : x \in A, f(x) = y\}$ is definable.*

Proof. (1) $\Rightarrow$ (2) Since A and f are definable, there exist $L(R)$-formulas $\varphi(x)$ and $\psi(x, y)$ such that $A = \varphi(\mathcal{R})$ and $\Gamma_<(f) = \psi(\mathcal{R})$. Let $\psi'(x, y) \equiv \varphi(x) \wedge \neg\psi(x, y) \wedge \forall z(z < y \to \psi(x, z))$. Then we obtain $\Gamma(f) = \psi'(\mathcal{R})$, as desired.

(2) $\Rightarrow$ (1) Since $\Gamma(f)$ is definable, there exists some $L(R)$-formula $\theta(x, y)$ such that $\Gamma(f) = \theta(\mathcal{R})$. Let $\theta'(x, y) \equiv \varphi(x) \wedge \exists z(y < z \wedge \theta(x, z))$. Then we obtain $\Gamma_<(f) = \theta'(\mathcal{R})$. Thus, the function f is definable. □

Let $n \in \mathbb{N}$ and $A \subset R^n$ an infinite definable set. The *dimension* of A, denoted by $\dim(A)$, is the largest r for which there exists some projection $\pi : R^n \to R^r$ such that $\pi(A)$ contains an open box, where an open box in R^n is the Cartesian product $(a_1, b_1) \times \cdots \times (a_n, b_n)$ of open intervals. Non-empty finite sets are said to have *dimension* 0. The empty set is said to have *dimension* $-\infty$. We will use the convention that if $d \in \mathbb{N} \cup \{0, -\infty\}$, then $d \geq -\infty$ and $d + (-\infty) = -\infty + d = -\infty$.

Lemma 2.2 [8, Fact 1.6]**.** *Let $\mathcal{R} = (R, <, \ldots)$ be a weakly o-minimal structure. Suppose that $m, n \in \mathbb{N}$ and $A, B \subset R^m$, $C \subset R^n$ are definable sets.*

(1) *If $A \subset B$, then* $\dim(A) \leq \dim(B)$.

(2) *If $k \in \{1, \ldots, m\}$ and $\pi : R^m \to R^k$ is a projection, then*

$$\dim(A) - (m - k) \leq \dim(\pi(A)) \leq \dim(A).$$

(3) *If $f : R^m \to R^m$ is a permutation of variables, then*

$$\dim(f(A)) = \dim(A).$$

(4) $\dim(A \times C) = \dim(A) + \dim(C)$.

(5) $\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$.

Let $\mathcal{R} = (R, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered group $(R, <, +)$. By [6, Theorem 5.1), the structure $\mathcal{R}$ is divisible and abelian. A cut $\langle C, D \rangle$ is said to be *non-valuational* if $\inf\{y - x : x \in C, y \in D\} = 0$. We say that the structure $\mathcal{R}$ is *non-valuational* if all cuts definable in $\mathcal{R}$ are non-valuational.

Let $\mathcal{R} = (R, <, +, \cdot, \ldots)$ be a non-valuational weakly o-minimal expansion of an ordered field $(R, <, +, \cdot)$. By [6, Theorem 5.3], the structure $\mathcal{R}$ is real closed. For any subsets A, B of R, we define $A + B := \{x + y : x \in A, y \in B\}$, $A - B := \{x - y : x \in A, y \in B\}$, $A \cdot B := \{x \cdot y : x \in A, y \in B\}$ and $-A := \{-x : x \in A\}$. Note that $C - D = (-\infty, 0)$ and

$D - C = (0, +\infty)$ for any element $\langle C, D\rangle$ of $\overline{R}$. Also notice that for any positive element $\langle C, D\rangle$ of $\overline{R}$, we have $\inf\{x^{-1} \cdot y : x \in C \cap (0, +\infty), y \in D\} = 1$, and for any negative element $\langle C, D\rangle$ of $\overline{R}$, we have $\inf\{x \cdot y^{-1} : x \in C, y \in D \cap (-\infty, 0)\} = 1$. For any elements $\langle C_1, D_1\rangle$, $\langle C_2, D_2\rangle$ of $\overline{R}$, we define

$$\langle C_1, D_1\rangle + \langle C_2, D_2\rangle := \langle R\setminus(D_1 + D_2), D_1 + D_2\rangle,$$

