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TIGHT WAVELET FRAMES THEORY BASED DECOMPOSITION AND RECONSTRUCTION ALGORITHMS AND PERFECT RECONSTRUCTED SYMMETRIC EXTENSION TRANSFORMS TECHNOLOGY

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Abstract

In this paper, we primarily prove decomposition and reconstruction algorithms taking advantage of tight wavelet frame transforms and thrash out the corresponding extension technology of the perfect reconstruction of image, which is testified additionally by experiments.

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1. Introduction

Wavelet transform has brought about great success when used in the field of image processing. Currently, another great pace has been taken in the work on wavelet frames. The redundant representation offered by wavelet frames has already been put to good use for signal denoising ([19]-[21]), and is currently applied to image compression. The definition of the frame is provided as below.

Definition 1.1 [7]**.** Let H be a Hilbert space with inner product $\langle ., . \rangle$, and norm $\| . \| = (\langle ., . \rangle)^{\frac{1}{2}}$. A system $X \subset H$ is called a *frame* of H if there are two positive constants A, B such that

$$A\| f \|^2 \leq \sum_{x \subset H} | \langle f, x \rangle |^2 \leq B\| f \|^2, \quad \forall f \in H. \tag{1.1}$$

The constants A and B are called *bounds* of the frame. If $A = B$, then X is called a *tight frame*.

We are interested in the study of wavelet frames via multiresolution analysis (MRA). Of particular interest to us are tight wavelet frames. Let $\phi \subset L_2\{R\}$ be given. Let D be the operator of dyadic dilation: $(Df)(y) = 2^{\frac{1}{2}} f(2y)$ and the translation $(T_t f)(y) = f(x - t)$. Set $V_j = D^j V_0$, $j \in Z$.

Definition 1.2 [2]**.** $\phi \in L^2(R)$ is said to *generate an MRA* $\{V_k, k \in Z\}$, if ϕ satisfies the following conditions:

(1) $V_k \subset V_{k+1}$, $k \in Z$;

(2) $\overline{\bigcup V_k} = L^2(R)$, $\bigcap V_k = 0$;

(3) $D(V_k) = V_{k+1}$, $k \in Z$;

(4) $T_1(V_0) = V_0$;

(5) $T_k \phi$, $k \in Z$ is an orthonormal basis for V_0.

The generator ϕ of the MRA is known as a scaling function or a refinable function. Let $\Psi = \{\psi_1, \psi_2, ..., \psi_r\}$ be a subset of $L_2(R)$. Then the dyadic wavelet system generated by the mother wavelet Ψ is the family

$$X(\Psi) = \{\psi_{j,k} = 2^{\frac{j}{2}}(2^j x - k) : \psi \subset \Psi,\ j, k \in Z\}. \tag{1.2}$$

Definition 1.3 [9]**.** A wavelet system $X(\Psi)$ is said to be *MRA-based* if there exists an MRA $\{V_j, j \subset Z\}$ such that the condition $\Psi \subset V_1$ holds. If, in addition, the system $X(\Psi)$ is a frame, then we refer to its elements as framelets.

Definition 1.4 [9]**.** Suppose that $V_j, j \in Z$ is an MRA induced by a refinable function ϕ. Let $\Psi = \{\psi_1, \psi_2, ..., \psi_r\}$ be a finite subset of V_1, $\hat{\phi} = (\tau_0\hat{\phi})\left(\frac{\cdot}{2}\right)$, $\hat{\psi}_i = (\tau_i\hat{\psi}_i)\left(\frac{\cdot}{2}\right)$, $i = 1, 2, ..., r$. Then we introduce the notations $\tau_0, \tau_1, ..., \tau_r$ for the combined MRA mask.

In [16], the sufficient conditions of construction of wavelet frames are provided.

