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THE WEYL GROUP OF THE 3-EXTENDED AFFINE ROOT SYSTEM $A_1^{(1,1,1)}$

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Abstract

We describe the Weyl group associated to the 3-extended affine root system $A_1^{(1,1,1)}$ [1, 5] in terms of the 3-extended affine diagram.

1. Introduction

In 1985, Saito [5] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belong to a positive semi-definite quadratic form I whose radical has rank two. Therefore 2-extended affine root systems are also called *elliptic root systems*. In 1997, Allison et al. [1] also introduced the extended affine root systems associated to the extended affine Lie algebras and gave a complete description of them by using the concept of a semilattice. The generators and their relations of elliptic Weyl groups associated to the elliptic root systems were described from the viewpoint of a

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generalization of Coxeter groups by Saito and Takebayashi [6]. In the cases of the simply-laced extended affine root systems, Azam and Shahsanaei [4] have given a presentation of the corresponding Weyl groups. In [7], in the cases of simply-laced 3-extended affine root systems except for $A_1^{(1,1,1)}$, similarly to the cases of the elliptic root systems, we described the 3-extended affine Weyl groups in terms of the 3-extended affine diagrams. In this paper, we describe the Weyl group of the 3-extended affine root system $A_1^{(1,1,1)}$ in terms of the 3-extended affine diagram.

2. The 3-extended Affine Root System $A_1^{(1,1,1)}$

We recall the 3-extended affine root system $A_1^{(1,1,1)}$ [1, 5], which is given as follows:

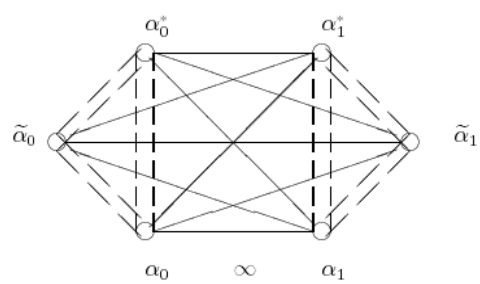
$$R = \{\pm(\varepsilon_1 - \varepsilon_2) + nb + ma + kc\ (n, m, k \in \mathbb{Z})\}.$$

We set

$$\alpha_0 = \varepsilon_2 - \varepsilon_1 + b, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2,$$

$$\alpha_i^* = \alpha_i + a \quad (i = 0, 1), \quad \widetilde{\alpha}_i = \alpha_i + c \quad (i = 0, 1).$$

The 3-extended affine diagram $\Gamma(R)$ of $A_1^{(1,1,1)}$ is defined as follows:



3. The Weyl Group of the 3-extended Affine Root System

The Weyl group of the 3-extended affine root system is defined as follows [1, 5]. Let V be an $(l+3)$-dimensional real vector space equipped with a positive semi-definite bilinear form. Let V^0 be the 3-dimensional radical of the form $\langle , \rangle$ and $(V^0)^*$ be the dual space of V^0. Set $V = \dot{V} \oplus V^0$, and $\widetilde{V} = \dot{V} \oplus V^0 \oplus (V^0)^*$. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the standard basis of $\dot{V}$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for all $i, j = 1, \ldots, l$. Define the bilinear form $\langle , \rangle$ on $\widetilde{V}$ so that $\langle , \rangle$ extends the form on V and $\langle , \rangle$ is nondegenerate on $\widetilde{V}$. For $\alpha \in R$, we define the reflection $w_\alpha \in GL(\widetilde{V})$ by $w_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha \; (u \in \widetilde{V})$ with $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Set $\widetilde{W}_R = \langle w_\alpha \mid \alpha \in R \rangle \subseteq GL(\widetilde{V})$. Then $\widetilde{W}_R$ is the Weyl group of the 3-extended affine root system R. In the case of $A_1^{(1,1,1)}$, we set $X = w_{\alpha_1} w_{\alpha_1 + b}$, $Y = w_{\alpha_1} w_{\alpha_1 + a}$, $Z = w_{\alpha_1} w_{\alpha_1 + c}$, and define the central elements η_1, η_2, η_3, by $\eta_1(u) := u + \langle u, a \rangle b - \langle u, b \rangle a$, $\eta_2(u) := u + \langle u, c \rangle b - \langle u, b \rangle c$, $\eta_3(u) := u + \langle u, 2c \rangle a - \langle u, 2a \rangle c$, then the following has been given in [4].

