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THE WEYL GROUP OF THE 3-EXTENDED AFFINE ROOT SYSTEM $A_1^{(1,1,1)*}(1)$

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Abstract

We describe the Weyl group associated to 3-extended affine root system $A_1^{(1,1,1)*}(1)$ [1, 5] in terms of the 3-extended affine diagram.

1. Introduction

In 1985, Saito [5] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belong to a positive semi-definite quadratic form I whose radical has rank two. Therefore, 2-extended affine root systems are also called *elliptic root systems*. In 1997, Allison et al. [1] also introduced the extended affine root systems associated to the extended affine Lie algebras and gave a complete description of them by using the concept of a semilattice. The generators and their relations of elliptic Weyl groups associated to the elliptic root systems were described from the viewpoint of a

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generalization of Coxeter groups by Saito and Takebayashi [6]. In cases of the simply-laced extended affine root systems, Azam and Shahsanaei [4] have given a presentation of the corresponding Weyl groups. In [7] and [8], in cases of the simply-laced 3-extended affine root systems we described the 3-extended affine Weyl groups in terms of the 3-extended affine diagrams. In this paper, we describe the Weyl group of the 3-extended affine root system $A_1^{(1,1,1)*}(1)$ in terms of the 3-extended affine diagram.

2. The 3-extended Affine Root System $A_1^{(1,1,1)*}(1)$

We recall the 3-extended affine root system $A_1^{(1,1,1)*}(1)$ [1, 5], which is given as follows:

$$R = \{\pm(\varepsilon_1 - \varepsilon_2) + 2nb + ma + kc \; (n, m, k \in \mathbb{Z} \text{ s.t. } mk \equiv 0 \, (\mathrm{mod}\, 2)),$$

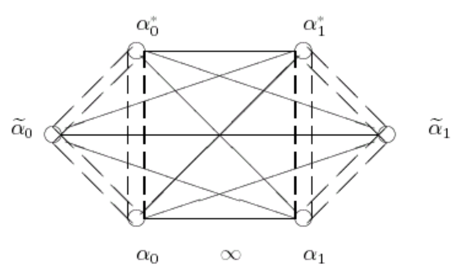
$$\pm(\varepsilon_1 - \varepsilon_2) + (2n+1)b + 2ma + 2kc \; (n, m, k \in \mathbb{Z})\}.$$

We set;

$$\alpha_0 = \varepsilon_2 - \varepsilon_1 + b, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_0^* = \alpha_0 + 2a,$$

$$\alpha_1^* = \alpha_1 + a, \quad \tilde{\alpha}_0 = \alpha_0 + 2c, \quad \tilde{\alpha}_1 = \alpha_1 + c.$$

The 3-extended affine diagram $\Gamma(R)$ of $A_1^{(1,1,1)*}(1)$ is given as follows:



3. The Weyl Group of the 3-extended Affine Root System

The Weyl group of the 3-extended affine root system is defined as follows [1, 5]. Let V be an $(l+3)$-dimensional real vector space equipped with a positive semi-definite bilinear form. Let V^0 be the 3-dimensional radical of the form $\langle , \rangle$ and $(V^0)^*$ be the dual space of V^0. Set $V = \dot{V} \oplus V^0$, and $\widetilde{V} = \dot{V} \oplus V^0 \oplus (V^0)^*$. Let $\{\varepsilon_1, ..., \varepsilon_l\}$ be the standard basis of $\dot{V}$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for all $i, j = 1, ..., l$. Define the bilinear form $\langle , \rangle$ on $\widetilde{V}$ so that $\langle , \rangle$ extends the form on V and $\langle , \rangle$ is nondegenerate on $\widetilde{V}$. For $\alpha \in R$, we define the reflection $w_\alpha \in GL(\widetilde{V})$ by $w_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha \, (u \in \widetilde{V})$ with $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Set $\widetilde{W}_R = \langle w_\alpha \mid \alpha \in R \rangle \subseteq GL(\widetilde{V})$. Then $\widetilde{W}_R$ is the Weyl group of the 3-extended affine root system R. In the case of $A_1^{(1,1,1)*}(1)$, we set $X = w_{\alpha_1} w_{\alpha_1 + b}$, $Y = w_{\alpha_1} w_{\alpha_1 + a}$, $Z = w_{\alpha_1} w_{\alpha_1 + c}$, and define the central elements η_1, η_2, η_3, by $\eta_1(u) := u + \langle u, 2a \rangle b - \langle u, 2b \rangle a$, $\eta_2(u) := u + \langle u, 2c \rangle b - \langle u, 2b \rangle c$, $\eta_3(u) := u + \langle u, 2c \rangle a - \langle u, 2a \rangle c$, then the following has been given in [4].

