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A LIE ALGEBRA AND ITS APPLICATIONS

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Abstract

A new Lie algebra R^6 is constructed by introducing the cycled numbers, whose resulting loop algebra $\widetilde{R}^6$ is also presented, which is used to establish a linear isospectral problem. It follows that an integrable hierarchy of soliton equations with 8-potential functions is obtained, from which the well-known KN integrable system is produced as a reduced case. Therefore, a type of expanding integrable system of the KN hierarchy is worked out. The method proposed in the paper can deduce a great many of other integrable soliton hierarchies.

1. Introduction

As we all know that it is an important topic to look for new integrable solitary hierarchies [11]. One has taken various approaches to obtain a host of interesting hierarchies of soliton equations such as the results in [1-3, 6-8, 10]. While integrable couplings are a quite new aspect of soliton theory, which are introduced by the study of the Virasoro symmetry algebra and the solitary solutions [4]. In terms of the related concepts of integrable couplings [9], we find they belong to the scope of integrable

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systems. Based on some associated theories on integrable couplings, we construct a new Lie algebra R^6 by introducing the cycled numbers. The resulting loop algebra $\widetilde{R}^6$ is given as well. It follows that the linear isospectral Lax pairs are constructed. By employing Tu scheme, an integrable system, as a matter of fact, an integrable coupling of the KN hierarchy, is obtained, which is a type of expanding integrable model of the KN hierarchy. The method given in this paper can be used to produce a lot of other integrable solitary hierarchies with multi-component potential functions.

2. An Integrable Solitary Hierarchy

Definition 1. The number set $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_s\}$ is called *cycled*, if the relations hold

$$\varepsilon_i\varepsilon_j = \begin{cases}\varepsilon_{i+j}, & i+j \le s,\\ \varepsilon_{i+j-s-1}, & i+j \ge s+1,\end{cases} \tag{1}$$

where $\varepsilon_i \neq 0$, $0 \le i \le s$; $\varepsilon_k \neq \varepsilon_j$, $k \neq j$.

Definition 2. As for two vectors $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} \in R^6$, a *commuting operation* in R^6 is defined by

$$[\varepsilon_i a, \varepsilon_k b] = [a, b]\varepsilon_i\varepsilon_k, \quad 0 \le i, k \le s, \tag{2}$$

where

$$[a, b] = \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \\ a_3b_5 - a_5b_3 + a_6b_2 - a_2b_6 \\ a_1b_6 - a_6b_1 + a_4b_3 - a_3b_4 \\ a_1b_5 - a_5b_1 + a_4b_2 - a_2b_4 \end{pmatrix}. \tag{3}$$

A resulting loop algebra $\widetilde{R}^6$ is given by

$$\widetilde{R}^6 = \left\{ \varepsilon_i \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \right\} \lambda^m, \tag{4}$$

with a commuting operation presented as

$$[\varepsilon_j a \lambda^m, \varepsilon_k b \lambda^n] = [\varepsilon_j a, \varepsilon_k b] \lambda^{m+n}, \tag{5}$$

where $a, b \in R^6$.

Consider the linear isospectral Lax pair as follows:

$$\begin{cases} \varphi_x = [U, \varphi], \\ \varphi_t = [V, \varphi], \quad \lambda_t = 0, \varphi, U, V \in \widetilde{R}^6, \end{cases} \tag{6}$$

whose compatibility leads to the zero-curvature equation just as showed in [5]

$$U_t - V_x + [U, V] = 0. \tag{7}$$

In what follows, we take spectral U and V in (6) and only consider $s = 0, 1$ for calculational convenience.