$$\langle C_1, D_1\rangle \cdot \langle C_2, D_2\rangle$$

$$:= \begin{cases} \langle R\setminus(D_1 \cdot D_2), D_1 \cdot D_2\rangle & \text{if } 0 \in C_1 \text{ and } 0 \in C_2 \\ \langle \mathrm{cl}(-((-C_1)\cdot D_2)), R\setminus\mathrm{cl}(-((-C_1)\cdot D_2))\rangle & \text{if } 0 \in D_1 \text{ and } 0 \in C_2 \\ \langle \mathrm{cl}(-(D_1 \cdot (-C_2))), R\setminus\mathrm{cl}(-(D_1 \cdot (-C_2)))\rangle & \text{if } 0 \in C_1 \text{ and } 0 \in D_2 \\ \langle R\setminus\mathrm{int}(C_1 \cdot C_2), \mathrm{int}(C_1 \cdot C_2)\rangle & \text{if } 0 \in D_1 \text{ and } 0 \in D_2, \end{cases}$$

where for each $n \in \mathbb{N}$ the topological closure in R^n of a set $A \subset R^n$ is denoted by $\mathrm{cl}(A)$, and its topological interior in R^n by $\mathrm{int}(A)$. Then the structure $(\overline{R}, <, +, \cdot)$ is an ordered field and the structure $(R, <, +, \cdot)$ is a subfield of it.

Recall the notion of the strong monotonicity from [7].

Definition 2.3. We say that a weakly o-minimal structure $\mathcal{R} = (R, <, \ldots)$ has *the strong monotonicity* if for each definable set $I \subset R$ and each definable function $f : I \to \overline{R}$, there exists a partition of I into a finite set X and definable convex open sets $I_1, \ldots, I_k$ such that for each $i \in \{1, \ldots, k\}$, one of the following conditions holds.

(1) $f | I_i$ is constant.

(2) $f | I_i$ is strictly increasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in R$ with $f(a) < c < d < f(b)$, there exists some $x \in (a, b)$ such that $c < f(x) < d$; in particular $f | I_i$ is continuous.

(3) $f | I_i$ is strictly decreasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in R$ with $f(a) > c > d > f(b)$, there exists some $x \in (a, b)$ such that $c > f(x) > d$; in particular $f | I_i$ is continuous.

Theorem 2.4 [7, Lemma 1.4]. *Suppose that* $\mathcal{R} = (R, <, +, \ldots)$ *is a weakly o-minimal expansion of an ordered group* $(R, <, +)$. *Then the following conditions are equivalent.*

(1) $\mathcal{R}$ *is non-valuational.*

(2) $\mathcal{R}$ *has the strong monotonicity.*

Let $\mathcal{R} = (R, <, +, \cdot, \ldots)$ be a non-valuational weakly o-minimal expansion of a real closed field $(R, <, +, \cdot)$. Let $m, n \in \mathbb{N}$. Suppose that $U \subset R^n$ is a definable open set and $f_i : U \to \overline{R}$ is definable for each $i \in \{1, \ldots, m\}$. Let $f := (f_1, \ldots, f_m) : U \to (\overline{R})^m$. We call f a C^1 *map* if f is continuous and for each $i \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, n\}$, the function $\partial f_i / \partial x_j$ is defined as a $\overline{R}$-valued function on U and is continuous. For each integer $r > 1$, the map f is called a C^r *map* if f is a C^1 map and $\partial f_i / \partial x_j$ is a C^{r-1} map for each $i \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, n\}$. Suppose that $A \subset R^n$ is definable (not necessarily open) and $g_i : A \to \overline{R}$ is definable for each $i \in \{1, \ldots, m\}$. For each positive integer r, we call $g := (g_1, \ldots, g_m)$ a C^r *map* if there exist a definable open set $U \subset R^n$ containing A and definable C^r functions $G_1, \ldots, G_m : U \to \overline{R}$ such that $G_i | A = g_i$ for each $i \in \{1, \ldots, m\}$.

Recall the notion of strong cells from [7].

Definition 2.5. Suppose that $\mathcal{R} = (R, <, \ldots)$ is a weakly o-minimal structure. For each $n \in \mathbb{N}$, we inductively define *strong cells* in R^n and their completions in $(\overline{R})^n$.