Proposition 3.1 [4]. *The Weyl group of the* 3-*extended affine root system* $A_1^{(1,1,1)}$ *is described as follows*:

Generators: $w_1 := w_{\alpha_1}$, *X, Y, Z and the central elements* η_1, η_2, η_3.

Relations: $w_1^2 = 1$, $\begin{cases} w_1 X w_1 X = 1 \\ w_1 Y w_1 Y = 1 \\ w_1 Z w_1 Z = 1, \end{cases}$ $\begin{cases} YX = XY\eta_1 \\ ZX = XZ\eta_2 \\ ZY = YZ\eta_3. \end{cases}$

From Proposition 3.1, we obtain the following.

Theorem 3.2. *The Weyl group of the* 3-*extended affine root system* $A_1^{(1,1,1)}$ *is described as follows*:

Generators: *for each* $\alpha \in \Gamma(R)$, *we attach a generator* $a_\alpha := w_\alpha$. *For simplicity, we shall write* $a, a^*, \tilde{a}, b, b^*, \tilde{b}$ *instead of* a_α, a_{α^*}, $a_{\tilde{\alpha}}$, a_β, a_{β^*}, $a_{\tilde{\beta}}$.

Relations:

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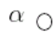
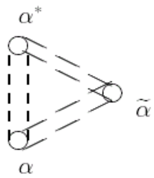


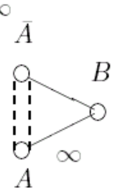
$\Rightarrow a^2 = 1$

$I.0$



$\Rightarrow (aa^*\tilde{a})^2 = (a^*\tilde{a}a)^2 = (\tilde{a}aa^*)^2$

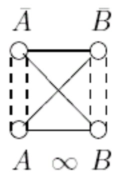
$I.\infty$



$\Rightarrow (A\overline{A}B)^2 = (\overline{A}BA)^2 = (BA\overline{A})^2$

where $A \neq \overline{A} \in \{\alpha, \alpha^*, \tilde{\alpha}\}$, *and* $B = \beta, \beta^*, \tilde{\beta}$

$II.\infty$



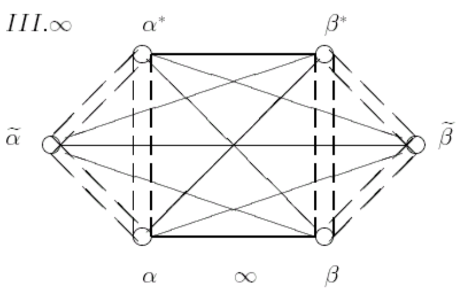
$\Rightarrow A\overline{A}B\overline{B} = \overline{A}B\overline{B}A = B\overline{B}A\overline{A} = \overline{B}A\overline{A}B$

where $\{A = \alpha, \overline{A} = \alpha^*, B = \beta, \overline{B} = \beta^*\}$,

$\{A = \alpha^*, \overline{A} = \tilde{\alpha}, B = \beta^*, \overline{B} = \tilde{\beta}\}$ *or*

$\{A = \alpha, \overline{A} = \tilde{\alpha}, B = \beta, \overline{B} = \tilde{\beta}\}$

$III.\infty$



$$\Rightarrow \begin{array}{l} aa^*bb^*\tilde{a} = \tilde{a}aa^*bb^* \\ aa^*bb^*\tilde{b} = \tilde{b}aa^*bb^* \\ a\tilde{a}b\tilde{b}a^* = a^*a\tilde{a}b\tilde{b} \\ a\tilde{a}b\tilde{b}b^* = b^*a\tilde{a}b\tilde{b} \end{array}$$

Proof. In the case of $I.0$ is the same as in [7], so we check the cases of $I.\infty$, $II.\infty$, and $III.\infty$.