Taking $$U = \begin{pmatrix} \varepsilon_0 \lambda^2 \\ (\varepsilon_0 q_0 + \varepsilon_1 q_1)\lambda \\ (\varepsilon_0 r_0 + \varepsilon_1 r_1)\lambda \\ 0 \\ (\varepsilon_0 p_0 + \varepsilon_1 p_1)\lambda \\ (\varepsilon_0 s_0 + \varepsilon_1 s_1)\lambda \end{pmatrix}, V = \sum_{m=0}^{\infty} \begin{pmatrix} \varepsilon_0 a(m, 0) + \varepsilon_1 a(m, 1) \\ \lambda[\varepsilon_0 b(m, 0) + \varepsilon_1 b(m, 1)] \\ \lambda[\varepsilon_0 c(m, 0) + \varepsilon_1 c(m, 1)] \\ \varepsilon_0 d(m, 0) + \varepsilon_1 d(m, 1) \\ \lambda[\varepsilon_0 f(m, 0) + \varepsilon_1 f(m, 1)] \\ \lambda[\varepsilon_0 e(m, 0) + \varepsilon_1 e(m, 1)] \end{pmatrix} \lambda^{-2m},$$

solving the stationary zero-curvature equation

$$V_x = [U, V] \tag{8}$$

yields the recursion relations for V as follows:

$$\begin{cases} a_x(m,\ 0) = r_0 b(m+1,\ 0) + r_1 b(m+1,\ 1) - q_0 c(m+1,\ 0) - q_1 c(m+1,\ 1) \\ = r_0 c_x(m,\ 0) + r_1 c_x(m,\ 1) - q_0 b_x(m,\ 0) - q_1 b_x(m,\ 1), \\ a_x(m,\ 1) = r_0 b(m+1,\ 1) + r_1 b(m+1,\ 0) - q_0 c(m+1,\ 1) - q_1 c(m+1,\ 0), \\ = r_0 c_x(m,\ 1) + r_1 c_x(m,\ 0) - q_0 b_x(m,\ 1) - q_1 b_x(m,\ 0), \\ c(m+1,\ 0) = b_x(m,\ 0) + r_0 a(m,\ 0) + r_1 a(m,\ 1), \\ c(m+1,\ 1) = b_x(m,\ 1) + r_0 a(m,\ 1) - r_1 a(m,\ 0), \\ b(m+1,\ 0) = c_x(m,\ 0) + q_0 a(m,\ 0) + q_1 a(m,\ 1), \\ b(m+1,\ 1) = c_x(m,\ 1) + q_0 a(m,\ 1) + q_1 a(m,\ 0), \\ d_x(m,\ 0) = r_0 f(m+1,\ 0) + r_1 f(m+1,\ 1) - p_0 c(m+1,\ 0) - p_1 c(m+1,\ 1) \\ + s_0 b(m+1,\ 0) + s_1 b(m+1,\ 1) - q_0 e(m+1,\ 0) - q_1 e(m+1,\ 1), \\ d_x(m,\ 1) = r_0 f(m+1,\ 1) + r_1 f(m+1,\ 0) - p_0 c(m+1,\ 1) - p_1 c(m+1,\ 0) \\ + s_0 b(m+1,\ 1) + s_1 b(m+1,\ 0) - q_0 e(m+1,\ 1) - q_1 e(m+1,\ 0), \\ e(m+1,\ 0) = f_x(m,\ 0) + s_0 a(m,\ 0) + s_1 a(m,\ 1) + r_0 d(m,\ 0) + r_1 d(m,\ 1), \\ e(m+1,\ 1) = f_x(m,\ 1) + s_0 a(m,\ 1) + s_1 a(m,\ 0) + r_0 d(m,\ 1) + r_1 d(m,\ 0), \\ f(m+1,\ 0) = e_x(m,\ 0) + p_0 a(m,\ 0) + p_1 a(m,\ 1) + q_0 d(m,\ 0) + q_1 d(m,\ 1), \\ f(m+1,\ 1) = e_x(m,\ 1) + p_0 a(m,\ 1) + p_1 a(m,\ 0) + q_0 d(m,\ 1) + q_1 d(m,\ 0), \end{cases} \tag{9}$$

with the following initials:

$$a(0,\ 0) = a(0,\ 1) = \alpha,\ b(0,\ 0) = c(0,\ 0) = d(0,\ 0) = f(0,\ 0) = b(0,\ 1)$$

$$= c(0,\ 1) = f(0,\ 1) = e(0,\ 1) = a(1,\ 0) = 0,\ b(1,\ 0) = \alpha(q_0 + q_1),$$

$$c(1,\ 0) = \alpha(r_0 + r_1),\ e(1,\ 0) = \alpha(s_0 + s_1),\ f(1,\ 0) = \alpha(p_0 + p_1),$$