(1) A one-element subset of R is called a *strong* $\langle 0 \rangle$*-cell* in R. If $C \subset R$ is a strong $\langle 0 \rangle$-cell, then its completion $\overline{C} := C$.

(2) A non-empty definable convex open subset of R is called a *strong* $\langle 1 \rangle$*-cell* in R. If $C \subset R$ is a strong $\langle 1 \rangle$-cell, then its completion $\overline{C} := \{x \in \overline{R} : \text{there exist } a, b \in C \text{ such that } a < x < b\}$.

Assume that $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \{0, 1\}$, and strong $\langle i_1, \ldots, i_n \rangle$-cells in R^n and their completions in $(\overline{R})^n$ are already defined.

(3) Let $C \subset R^n$ be a strong $\langle i_1, \ldots, i_n \rangle$-cell in R^n and $f : C \to R$ is a definable continuous function which has a continuous extension $\bar{f} : \overline{C} \to \overline{R}$. Then the graph $\Gamma(f)$ is called a *strong* $\langle i_1, \ldots, i_n, 0 \rangle$-*cell* in R^{n+1} and its completion $\overline{\Gamma(f)} := \Gamma(\bar{f})$.

(4) Let $C \subset R^n$ be a strong $\langle i_1, \ldots, i_n \rangle$-cell in R^n and $g, h : C \to \overline{R} \cup \{-\infty, +\infty\}$ are definable continuous functions which have continuous extensions $\overline{g}, \overline{h} : \overline{C} \to \overline{R} \cup \{-\infty, +\infty\}$ such that $\overline{g}(x) < \overline{h}(x)$ for all $x \in \overline{C}$. Then the set

$$(g, h)_C := \{\langle a, b \rangle \in R^{n+1} : a \in C,\, g(a) < b < h(a)\}$$

is called a *strong* $\langle i_1, \ldots, i_n, 1 \rangle$-*cell* in R^{n+1}. The completion of $(g, h)_C$ is defined as

$$\overline{(g, h)_C} := \{\langle a, b \rangle \in (\overline{R})^{n+1} : a \in \overline{C},\, \overline{g}(a) < b < \overline{h}(a)\}.$$

(5) Let C be a subset of R^n. The set C is called a *strong cell* in R^n if there exist $i_1, \ldots, i_n \in \{0, 1\}$ such that C is a strong $\langle i_1, \ldots, i_n \rangle$-cell in R^n. If additionally $r \in \mathbb{N}$, the structure $\mathcal{R}$ is a non-valuational weakly o-minimal expansion of a real closed field and f, g, h of (3), (4) are C^r functions, then the set C is called a C^r *strong cell* in R^n.

Let $\mathcal{R} = (R, <, \ldots)$ be a weakly o-minimal structure, $n \in \mathbb{N}$ and C be a strong cell of R^n. A definable function $f : C \to \overline{R}$ is said to be *strongly continuous* if f has a continuous extension $\bar{f} : \overline{C} \to \overline{R}$. A function which is identically equal to $-\infty$ or $+\infty$, and whose domain is a strong cell is also said to be *strongly continuous*.

Definition 2.6. Let $\mathcal{R} = (R, <, \ldots)$ be a weakly o-minimal structure. For each $n \in \mathbb{N}$, we inductively define a *strong cell decomposition* (or a *decomposition into strong cells* in R^n) of a non-empty definable set $A \subset R^n$.

(1) If $A \subset R$ is a non-empty definable set and $\mathcal{D} = \{C_1, \dots, C_k\}$ is a partition of A into strong cells in R, then $\mathcal{D}$ is called a *decomposition of A into strong cells* in R.

(2) Suppose that $A \subset R^{n+1}$ is a non-empty definable set and $\mathcal{D} = \{C_1, \dots, C_k\}$ is a partition of A into strong cells in R^{n+1}. Then $\mathcal{D}$ is called a *decomposition of A into strong cells* in R^{n+1} if $\{\pi(C_1), \dots, \pi(C_k)\}$ is a decomposition of $\pi(A)$ into strong cells in R^n, where $\pi : R^{n+1} \to R^n$ is the projection on the first n coordinates.