$(I.\infty)$ $(A\overline{A}B)^2(u) = u - \langle u, 2A - 2\overline{A}\rangle B - \langle u, -2\overline{A} - 2B\rangle A$

$$- \langle u, 2A + 2B\rangle \overline{A} = (\overline{A}BA)^2(u) = (BA\overline{A})^2(u).$$

$(II.\infty)$ $A\overline{A}B\overline{B}(u) = u + \langle u, d\rangle (A + B) - \langle u, A + B\rangle d = \overline{A}B\overline{B}A(u)$

$$= B\overline{B}A\overline{A}(u) = \overline{B}A\overline{A}B(u). \ (\overline{A} =: A + d, \overline{B} =: B + d)$$

$(III.\infty)$ $aa^*bb^*\widetilde{a}(u) = u + \langle u, a\rangle (\alpha + \beta) - \langle u, \alpha + \beta\rangle a - \langle u, \widetilde{\alpha}\rangle \widetilde{\alpha}$

$$= \widetilde{a}aa^*bb^*(u).$$

The others are similarly checked.

Next, we show that the relations in Theorem 3.2 are the defining relations of $\widetilde{W}_R$. We denote by $\widetilde{W}(\Gamma(R))$ the group defined by the generators and relations in Theorem 3.2. Let $N(R)$ be the smallest normal subgroup of $\widetilde{W}(\Gamma(R))$ containing $a_\alpha \widetilde{a}_\alpha$ for $\alpha \in \{\alpha_0, \alpha_1\}$. Then one has a natural isomorphism

$$\widetilde{W}(\Gamma(R))/N(R) \cong \widetilde{W}(R_{el}).$$

The left hand side is a group obtained from $\widetilde{W}(\Gamma(R))$ by substituting $\widetilde{a}$, $\widetilde{b}$, by a, b. Therefore, it is isomorphic to the central extension $\widetilde{W}(R_{el})$ of the elliptic Weyl group associated to the elliptic root system R_{el} [6]. For the proof of Theorem 3.2, we prepare the following.

Lemma 3.3. *We set* $\gamma_3 := (w_1 w_1^* \widetilde{w}_1)^2$, *then*

(i) $\gamma_3 = (w_0 w_0^* \widetilde{w}_0)^2$,

(ii) γ_3 *is a central element in* $\widetilde{W}(\Gamma(R))$,

(iii) $N(R)$ *is an abelian group generated by* $T_\alpha := a_\alpha \widetilde{a}_\alpha$ *for* $\alpha \in \{\alpha_0, \alpha_1\}$ *and* γ_3.

Proof.

(i) $\gamma_3 = (w_1 w_1^* \widetilde{w}_1)^2$

$$= w_1 w_1^* \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_1^* \widetilde{w}_1$$

$$= w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0 \text{ (by } \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 w_1^* = w_1^* \widetilde{w}_1 w_1 \widetilde{w}_0 w_0)$$

$$= w_1 w_1^* w_0 w_0^* w_0^* \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0$$

$$= w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \text{ (by } w_1 w_1^* w_0 w_0^* \widetilde{w}_0 = \widetilde{w}_0 w_1 w_1^* w_0 w_0^*)$$

$$= (w_0 w_0^* \widetilde{w}_0)^2.$$

(ii) It is trivial from the relations

$$(w_1 w_1^* \widetilde{w}_1)^2 = (w_1^* \widetilde{w}_1 w_1)^2 = (\widetilde{w}_1 w_1 w_1^*)^2 = (w_0 w_0^* \widetilde{w}_0)^2$$