Proposition 3.1 [4]. *The Weyl group of the* 3*-extended affine root system* $A_1^{(1,1,1)*}(1)$ *is described as follows*:

Generators: $w_1 := w_{\alpha_1}$, *X, Y, Z and the central elements* η_1, η_2, η_3.

$$\textit{Relations}: \; w_1^2 = 1, \quad \begin{cases} w_1 X w_1 X = 1 \\ w_1 Y w_1 Y = 1 \\ w_1 Z w_1 Z = 1, \end{cases} \quad \begin{cases} YX = XY\eta_1 \\ ZX = XZ\eta_2 \\ ZY = YZ\eta_3. \end{cases}$$

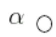
From Proposition 3.1, we obtain the following.

Theorem 3.2. *The Weyl group of the* 3*-extended affine root system* $A_1^{(1,1,1)*}(1)$ *is described as follows*:

Generators: *for each* $\alpha \in \Gamma(R)$, *we attach a generator* $a_\alpha := w_\alpha$. *For simplicity, we shall write* $a, a^*, \widetilde{a}, b, b^*, \widetilde{b}$ *instead of* a_α, a_{α^*}, $a_{\widetilde{\alpha}}$, a_β, a_{β^*}, $a_{\widetilde{\beta}}$.

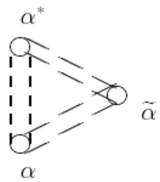
Relations:

0



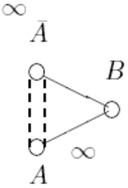
$$\Rightarrow a^2 = 1$$

$I.0$



$$\Rightarrow (aa^*\tilde{a})^2 = (a^*\tilde{a}a)^2 = (\tilde{a}aa^*)^2$$

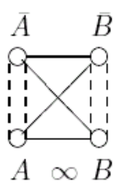
$I.\infty$



$$\Rightarrow (A\overline{A}B)^2 = (\overline{A}BA)^2 = (BA\overline{A})^2$$

where $A \neq \overline{A} \in \{\alpha, \alpha^*, \tilde{\alpha}\}$, *and* $B = \beta, \beta^*, \tilde{\beta}$

$II.\infty$

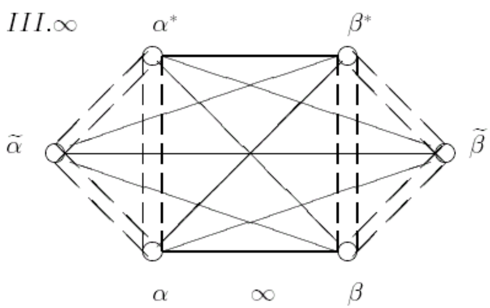


$$ba^*b = b^*ab^*,\ \ b\tilde{a}b = \tilde{b}a\tilde{b},\ \ \tilde{b}a^*\tilde{b} = b^*\tilde{a}b^*$$

$$\Rightarrow aa^*bb^* = bb^*aa^*,\ a\tilde{a}b\tilde{b} = b\tilde{b}a\tilde{a},$$

$$a^*\tilde{a}b^*\tilde{b} = b^*\tilde{b}a^*\tilde{a}$$

$III.\infty$



$$\Rightarrow \begin{aligned} aa^*(bb^*)^2\tilde{a} &= \tilde{a}aa^*(bb^*)^2 \\ aa^*(bb^*)^2\tilde{b} &= \tilde{b}aa^*(bb^*)^2 \\ a\tilde{a}(b\tilde{b})^2a^* &= a^*a\tilde{a}(b\tilde{b})^2 \\ a\tilde{a}(b\tilde{b})^2b^* &= b^*a\tilde{a}(b\tilde{b})^2 \end{aligned}$$

Proof. The case of $I.\infty$ is the same as [8], so we check the cases of $II.\infty$ and $III.\infty$.