Let $\mathcal{R} = (R, <, +, \cdot, \dots)$ be a non-valuational weakly o-minimal expansion of a real closed field $(R, <, +, \cdot)$ and $n, r \in \mathbb{N}$. Then in a similar way to the above definition, we define a C^r *strong cell decomposition* of a non-empty definable set $A \subset R^n$.

Definition 2.7. Let $\mathcal{R} = (R, <, \dots)$ be a weakly o-minimal structure and $n \in \mathbb{N}$. Suppose that $A, B \subset R^n$ are definable sets, $A \neq \emptyset$ and $\mathcal{D}$ is a decomposition of A into strong cells in R^n. We say that $\mathcal{D}$ *partitions* B if for each strong cell $C \in \mathcal{D}$, we have either $C \subset B$ or $C \cap B = \emptyset$.

Definition 2.8. A weakly o-minimal structure $\mathcal{R} = (R, <, \dots)$ is said to have the *strong cell decomposition* if for any $k, n \in \mathbb{N}$ and any definable sets $A_1, \dots, A_k \subset R^n$, there exists a decomposition of R^n into strong cells partitioning each of the sets $A_1, \dots, A_k$.

Let $\mathcal{R} = (R, <, +, \cdot, \dots)$ be a non-valuational weakly o-minimal expansion of a real closed field $(R, <, +, \cdot)$ and $r \in \mathbb{N}$. Then in a similar way to the above definition, we define the C^r *strong cell decomposition*.

Theorem 2.9 [7, Theorem 2.14]**.** *Let* $\mathcal{R} = (R, <, +, \dots)$ *be a non-valuational weakly o-minimal expansion of an ordered group* $(R, <, +)$ *and* $n \in \mathbb{N}$.

(1) *For any* $k \in \mathbb{N}$ *and any definable sets* $A_1, \ldots, A_k \subset R^n$, *there exists a decomposition of* R^n *into strong cells partitioning each of the sets* $A_1, \ldots, A_k$.

(2) *For any definable set* $A \subset R^n$ *and any definable function* $f : A \to \overline{R}$, *there exists a decomposition of A into strong cells such that the restriction* $f \mid C : C \to \overline{R}$ *is strongly continuous for each* $C \subset A$ *of the decomposition.*

The following is the main theorem of this paper.

Theorem 2.10 (C^r strong cell decompositions)**.** *Let* $\mathcal{R} = (R, <, +, \cdot, \ldots)$ *be a non-valuational weakly o-minimal expansion of a real closed field* $(R, <, +, \cdot)$ *and* $n, r \in \mathbb{N}$.

(1) *For any* $k \in \mathbb{N}$ *and any definable sets* $A_1, \ldots, A_k \subset R^n$, *there exists a decomposition of* R^n *into* C^r *strong cells partitioning each of the sets* $A_1, \ldots, A_k$.

(2) *For any definable set* $A \subset R^n$ *and any definable function* $f : A \to \overline{R}$, *there exists a decomposition of A into* C^r *strong cells such that the restriction* $f \mid C : C \to \overline{R}$ *is* C^r *and strongly continuous for each* $C \subset A$ *of the decomposition.*

The o-minimal version of our main theorem was proved by L. van den Dries [5, pp. 115-116].

3. C^r Strong Cell Decompositions

Throughout this section, we assume that $\mathcal{R} = (R, <, +, \cdot, \ldots)$ is a non-valuational weakly o-minimal expansion of a real closed field $(R, <, +, \cdot)$. In this section, we prove our main theorem.

We first prove the following proposition.

Proposition 3.1. *Let I be a non-empty definable subset of R. Let*

$f : I \to \overline{R}$ be a definable function. Then f is differentiable at all but finitely many points of I.

This requires several lemmas.