$$d(1,\ 0) = \alpha(r_1 s_1 + r_1 s_0 + r_0 s_0 + r_0 s_1 - p_0 q_0 - p_0 q_1 - p_1 q_1 - p_1 q_0),$$

$$a(1,\ 1) = 0,\ b(1,\ 1) = \alpha(q_1 + q_0),\ c(1,\ 1) = \alpha(r_0 + r_1),$$

$$e(1,\ 1) = \alpha(s_1 + s_0),\ f(1,\ 1) = \alpha(p_1 + p_0),$$

$$d(1,\ 1) = \alpha(r_0 s_1 + r_0 s_0 + r_1 s_0 + r_1 s_1 - p_0 q_1 - p_0 q_0 - p_1 q_0 - p_1 q_1).$$

Denote $V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} \varepsilon_0 a(m, 0) + \varepsilon_1 a(m, 1) \\ \lambda[\varepsilon_0 b(m, 0) + \varepsilon_1 b(m, 1)] \\ \lambda[\varepsilon_0 c(m, 0) + \varepsilon_1 c(m, 1)] \\ \varepsilon_0 d(m, 0) + \varepsilon_1 d(m, 1) \\ \lambda[\varepsilon_0 f(m, 0) + \varepsilon_1 f(m, 1)] \\ \lambda[\varepsilon_0 e(m, 0) + \varepsilon_1 e(m, 1)] \end{pmatrix} \lambda^{2n-2m}, \quad V_+^{(n)} + V_-^{(n)} = \lambda^{2n} V,$

then Eq. (8) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{10}$$

It is easy to verify that the terms on the left-hand side in above formula are of degree ≥ 0, however, the terms on the right-hand side are of degree ≤ 1. Therefore, the terms on the both sides are of degrees 0, 1. Thus we have

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -\begin{pmatrix} \varepsilon_0 a_x(n, 0) + \varepsilon_1 a_x(n, 1) \\ \lambda[\varepsilon_0 c(n+1, 0) + \varepsilon_1 c(n+1, 1)] \\ \lambda[\varepsilon_0 b(n+1, 0) + \varepsilon_1 b(n+1, 1)] \\ \varepsilon_0 d_x(n, 0) + \varepsilon_1 d_x(n, 1) \\ \lambda[\varepsilon_0 e(n+1, 0) + \varepsilon_1 e(n+1, 1)] \\ \lambda[\varepsilon_0 f(n+1, 0) + \varepsilon_1 f(n+1, 1)] \end{pmatrix}.$$

Taking a modified term Δ_n for $V_+^{(n)}$ and denoting by $V^{(n)} = V_+^{(n)} + \Delta_n$, $\Delta_n = (-\varepsilon_0 a(n, 0) - \varepsilon_1 a(n, 1), 0, 0, -\varepsilon_0 d(n, 0) - \varepsilon_1 d(n, 1), 0, 0)^T$, a direct calculation reads that

$$-V_x^{(n)} + [U, V^{(n)}] =$$

$$\begin{pmatrix} 0 \\ \varepsilon_0 \lambda[r_0 a(n, 0) + r_1 a(n, 1) - c(n+1, 0)] + \varepsilon_1 \lambda[r_0 a(n, 1) + r_1 a(n, 0) - c(n+1, 1)] \\ \varepsilon_0 \lambda[q_0 a(n, 0) + q_1 a(n, 1) - b(n+1, 0)] + \varepsilon_1 \lambda[q_0 a(n, 1) + q_1 a(n, 0) - b(n+1, 1)] \\ 0 \\ A \\ B \end{pmatrix}$$

with

$$A = \varepsilon_0\lambda[s_0a(n, 0) + s_1a(n, 1) + r_0d(n, 0) + r_1d(n, 1) - e(n+1, 0)]$$
$$+ \varepsilon_1\lambda[s_0a(n, 1) + s_1a(n, 0) + r_0d(n, 1) + r_1d(n, 0) - e(n+1, 1)],$$
$$B = \varepsilon_0\lambda[p_0a(n, 0) + p_1a(n, 1) + q_0d(n, 0) + q_1d(n, 1) - f(n+1, 0)]$$
$$+ \varepsilon_1\lambda[p_0a(n, 1) + p_1a(n, 0) + q_0d(n, 1) + q_1d(n, 0) - f(n+1, 1)].$$