$$= (w_0^* \widetilde{w}_0 w_0)^2 = (\widetilde{w}_0 w_0 w_0^*)^2.$$

(iii) We show that the subgroup generated by T_α (for $\alpha \in \{\alpha_0, \alpha_1\}$) and γ_3 is closed under the adjoint action Ad_{α_α}, $\forall \alpha \in \Gamma(R)$,

$$Ad_{\alpha_0}(T_{\alpha_1}) = w_0 w_1 \widetilde{w}_1 w_0 = w_0 w_1 \widetilde{w}_1 w_0 \widetilde{w}_0 \widetilde{w}_0$$

$$= w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 w_0 \widetilde{w}_0 = T_{\alpha_0}^2 T_{\alpha_1},$$

$$Ad_{\alpha_0^*}(T_{\alpha_1}) = w_0^* w_1 \widetilde{w}_1 w_0^* = w_0^* \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 w_0^*$$

$$= w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 = \gamma_3 T_{\alpha_0}^2 T_{\alpha_1},$$

$$Ad_{\widetilde{\alpha}_0}(T_{\alpha_1}) = \widetilde{w}_0 w_1 \widetilde{w}_1 \widetilde{w}_0 = w_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 \widetilde{w}_0$$

$$= w_0 \widetilde{w}_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 = T_{\alpha_0}^2 T_{\alpha_1},$$

$$Ad_{\alpha_1^*}(T_{\alpha_1}) = w_1^* w_1 \widetilde{w}_1 w_1^* = w_1^* w_1 \widetilde{w}_1 w_1^* w_1 \widetilde{w}_1 \widetilde{w}_1 w_1 = \gamma_3^{-1} T_{\alpha_1}^{-1}.$$

Similarly,

$$Ad_{\alpha_1}(T_{\alpha_0}) = T_{\alpha_0}T_{\alpha_1}^2, \qquad Ad_{\alpha_1^*}(T_{\alpha_0}) = \gamma_3 T_{\alpha_1}^2 T_{\alpha_0},$$

$$Ad_{\tilde{\alpha}_1}(T_{\alpha_0}) = T_{\alpha_0}T_{\alpha_1}^2, \qquad Ad_{\alpha_0^*}(T_{\alpha_0}) = \gamma_3^{-1} T_{\alpha_0}^{-1}.$$

The Proof of Theorem 3.2. Let N be a subgroup generated by Z, η_2 and η_3. Then N is a normal subgroup of $\widetilde{W}_R$ and there is an isomorphism $\widetilde{W}_R/N \cong \widetilde{W}(R_{el})$, where $\widetilde{W}(R_{el}) \cong \langle w_1, X, Y, \eta_1\rangle$. So we have the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & N(R) & \to & \widetilde{W}(\Gamma(R)) & \to & \widetilde{W}(R_{el}) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & \widetilde{W}_R & \to & \widetilde{W}(R_{el}) & \to & 1 \end{array}$$

By the same argument in the case of the elliptic Weyl group [6], noting the expression of η_2 is generated by Y, we see that the first arrow is an isomorphism. Therefore the middle arrow is also an isomorphism.

Let us denote $\dot{w}_\alpha$ be the reflection in $GL(V)$ such that $w_\alpha|_V = \dot{w}_\alpha$, and set $W_R = \langle \dot{w}_\alpha \,|\, \alpha \in R\rangle$. Then, from the same argument in the elliptic case [6] and Proposition 3.1, we see the following.

Proposition 3.4. (i) *The central elements* γ_1 *and* $\gamma_2 \in \widetilde{W}(\Gamma(R))$ *corresponding to* η_1 *and* η_2 *are given as follows*:

$$\gamma_1 = w_0 w_0^* w_1 w_1^*, \qquad \gamma_2 = w_0 \tilde{w}_0 w_1 \tilde{w}_1.$$

(ii) *We have an isomorphism* $\widetilde{W}(\Gamma(R))/\langle \gamma_1, \gamma_2, \gamma_3\rangle \cong W_R$.

Proof. (i) is directly checked and (ii) is trivial.

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