Lemma 3.2 [7, Lemma 1.2]. *Let $I \subset R$ be a non-empty definable convex open set and $f : I \to \overline{R}$ be a definable function. Then the limits $\lim_{x \to \sup I - 0} f(x)$ and $\lim_{x \to \inf I + 0} f(x)$ exist in $\overline{R} \cup \{-\infty, +\infty\}$.*

Let I be a non-empty definable open subset of R and $f : I \to \overline{R}$ be a definable function. For each $x \in I$, we define the limits

$$f'_+(x) := \lim_{t \to +0} \frac{f(x+t) - f(x)}{t},$$

$$f'_-(x) := \lim_{t \to -0} \frac{f(x+t) - f(x)}{t},$$

$$f'(x) := \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}.$$

Lemma 3.3. *Let $I \subset R$ be a non-empty definable convex open set and $f : I \to \overline{R}$ be a definable function. Then, for each $x \in I$ the limits $f'_+(x)$ and $f'_-(x)$ exist in $\overline{R} \cup \{-\infty, +\infty\}$.*

Proof. Suppose that $x \in I$. We define $g(t) := t^{-1}(f(x+t) - f(x))$ on an open interval $(0, \varepsilon)$. Then by Lemma 3.2, the limit $f'_+(x) = \lim_{t \to +0} g(t)$ exists in $\overline{R} \cup \{-\infty, +\infty\}$. Similarly, the limit $f'_-(x)$ exists in $\overline{R} \cup \{-\infty, +\infty\}$. □

Let I be a non-empty definable convex open subset of R. Let $f : I \to R$ be a definable continuous function and $f'_+(x) > 0$ for each $x \in I$. If $\mathcal{R}$ is o-minimal, then the function *f* is strictly increasing on *I*, see [5, p. 109]. However in the weakly o-minimal setting, the function *f* is not necessarily strictly increasing on *I*.

Example 3.4. Let R_{alg} be the real closure of $\mathbb{Q}$. We consider $\mathcal{R}_1 = (R_{\text{alg}}, <, +, \cdot, P)$, where the unary predicate symbol *P* is interpreted by the convex set $(-\pi, \pi) \cap R_{\text{alg}}$. By [1], the structure $\mathcal{R}_1$ is weakly o-minimal. Suppose that a definable function $f : (0, 4) \to R_{\text{alg}}$ is defined by $f(x) = x$ for all $x \in (0, \pi) \cap R_{\text{alg}}$ and $f(x) = x - 1$ for all $x \in (\pi, 4) \cap R_{\text{alg}}$. Then $f'_+(x) > 0$ for each $x \in (0, 4)$ and f is continuous on the open interval $(0, 4)$. However f is not strictly increasing on the open interval $(0, 4)$.

Let $I \subset R$ be a closed bounded interval. Let $f : I \to \overline{R}$ be a definable strongly continuous function. If $\mathcal{R}$ is o-minimal, then the function f takes a maximum and a minimum value on I, see [5, p. 46]. However in the weakly o-minimal setting, the function f does not necessarily attain a maximum and a minimum value on I.

Example 3.5. We consider $\mathcal{R}_1 = (R_{\text{alg}}, <, +, \cdot, P)$ of Example 3.4. Suppose that a definable function $f : [0, 4] \to \overline{R}_{\text{alg}}$ is defined by $f(x) = x$ for all $x \in [0, \pi) \cap R_{\text{alg}}$ and $f(x) = -x + 2\pi$ for all $x \in (\pi, 4] \cap R_{\text{alg}}$. Then f is strongly continuous on the closed interval $[0, 4]$. However f does not have a maximum value on the closed interval $[0, 4]$.

Examples 3.4 and 3.5 also show that the Mean Value Theorem does not hold in general in the weakly o-minimal setting.

Lemma 3.6. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable strongly continuous function. Suppose that either $f'_+(x) > 0$ for each $x \in I$ or $f'_-(x) > 0$ each $x \in I$. Then f is strictly increasing on I. Similarly, if either $f'_+(x) < 0$ for each $x \in I$ or $f'_-(x) < 0$ for each $x \in I$, then f is strictly decreasing on I.*

Proof. We will show that if $f'_+(x) > 0$ for each $x \in I$, then f is strictly increasing on I. The other cases are similar. Suppose for a contradiction that there exist some $a, b \in I$ with $a < b$ such that $f(a) \geq f(b)$. Since $f'_+(x) > 0$ for each $x \in I$, by Theorem 2.4, there exists a partition of I into a finite set X and definable convex open sets $I_1, \ldots, I_k$ such that for each $i \in \{1, \ldots, k\}$, the restriction $f \mid I_i$ is strictly increasing. Because f is continuous, we may assume that $X = \varnothing$. We may also assume that