Hence the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{11}$$

admits the following integrable system:

$$u_t = \begin{pmatrix} q_0 \\ q_1 \\ r_0 \\ r_1 \\ p_0 \\ p_1 \\ s_0 \\ s_1 \end{pmatrix}_t = \begin{pmatrix} c(n+1, 0) - r_0a(n, 0) - r_1a(n, 1) \\ c(n+1, 1) - r_0a(n, 1) - r_1a(n, 0) \\ b(n+1, 0) - q_0a(n, 0) - q_1a(n, 1) \\ b(n+1, 1) - q_0a(n, 1) - q_1a(n, 0) \\ e(n+1, 0) - s_0a(n, 0) - s_1a(n, 1) - r_0d(n, 0) - r_1d(n, 1) \\ e(n+1, 1) - s_0a(n, 1) - s_1a(n, 0) - r_0d(n, 1) - r_1d(n, 0) \\ f(n+1, 0) - p_0a(n, 0) - p_1a(n, 1) - q_0d(n, 0) - q_1d(n, 1) \\ f(n+1, 1) - p_0a(n, 1) - p_1a(n, 0) - q_0d(n, 1) - q_1d(n, 0) \end{pmatrix}$$

$$= \begin{pmatrix} b_x(n, 0) \\ b_x(n, 1) \\ c_x(n, 0) \\ c_x(n, 1) \\ f_x(n, 0) \\ f_x(n, 1) \\ e_x(n, 0) \\ e_x(n, 1) \end{pmatrix} = \begin{pmatrix} 0 & \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial & 0 \end{pmatrix} \begin{pmatrix} b(n, 1) \\ b(n, 0) \\ c(n, 1) \\ c(n, 0) \\ f(n, 1) \\ f(n, 0) \\ e(n, 1) \\ e(n, 0) \end{pmatrix} = JG_n, \tag{12}$$

where J is a Hamiltonian operator.

From the relations (9), a recursion operator satisfies

$$G_{n+1} = (l_{ij})_{8\times 8}G_n = LG_n, \tag{13}$$

where

$$l_{11} = -q_0\partial^{-1}q_0 - q_1\partial^{-1}q_1\partial,\ l_{12} = -q_0\partial^{-1}q_1\partial - q_1\partial^{-1}q_0\partial,$$

$$l_{13} = \partial + q_0\partial^{-1}r_0\partial + q_1\partial^{-1}r_1\partial,\ l_{14} = q_0\partial^{-1}r_1\partial + q_1\partial^{-1}r_0\partial,$$

$$l_{15} = l_{16} = l_{17} = l_{18} = 0,\ l_{21} = -q_0\partial^{-1}q_1\partial - q_1\partial^{-1}q_0\partial,$$

$$l_{22} = -q_0\partial^{-1}q_0\partial - q_1\partial^{-1}q_1\partial,\ l_{23} = q_0\partial^{-1}r_1\partial + q_1\partial^{-1}r_0\partial,$$

$$l_{24} = \partial + q_0\partial^{-1}r_0\partial + q_1\partial^{-1}r_1\partial,\ l_{25} = l_{26} = l_{27} = l_{28} = 0,$$

$$l_{31} = \partial - r_0\partial^{-1}q_0\partial - r_1\partial^{-1}q_1\partial,\ l_{32} = -r_0\partial^{-1}q_1\partial - r_1\partial^{-1}q_0\partial,$$

$$l_{33} = r_0\partial^{-1}r_0\partial + r_1\partial^{-1}r_1\partial,\ l_{34} = r_0\partial^{-1}r_1\partial + r_1\partial^{-1}r_0\partial,$$

$$l_{35} = l_{36} = l_{37} = l_{38} = 0,\ l_{41} = -r_0\partial^{-1}q_1\partial - r_1\partial^{-1}q_0\partial,$$

$$l_{42} = \partial - r_0\partial^{-1}q_0\partial - r_1\partial^{-1}q_1\partial,\ l_{43} = l_{34},$$