$(II.\infty)$ $ba^*b(u) = u - \langle u, \alpha + 2\beta + 2a\rangle(\alpha + 2\beta + 2a) = b^*ab^*(u).$

The others are similarly checked.

$$(III.\infty)\ \ aa^*(bb^*)^2\widetilde{a}(u) = u + \langle u,\, 2a\rangle(\alpha + \beta) - \langle u,\, 2\alpha + 2\beta\rangle a - \langle u,\, \widetilde{\alpha}\rangle\widetilde{\alpha}$$

$$= \widetilde{a}aa^*(bb^*)^2(u).$$

The others are similarly checked.

Next, we show that the relations in Theorem 3.2 are the defining relations of $\widetilde{W}_R$. We denote by $\widetilde{W}(\Gamma(R))$ the group defined by the generators and relations in Theorem 3.2. Let $N(R)$ be the smallest normal subgroup of $\widetilde{W}(\Gamma(R))$ containing $a_\alpha\widetilde{a}_\alpha$ for $\alpha \in \{\alpha_0, \alpha_1\}$. Then one has a natural isomorphism

$$\widetilde{W}(\Gamma(R))/N(R) \cong \widetilde{W}(R_{el}).$$

The left hand side is a group obtained from $\widetilde{W}(\Gamma(R))$ by substituting $\widetilde{a},\ \widetilde{b}$ by a, b. Therefore, it is isomorphic to the central extension $\widetilde{W}(R_{el})$ of the elliptic Weyl group associated to the elliptic root system R_{el} [6]. For the proof of Theorem 3.2, we prepare the following.

Lemma 3.3. *We set* $\gamma_3 := (w_1w_1^*\widetilde{w}_1)^2$, *then*

(i) γ_3 *is a central element in* $\widetilde{W}(\Gamma(R))$,

(ii) $\gamma_3^2 = w_0w_0^*w_1\widetilde{w}_1w_0^*w_0\widetilde{w}_1w_1 = w_1w_1^*w_0\widetilde{w}_0w_1^*w_1\widetilde{w}_0w_0$, *and* $\gamma_3^4 = (w_0w_0^*\widetilde{w}_0)^2$.

(iii) $N(R)$ *is an abelian group generated by* $T_\alpha := a_\alpha\widetilde{a}_\alpha$ *for* $\alpha \in \{\alpha_0, \alpha_1\}$ *and* γ_3.

Proof. (i) $w_0\gamma_3 = w_0w_1w_1^*\widetilde{w}_1w_1w_1^*\widetilde{w}_1$

$$= w_0\widetilde{w}_1w_1w_1^*\widetilde{w}_1w_1w_1^* \ \ (\text{by } (w_1w_1^*\widetilde{w}_1)^2 = (\widetilde{w}_1w_1w_1^*)^2)$$

$$= \widetilde{w}_1w_1\widetilde{w}_0w_1^*\widetilde{w}_1w_1w_1^* \ \ (\text{by } w_0\widetilde{w}_1w_1 = \widetilde{w}_1w_1\widetilde{w}_0)$$

$$= \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_0^* w_1 w_1^* \ \left(\text{by } \widetilde{w}_0 w_1^* \widetilde{w}_1 = w_1^* \widetilde{w}_1 w_0^*\right)$$

$$= \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1 w_1^* w_0 = \gamma_3 w_0 \ \left(\text{by } w_0^* w_1 w_1^* = w_1 w_1^* w_0\right).$$

The others are similar.

