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GRAPH REPRESENTATION OF PERMUTATION GROUPS-I

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Abstract

The aim of this paper is to study how to construct a graph representing a permutation group, and illustrate much of the basic relations between its elements by graphs. Finally, some applications are presented.

Introduction

The aim of this paper is to study how to construct a graph representing a permutation group, and illustrate much of the basic relations between its elements by graphs. In order to do this we shall state much of the preliminaries related to both permutations and graphs in Section 1. Section 2 is devoted to constructing the known expression representing a given permutation. Also we define the product $(\Gamma \times \Delta, (u_0, v_0))$ of the known expressions (Γ, u_0) and (Δ, v_0), which

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represents the product (composition) $\alpha\beta$, of the permutations α and β acting on the same points, respectively. Finally we get the minimal graph of $(\Gamma \times \Delta, (u_0, v_0))$. Furthermore, some applications are presented in Section 3.

1. Preliminaries

In this section we give some basic definitions on permutation groups and graphs.

Let N denote a set. A bijection mapping α of N onto itself, for short $\alpha : N \to N$, having the property that $\{K \mid K \in N \text{ and } \alpha(K) \neq K\}$, is called a *permutation* of N. The order $|N|$ of N is called the *degree of the permutation* α.

If both α and β are permutations of N, then their composition is denoted by $\alpha\beta$ and defined by $\alpha\beta(K) = \alpha(\beta(K))\ \forall K \in N$. This composition is bijective, keeps almost all the points fixed, and is therefore also a permutation.

If a set P of permutations of N forms a group under this composition, then we call P a *permutation group* and say that P is acting on N. $|N|$ is then called the *degree of* P. The set of all permutations of N, i.e., the set $\{\alpha | \alpha : N \to N \text{ and } \alpha(K) \neq K \text{ for finitely many } k \in N\}$ is a permutation group, it is called the *symmetric group* on N. If N has cardinality n, it is denoted by S_n, then the elements of N are called *points*.

Two permutation groups, say P on N and P' on N', i.e., subgroups of S_n and $S_{n'}$ respectively, are similar if and only if there exists a bijective, $\gamma : N \to N'$ and an isomorphism $\phi : P \simeq P'$ such that the following holds:

$$\forall \alpha \in P, \quad K \in N, \quad \phi(\alpha)(\gamma(K)) = \gamma(\alpha(K))$$

(this means that by renaming the elements of P by ϕ and the points of N by γ, we obtain P'). It is easy to check that two symmetric groups are similar if and only if they are of the same degree, i.e., if they act on the same number of points. An excellent account of this material can be found in [4].

We may write a permutation α as a succession of disjoint cycles and the order of α is the least common multiple of the lengths of its cycles. Then two permutations are conjugate in a symmetric group if and only if they have the same number of cycles of each length [3].

An oriented graph Γ consists of two disjoint sets, a non-empty set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, together with a function $t : E(\Gamma) \to V(\Gamma)$ and a fixed involution $E(\Gamma) \to E(\Gamma)$ denoted by $e \to \bar{e}$, where $i(e)$ and $t(\bar{e})$ are called the *initial point* of e and the *terminal point* of $\bar{e}$ respectively ($\bar{e}$ is the inverse of e). A morphism of graphs [1, 2] $f : \Gamma \to \Gamma'$ is a map taking each vertex to a vertex and each edge to an edge or a vertex such that

$$f(\bar{e}) = \overline{f(e)} \quad \text{and} \quad t(f(e)) = f(t(e)).$$

A path P in the graph Γ is an ordered n-tuple $(n \geq 1)$, $P = (e_1, e_2, \ldots, e_n)$; $e_i \in E(\Gamma)$; $1 \leq i \leq n$ such that $t(e_j) = i(e_{j+1}), 1 \leq j \leq n-1$; $i(e_1)$ and $t(e_n)$ are called the *initial* and *terminal points* of P, written $i(P)$ and $t(P)$ respectively. P is called *reduced path* [1], if it admits no elementary reductions, and it will be a loop at V if it is not necessarily reduced with $i(P) = t(P) = V$, otherwise it is called a *circuit*.