$a \in I_1$ and $b \in I_2$. Now $\lim_{x \to \sup I_1 - 0} f(x) > f(a) \geq f(b) > \lim_{x \to \inf I_2 + 0} f(x)$. This contradicts strong continuity of f. □

Lemma 3.7. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable strongly continuous function. Suppose that the functions f'_+ and f'_- are $\overline{R}$-valued and continuous on I. Then f is differentiable at each point of I and $f' : I \to \overline{R}$ is continuous.*

Proof. It suffices to prove that $f'_+(a) = f'_-(a)$ for all $a \in I$. Suppose for a contradiction that there exists some $a \in I$ such that $f'_+(a) \neq f'_-(a)$. Without loss of generality, we may assume $f'_+(a) > f'_-(a)$. Then there exist $c \in R$ and an open subinterval J of I around a such that $f'_+(x) > c > f'_-(x)$ for each $x \in J$. Thus the definable function $g : J \to \overline{R}$ defined by $g(x) := f(x) - cx$ satisfies $g'_+(x) > 0$ and $g'_-(x) < 0$ for each $x \in J$. Also, the function g is strongly continuous on J. Hence, by Lemma 3.6, the function g is both strictly increasing and strictly decreasing on J, a contradiction. □

Lemma 3.8. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable strongly continuous function. Then there exist only finitely many $x \in I$ such that $f'_+(x)$, $f'_-(x) \in \{-\infty, +\infty\}$.*

Proof. Suppose that the definable set $\{x \in I : f'_+(x) = +\infty\}$ is infinite. Then by weak o-minimality of $\mathcal{R}$, it contains an open subinterval. After shrinking I, we may assume that $f'_+(x) = +\infty$ for all $x \in I$.

We take $a, b \in I$ with $a < b$. For all $x \in I$, we define

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $g'_+(x) = +\infty$ for all $x \in I$ and g is strongly continuous on I. Also, we have $g(a) = g(b)$. By Lemma 3.6, the function g is strictly increasing on I. We deduce $g(a) < g(b)$, which is impossible. □

Proof of Proposition 3.1. By Theorem 2.9, Lemmas 3.3 and 3.8, we may reduce to the case, where I is a convex open set, the function f is strongly continuous on I, and the functions f'_+, f'_- are $\overline{R}$-valued and continuous on I. Applying Lemma 3.7, we have the proposition. □

Let $m, n, r \in \mathbb{N}$. A C^r map f from a definable set $A \subset R^n$ to a definable set $B \subset R^m$ is called a C^r *diffeomorphism* if f is bijective and f^{-1} is a C^r map. Note that for each C^r strong cell C in R^n, there exist positive integers m, $i_1, \dots, i_m$ with $1 \le i_1 < \cdots < i_m \le n$ such that the map $(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_m}) : C \to R^m$ is a C^r diffeomorphism onto an open C^r strong cell in R^m. Thus, the following holds.

Lemma 3.9. *Let $n \in \mathbb{N}$. For each strong $\langle i_1, \dots, i_n \rangle$-cell C in R^n, we have* $\dim(C) = i_1 + \cdots + i_n$.

Let $n \in \mathbb{N}$ and $A \subset R^n$ definable. Suppose that $f : A \to \overline{R}$ is definable. If x is an interior point of A, we define

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right),$$

provided these partial derivatives exist at x. If some partial derivative is not defined at x, then ∇f is not defined at x. We define $A' := \{x \in A : x$ is an interior point of A at which ∇f is defined$\}$. Note that A' is definable.

Lemma 3.10. *For the above sets A and A', the set $A \setminus A'$ has empty interior.*

Proof. If $\mathrm{int}(A) = \emptyset$, then this lemma holds. Thus, assume that $\mathrm{int}(A) \neq \emptyset$. We proceed by induction on n.

Let $n = 1$. Then it follows from Proposition 3.1.