(ii) $\gamma_3^2 = w_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1,$

here $w_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1^* \widetilde{w}_1 w_1 w_1^*$

$$= \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1 \widetilde{w}_1 w_1 w_1^*$$

$$= w_1^* w_1 w_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1 \widetilde{w}_1 w_1 w_1^*$$

$$= w_1^* w_1 w_1^* \widetilde{w}_1 w_1 w_1^* w_1 w_1^*$$

$$= w_1^* w_1 w_1^* \widetilde{w}_1 w_1 w_1^* w_1 w_1^* w_0 w_0^* w_0^* w_0$$

$$= w_1 w_1^* w_0 w_0^* w_1^* \widetilde{w}_1 w_0^* w_0$$

$$= w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0,$$

so, $\gamma_3^2 = w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1.$

Further, $w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1$

$$= w_0 w_0^* w_1 w_1^* w_1^* \widetilde{w}_1 w_0^* \widetilde{w}_0 \widetilde{w}_0 w_0 \widetilde{w}_1 w_1$$

$$= w_1 w_1^* w_0 w_0^* w_1^* \widetilde{w}_1 w_0^* \widetilde{w}_0 \widetilde{w}_1 w_1 \widetilde{w}_0 w_0$$

$$\left(\text{by } w_0 w_0^* w_1 w_1^* = w_1 w_1^* w_0 w_0^*, \quad w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 = w_1 \widetilde{w}_1 w_0 \widetilde{w}_0\right)$$

$$= w_1 w_1^* w_0 w_0^* \widetilde{w}_0 w_1^* \widetilde{w}_1 \widetilde{w}_0 \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 \ \left(\text{by } w_1^* \widetilde{w}_1 w_0^* = \widetilde{w}_0 w_1^* \widetilde{w}_1\right)$$

$$= w_1 w_1^* w_0 w_1^* \widetilde{w}_1 w_0^* \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 \ \left(\text{by } w_0^* \widetilde{w}_0 w_1^* \widetilde{w}_1 = w_1^* \widetilde{w}_1 w_0^* \widetilde{w}_0\right)$$

$$= w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0 \ \left(\text{by } \widetilde{w}_1 w_0^* \widetilde{w}_1 = w_1^* \widetilde{w}_0 w_1^*\right).$$

$$\gamma_3^4 = \left(w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1\right)^2$$

$$= w_0 w_0^* w_1 \widetilde{w}_1 w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1 \widetilde{w}_1 w_1,$$

here,

$$w_1\widetilde{w}_1w_1\widetilde{w}_1w_0^*w_0\widetilde{w}_1w_1\widetilde{w}_1w_1$$

$$= \widetilde{w}_0w_0w_0\widetilde{w}_0w_1\widetilde{w}_1w_1\widetilde{w}_1w_0^*w_0\widetilde{w}_1w_1\widetilde{w}_1w_1$$

$$(\text{by } w_0\widetilde{w}_0(w_1\widetilde{w}_1)^2w_0^* = w_0^*w_0\widetilde{w}_0(w_1\widetilde{w}_1)^2)$$

$$= \widetilde{w}_0w_0w_0^*\widetilde{w}_0,$$

so, $\gamma_3^4 = (w_0w_0^*\widetilde{w}_0)^2$.

(iii) We show that the subgroup generated by T_α (for $\alpha \in \{\alpha_0, \alpha_1\}$) and γ_3 is closed under the adjoint action Ad_{a_α} $\forall\alpha \in \Gamma(R)$.

$$Ad_{\alpha_0}(T_{\alpha_1}) = w_0w_1\widetilde{w}_1w_0 = w_0\widetilde{w}_0w_1\widetilde{w}_1 = T_{\alpha_0}T_{\alpha_1}.$$

$$Ad_{\alpha_0^*}(T_{\alpha_1}) = w_0^*w_1\widetilde{w}_1w_0^* = w_0w_0w_0^*w_1\widetilde{w}_1w_0^*w_0\widetilde{w}_1w_1w_1\widetilde{w}_1w_0$$