A pair of edges (e_1, e_2) of Γ is said to be *admissible* [6] if $i(e_1) = i(e_2)$ and $e_1 \neq \bar{e}_2$ to obtain a graph denoted by $\Gamma/[e_1 = e_2]$. The morphism $\Gamma \to \Gamma/[e_1 = e_2]$ is called an *edge fold*.

Let Γ, Γ' be two graphs. An edge collapse is a morphism $f : \Gamma \to \Gamma'$ whose effect is to identify e, $\bar{e}$, $i(e)$ and $t(e)$ to one vertex.

A labelling is a map $f : E(\Gamma) \to N$, where $N = \{1, 2, \ldots, n\}$ such that $f(\bar{e}) = f(e)^{-1}\ \forall e \in E(\Gamma)$. If $P = e_1\ e_2 \ldots e_n$ is a path in Γ, then $f(P) = f(e_1) \cdot f(e_2) \ldots f(e_n)$. Now $f_1(P) = f_1(e_1) \cdot f_1(e_2) \ldots f_1(e_n)$ is the first label on the edges e_i. Similarly $f_2(P) = f_2(e_1) \cdot f_2(e_2) \ldots f_2(e_n)$ is the second label on the edges e_i. We use (k, ℓ) to denote an edge with first label k and second label ℓ, where $k, \ell \in N$, where $f_i(0) = 1$; $i = 1, 2$.

2. Permutations and Graphs

2.1. The based graph

First of all we shall define how we can construct a based graph which represents a given permutation. Now we will define from α a labelled graph Γ with base vertex u_0 as follows: For each $k \in N$, we have two cases:

(i) If $k \neq \alpha(k)$, then take a circuit with two edges e and e' at u_0 with labels $(k, 0)$ and $(0, \alpha(k))$ respectively; $i(e) = u_0 = t(e')$.

(ii) If $k = \alpha(k)$, then take a loop at u_0 labelled (k, k). The graph (Γ, u_0) is called a *based graph* representing α, where u_0 is the base vertex.

Note. $T_1 = \{e \mid e \text{ labelled } (0, x)\ \forall x \in N\}$,

$T_2 = \{e \mid e \text{ labelled } (x, 0)\ \forall x \in N\}$.

The identity graph which corresponds to the identity permutation e consists of n loops labelled (k, k); $1 \leq k \leq n$, and denoted by (I, u_0). Also, we can define the inverse graph $(\Gamma, u_0)^{-1}$ of the based graph (Γ, u_0) which corresponds to the inverse permutation α^{-1}, that takes the second label to the first label of every loop. The inverse graph will be abbreviated by (Γ^{-1}, u_0), such that

$$(\Gamma, u_0) \times (\Gamma^{-1}, u_0) = (I, u_0).$$

2.2. The product of based graphs

Let (Γ, u_0) and (Δ, v_0) be the based graphs representing α and β respectively. The product of these graphs is denoted by $(\Gamma \times \Delta, (u_0, v_0))$ defined as follows:

(1) The vertices of $(\Gamma \times \Delta, (u_0, v_0))$ are the set of all ordered pairs

$(u, v)\ \forall u \in V(\Gamma), v \in V(\Delta)$, and the base of $(\Gamma \times \Delta, (u_0, v_0))$ is the vertex (u_0, v_0).

(2) The edges of $(\Gamma \times \Delta, (u_0, v_0))$ are given by:

(i) If k is the second label of an edge in Δ joining $v \to v'$, say with first label 0, put $(0, k) : (u, v) \to (u, v')$ in $(\Gamma \times \Delta, (u_0, v_0))\ \forall$ vertices $u \in \Gamma$.