Given $n \ge 1$, assume the lemma proved for n. It suffices to show that for each open box $U \subset A$, we have $U \cap A' \neq \emptyset$. Let $U \subset A$ be an open box. We define $\widetilde{U} := \{x \in U : (\partial f / \partial x_{n+1})(x)$ is defined$\}$. Then the set $\widetilde{U}$ is definable. By Proposition 3.1, we have $\mathrm{int}(U \setminus \widetilde{U}) = \emptyset$. By Lemma 2.2, we have $\dim(U) = \max\{\dim(U \setminus \widetilde{U}), \dim(\widetilde{U})\} = \dim(\widetilde{U})$. Thus there exists a non-empty open box $V \times W \subset \widetilde{U}$ such that $V \subset R^n$ and $W \subset R$. We take $w \in W$. By the inductive hypothesis to the function $v \mapsto f(v, w) : V \to \overline{R}$, there exists $v_0 \in V$ such that for each $i \in \{1, \dots, n\}$ the partial derivative $(\partial f / \partial x_i)(v_0, w)$ is defined. Therefore we obtain $(v_0, w) \in A'$, as desired. □

Proof of Theorem 2.10. Without loss of generality, we may assume that $r = 1$. We proceed by induction on n.

By weak o-minimality of $\mathcal{R}$, the condition $(1)_1$ holds. The condition $(2)_1$ follows from Theorem 2.9 and Proposition 3.1. The condition $(1)_{n+1}$ follows from Theorem 2.9 and $(2)_n$.

We prove $(2)_{n+1}$. Let $A \subset R^{n+1}$ be definable and $f : A \to \overline{R}$ definable. We define A' as above. By $(1)_{n+1}$ and Theorem 2.9, we can take a decomposition $\mathcal{D}$ of A into C^1 strong cells in R^{n+1} partitioning A' such that f (respectively: ∇f) is strongly continuous on each C^1 strong cell of $\mathcal{D}$ (respectively: on each C^1 strong cell of $\mathcal{D}$ contained in A').

Let $C \in \mathcal{D}$ and $\dim C = d \leq n$. Then there exists a C^1 diffeomorphism p from C onto some C^1 strong cell $D \subset R^d$. By $(2)_d$, there exists a decomposition $\mathcal{D}_D$ of D into C^1 strong cells in R^d such that $f \circ p^{-1} | B$ is a C^1 function for each $B \in \mathcal{D}_D$. By composing with the C^1 map p, we obtain that $f | p^{-1}(B)$ is a C^1 function for each $B \in \mathcal{D}_D$. By $(1)_{n+1}$, there exists a decomposition $\mathcal{D}_C$ of C into C^1 strong cells in R^{n+1} partitioning $p^{-1}(B)$ for each $B \in \mathcal{D}_D$. Then $f | C'$ is a C^1 function for each $C' \in \mathcal{D}_C$.

Let $C \in \mathcal{D}$ and $\dim C = n + 1$. By Lemma 3.10, the C^1 strong cell C

intersects A'. Since $\mathcal{D}$ partitions A', we have $C \subset A'$. It follows that $f \mid C$ is a C^1 function.

This finishes the proof of $(2)_{n+1}$. □

References

[1] Y. Baisalov and B. Poizat, Paires de structures o-minimales, J. Symbolic Logic 63 (1998), 570-578.

[2] M. Coste, An introduction to o-minimal geometry, Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa, Istituti Editoriali e Poligrafici Internazionali, 2000.

[3] M. A. Dickmann, Elimination of quantifiers for ordered valuation rings, J. Symbolic Logic 52 (1987), 116-128.

[4] A. Dolich, Theories without the intermediate value property and weak o-minimality, preprint.

[5] L. van den Dries, Tame topology and o-minimal structures, London Math. Soc. Lecture Notes Series, 248, Cambridge Univ. Press, Cambridge, 1998.

[6] D. Macpherson, D. Marker and C. Steinhorn, Weakly o-minimal structures and real closed fields, Trans. Amer. Math. Soc. 352 (2000), 5435-5483.

[7] R. Wencel, Weakly o-minimal non-valuational structures, RAAG preprint n. 182, (http://ihp-raag.org/).

[8] R. Wencel, Topological properties of sets definable in weakly o-minimal structures, RAAG preprint n. 181 (http://ihp-raag.orgj/).

Department of Mathematics
Faculty of Science
Okayama University 1-1
Naka 3-chome, Tsushima
Okayama 700-8530, Japan
e-mail: htanaka@math.okayama-u.ac.jp

Department of Mathematics
Faculty of Education
Wakayama University
Sakaedani, Wakayama 640-8510, Japan
e-mail: kawa@center.wakayama-u.ac.jp