$$= \gamma_3^2w_0w_1\widetilde{w}_1w_0 = \gamma_3^2T_{\alpha_0}T_{\alpha_1}.$$

$$Ad_{\widetilde{\alpha}_0}(T_{\alpha_1}) = \widetilde{w}_0w_1\widetilde{w}_1\widetilde{w}_0 = w_1\widetilde{w}_1w_0\widetilde{w}_0 = T_{\alpha_0}T_{\alpha_1}.$$

$$Ad_{\alpha_1^*}(T_{\alpha_1}) = w_1^*w_1\widetilde{w}_1w_1^* = w_1^*w_1\widetilde{w}_1w_1^*w_1\widetilde{w}_1\widetilde{w}_1w_1 = \gamma_3^{-1}T_{\alpha_1}^{-1}.$$

$$Ad_{\alpha_1}(T_{\alpha_0}) = w_1w_0\widetilde{w}_0w_1 = w_1w_0\widetilde{w}_0w_1\widetilde{w}_1w_1\widetilde{w}_1\widetilde{w}_1w_1\widetilde{w}_1$$

$$= w_1\widetilde{w}_1w_1\widetilde{w}_1w_0\widetilde{w}_0w_1\widetilde{w}_1w_1\widetilde{w}_1 = T_{\alpha_0}T_{\alpha_1}^4.$$

$$Ad_{\alpha_1^*}(T_{\alpha_0}) = w_1^*w_0\widetilde{w}_0w_1^* = w_1w_1w_1^*w_0\widetilde{w}_0w_1^*w_1\widetilde{w}_0w_0w_0\widetilde{w}_0w_1$$

$$= \gamma_3^2w_1w_0\widetilde{w}_0w_1 = \gamma_3^2T_{\alpha_0}T_{\alpha_1}^4.$$

$$Ad_{\widetilde{\alpha}_1}(T_{\alpha_0}) = \widetilde{w}_1w_0\widetilde{w}_0\widetilde{w}_1 = w_1w_1\widetilde{w}_1w_0\widetilde{w}_0\widetilde{w}_1$$

$$= w_1w_0\widetilde{w}_0w_1 = T_{\alpha_0}T_{\alpha_1}^4.$$

$$Ad_{\alpha_0^*}(T_{\alpha_0}) = w_0^*w_0\widetilde{w}_0w_0^* = w_0^*w_0\widetilde{w}_0w_0^*w_0\widetilde{w}_0\widetilde{w}_0w_0 = \gamma_3^{-4}T_{\alpha_0}^{-1}.$$

The Proof of Theorem 3.2. Let N be a subgroup generated by Z, η_2 and η_3. Then N is a normal subgroup of $\widetilde{W}_R$ and there is an isomorphism $\widetilde{W}_R/N \cong \widetilde{W}(R_{el})$, where $\widetilde{W}(R_{el}) \cong \langle w_1, X, Y, \eta_1 \rangle$. So we have the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & N(R) & \to & \widetilde{W}(\Gamma(R)) & \to & \widetilde{W}(R_{el}) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & \widetilde{W}_R & \to & \widetilde{W}(R_{el}) & \to & 1 \end{array}$$

By the same argument in the case of the elliptic Weyl group [6], noting the expression of η_2, we see that the first arrow is an isomorphism. Therefore, the middle arrow is also an isomorphism.

Let us denote $\dot{w}_\alpha$ be the reflection in $GL(V)$ such that $w_\alpha|_V = \dot{w}_\alpha$, and set $W_R = \langle \dot{w}_\alpha \mid \alpha \in R \rangle$. Then, from the same argument in the elliptic case [6] and Proposition 3.1, we see the following.

Proposition 3.4. (i) *The central elements* γ_1 *and* $\gamma_2 \in \widetilde{W}(\Gamma(R))$ *corresponding to* η_1 *and* η_2 *are given as follows*:

$$\gamma_1 = w_0 w_0^* (w_1 w_1^*)^2, \quad \gamma_2 = w_0 \widetilde{w}_0 (w_1 \widetilde{w}_1)^2.$$

(ii) *We have an isomorphism* $\widetilde{W}(\Gamma(R))/\langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong W_R$.

Proof. (i) is directly checked and (ii) is trivial.

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