$$l_{44} = r_0\partial^{-1}r_0\partial + r_1\partial^{-1}r_1\partial,\ l_{45} = l_{46} = l_{47} = l_{48} = 0,$$

$$l_{51} = -p_0\partial^{-1}q_0\partial - p_1\partial^{-1}q_1\partial - q_0\partial^{-1}p_0\partial - q_1\partial^{-1}p_1\partial,$$

$$l_{52} = -p_0\partial^{-1}q_1\partial - p_1\partial^{-1}q_0\partial - q_0\partial^{-1}p_1\partial - q_1\partial^{-1}p_0\partial,$$

$$l_{53} = p_0\partial^{-1}r_0\partial + p_1\partial^{-1}r_1\partial + q_0\partial^{-1}s_0\partial + q_1\partial^{-1}s_1\partial,$$

$$l_{54} = p_0\partial^{-1}r_1\partial + p_1\partial^{-1}r_0\partial + q_0\partial^{-1}s_1\partial + q_1\partial^{-1}s_0\partial,$$

$$l_{55} = -q_0\partial^{-1}q_0\partial - q_1\partial^{-1}q_1\partial,\ l_{56} = -q_0\partial^{-1}q_1\partial - q_1\partial^{-1}q_0\partial,$$

$$l_{57} = \partial + q_0\partial^{-1}r_0\partial + q_1\partial^{-1}r_1\partial,\ l_{58} = l_{14},\ l_{61} = l_{52},\ l_{62} = l_{51},$$

$$l_{63} = l_{54},\ l_{64} = l_{53},\ l_{65} = l_{56},\ l_{66} = l_{55},\ l_{67} = l_{58},\ l_{68} = l_{57},$$

$$l_{71} = -s_0\partial^{-1}q_0\partial - s_1\partial^{-1}q_1\partial - r_0\partial^{-1}p_0\partial - r_1\partial^{-1}p_1\partial = l_{82},$$

$$l_{72} = -s_0\partial^{-1}q_1\partial - s_1\partial^{-1}q_0\partial - r_0\partial^{-1}p_1\partial - r_1\partial^{-1}p_0\partial = l_{81},$$

$$l_{73} = s_0\partial^{-1}r_0\partial + s_1\partial^{-1}r_1\partial + r_0\partial^{-1}s_0\partial + r_1\partial^{-1}s_1\partial = l_{84},$$

$$l_{74} = s_0\partial^{-1}r_1\partial + s_1\partial^{-1}r_0\partial + r_0\partial^{-1}s_1\partial + r_1\partial^{-1}s_0\partial = l_{83},$$

$$l_{75} = \partial - r_0\partial^{-1}q_0\partial - r_1\partial^{-1}q_1\partial = l_{86},\ l_{76} = -r_0\partial^{-1}q_1\partial - r_1\partial^{-1}q_0\partial = l_{85},$$

$$l_{77} = l_{88} = l_{44},\ l_{78} = l_{87} = l_{43}.$$

Hence, the system (12) can be written as

$$u_t = \begin{pmatrix} q_0 \\ q_1 \\ r_0 \\ r_1 \\ p_0 \\ p_1 \\ s_0 \\ s_1 \end{pmatrix}_t = JL^{n-1}\begin{pmatrix} b(1,1) \\ b(1,0) \\ c(1,1) \\ c(1,0) \\ f(1,1) \\ f(1,0) \\ e(1,1) \\ e(1,0) \end{pmatrix}. \tag{14}$$

Taking $q_1 = r_1 = p_1 = s_1 = p_0 = s_0 = 0$, the system (14) is reduced to the KN hierarchy. Therefore, it is a type of expanding integrable system with 8-component potential functions of the KN.

Remark. The method presented in the paper for obtaining expanding integrable hierarchies of soliton equations, is far from any one in [5]. If taking $s = 0, 1, 2, \ldots, n,$ we could obtain integrable systems with more multiple component functions. However, the calculation would be too tedious to complete only by hand, except for using computer. In addition, there is an open problem, that is, how do we get the Hamiltonian structure, symmetries and conserved laws for the integrable systems obtained by the above approach? Which is worth studying in the future.

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