(ii) If k is the first label of an edge in Γ, we have two cases:

(a) If $(k, 0) : u \to u'$ in Γ, then put $(k, 0) : (u, v) \to (u', v)$ in $(\Gamma \times \Delta, (u_0, v_0))\ \forall$ vertices $v \in \Delta$;

(b) If $(k, \ell) : u \to u'$ in Γ; $\ell \neq 0$, then put $(k, z) : (u, v) \to (u', v')$ in $(\Gamma \times \Delta, (u_0, v_0))$, whenever $(\ell, z) : v \to v'$ in Δ.

2.3. Minimal graph

In this section we start with two definitions which are used later on.

If u is a vertex in (Γ, u_0) with valency 2, $u \neq u_0$ and if

(i) u is an extreme vertex of both T_1 and T_2,

(ii) the labels of the two edges in start $(\Gamma, u) = \{e \in (\Gamma, u_0) | i(e) = u\}$ occur as the labels of some other edges of (Γ, u_0), then u and its adjacent edges are redundant. If (ii) does not hold, then u and its adjacent edges are singular.

Now from the based graph, we get a minimal graph as follows:

M1. Collapsing all edges labelled $(0, 0)$ and folding all edges whose labels and initial vertex are the same, until no further such folds are possible;

M2. Remove all the redundant vertices and their adjacent edges;

M3. Combine every two edges e_1 and e_2, say, where e_1 is an edge labelled $(k, 0)$ in T_2 with $i(e_1) = u$, $t(e_1) = v$ and e_2 is an edge labelled $(0, \ell)$ in T_1 with $i(e_2) = v$, $t(e_2) = u_1$ into one edge e labelled (k, ℓ) with $i(e) = u$, $t(e) = v$, whenever v is singular.

3. Applications

(i) Let $\alpha = (1)(2)(3)(4)(5)$ and $\beta = (1\,2\,3\,4\,5)$ be two permutations in S_5. Then by using (2.1), we can present α and β by the based graphs (Γ, u_0) and (Δ, v_0), respectively. Also, by using (2.2) we will get $(\Gamma \times \Delta, (u_0, v_0))$ and its minimal graph which corresponds to $\alpha\beta$.

(Γ, u_0) :

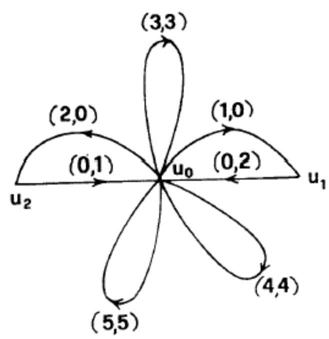

Figure 1

(Δ, v_0) :

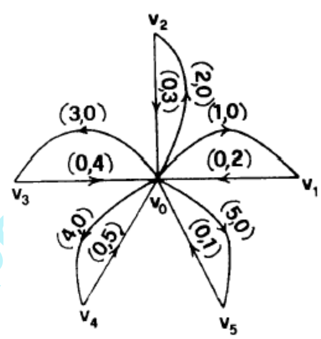

Figure 2

$(\Gamma \times \Delta, (u_0, v_0))$:

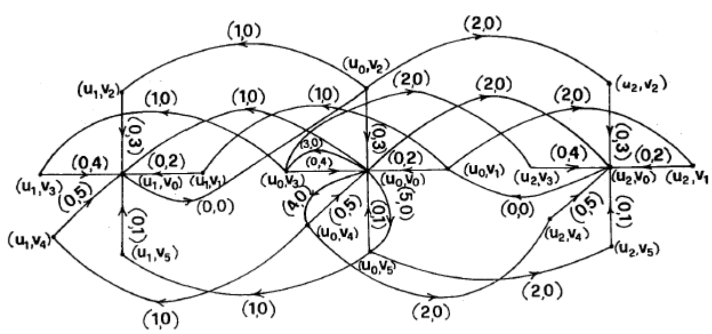

Figure 3

Applying M1 to Figure 3, we get

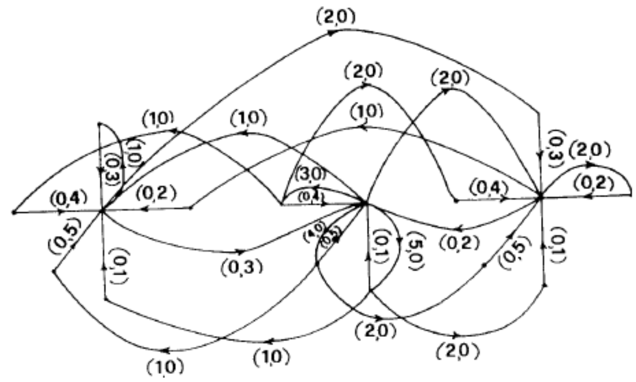

Figure 4

Applying M2 to Figure 4, we get

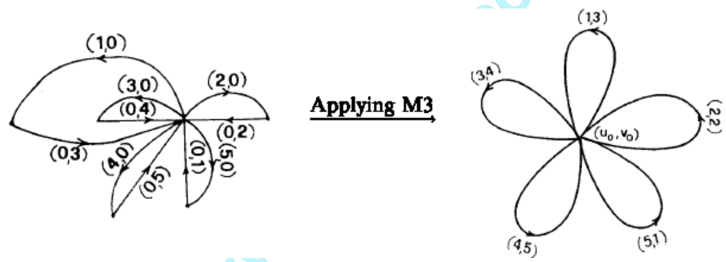

Figure 5 **Figure 6**

Similarly, the based graph $(\Delta \times \Gamma, (v_0, u_0))$ which corresponds to $\beta\alpha$ is:

$(\Delta \times \Gamma, (v_0, u_0))$:

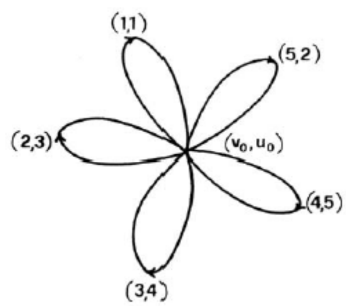

Figure 7

(ii) Consider the permutations of the group $s_3 \cdot P_0 = e$, $P_1 = (1\ 2)(3)$,

$P_2 = (1\ 3)(2), P_3 = (1)(2\ 3), P_4 = (1\ 2\ 3), P_5 = (1\ 3\ 2)$. Here we are interested in presenting it in terms of its multiplication table, or Cayley table.

Table 1

	P_0	P_1	P_2	P_3	P_4	P_5
P_0	P_0	P_1	P_2	P_3	P_4	P_5
P_1	P_1	P_0	P_4	P_5	P_2	P_3
P_2	P_2	P_5	P_0	P_4	P_3	P_1
P_3	P_3	P_4	P_5	P_0	P_1	P_2
P_4	P_4	P_3	P_1	P_2	P_5	P_0
P_5	P_5	P_2	P_3	P_1	P_0	P_4

Now let $I, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ be the graph representations corresponding to the set of permutations P_i, $i = 0, 1, \ldots, 5$ defined above. If we present these graphs in terms of its multiplication table, we get

Table 2

	I	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
I	I	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	I	Γ_4	Γ_5	Γ_2	Γ_3
Γ_2	Γ_2	Γ_5	I	Γ_4	Γ_3	Γ_1
Γ_3	Γ_3	Γ_4	Γ_5	I	Γ_1	Γ_2
Γ_4	Γ_4	Γ_3	Γ_1	Γ_2	Γ_5	I
Γ_5	Γ_5	Γ_2	Γ_3	Γ_1	I	Γ_4

It is clearly seen that Tables 1 and 2 are isomorphic.

References

[1] S. M. Gersten, Fixed points of automorphisms of free groups, unpublished paper.

[2] S. M. Gersten, Intersection of finitely generated subgroups of free groups and resolutions of graphs, Invent. Math. 71 (1983), 567-591.

[3] M. Hall, Jr., Theory of Groups, The MacMillan Company, 1959.

[4] G. D. James and A. Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications 16 (1981).

[5] J. P. Serre, Trees, Springer-Verlag, Berlin, New York, 1980.

[6] J. R. Stallings, Topology of finite groups, Invent. Math. 71 (1983), 551-565.

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