Lemma 1.1 (The unitary extension principle (UEP))**.** *Let* $\tau_0, \tau_1, ..., \tau_r$ *be the combined MRA mask that satisfies Definition* 1.4. $\hat{\phi}(0) = 1$, $|\hat{\phi}(\omega)| \le c(1 + |\omega|^{-\frac{1}{2}-\varepsilon})$, $\varepsilon > 0$

$$M(z) = \begin{pmatrix} \tau_0(z) & \tau_0(-z) \\ \tau_1(z) & \tau_1(-z) \\ \vdots & \vdots \\ \tau_r(z) & \tau_r(-z) \end{pmatrix} \quad for\ |z| = 1. \tag{1.3}$$

If

$$M^*(z)M(z) = E, \tag{1.4}$$

where we use standard $M^*(z)$ *to represent the complex conjugate of transpose of* $M(z)$, *then* $\Psi = \{\psi_1, \psi_2, ..., \psi_r\}$ *is a wavelet frame that is derived from a multiresolution analysis* (*MRA*), *and MRA is generated by* ϕ.

It was proved in [3], [14] that the decay assumption of $\hat{\phi}$ at infinity can be removed. Thus if $\hat{\phi}(0) = 1$, then we have filters τ_0, τ_i, $i = 1, 2, 3, \ldots, r$, such that τ_0 generates an $L^2(R)$ function ϕ with $\hat{\phi}(0) = 1$, and $M(z)$ satisfies (1.4), hence we have a tight affine frame. The larger the number r of framelets, the more is the freedom or redundancy.

Lemma 1.2 [9] (Oblique extension principle (OEP))**.** *Let* $\tau = \{\tau_0, \tau_1, \ldots, \tau_r\}$ *be the combined mask of an MRA. Suppose that there is a* 2π*-periodic function* Θ *that satisfies the following*:

(1) *Each mask* τ_i *in the combined MRA mask* τ *is measurable and* (*essentially*) *bounded*;

(2) *The refinable function* ϕ *satisfies* $\lim_{\omega \to 0} \hat{\phi}(\omega) = 1$;

(3) *The function* $[\hat{\phi}, \hat{\phi}] = \sum_{k \in 2\pi Z^d} |\hat{\phi}(. + k)|^2$ *essentially*;

(4) Θ *is nonnegative*, *essentially bounded*, *continuous at the origin and* $\Theta(0) = 1$;

(5) *If* $\omega \in \sigma(V_0)$ *such that* $\omega + \nu \in \sigma(V_0)$, *then*

$$\langle \tau(\omega), \tau(\omega + \nu) \rangle = \begin{cases} \Theta(\omega), & \text{if } v = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Based on UEP, OEP, [1]-[18] have offered many varied construction methods of tight frames. However, most of the methods of construction that are enumerated as above cannot achieve perfect reconstruction of image. As is shown by our work the perfect construction of image can be achieved only if the centers of symmetry of filters (whose length can simultaneously be either odd or even) superposes one another.

The present paper is divided into three Sections. Section 1 reviews very briefly major concepts that are important to our presentation and introduces relevant notations and definitions. Section 2 presents our algorithms of decomposition and reconstruction based on the frames theory. Section 3 thrashes out the extension method of the perfect reconstruction of image. The extension technology studied by us is of vast significance when applied to image processing.

2. Tight Wavelet Frames Theory based Decomposition and Reconstruction Algorithms

Tight wavelet frames theory based decomposition and reconstruction algorithms are considered in the following section.

Let $\Psi = \{\psi_1, \psi_2\}$ be mother wavelet functions, and $\phi \in L^2$ be compactly supported scaling function governed by a two-scale relation which can be written as

$$\phi(x) = 2\sum_k h_k \phi(2x - k), \tag{2.1}$$

such that its refinement symbol

$$h(z) = \sum_k h_k z^k. \tag{2.2}$$

Let V_j be the nested subspaces generated by ϕ, and consider $\Psi = \{\psi_1, \psi_2\} \subset V_1$, with

$$\psi^i(x) = 2\sum_k q_k^i \phi(2x - k), \quad i = 1, 2, \tag{2.3}$$

and the following two-scale symbols

$$q(z) = \sum_k g_k z^k, \quad f(z) = \sum_k f_k z^k. \tag{2.4}$$

By (2.1) and (2.3), we have

$$\phi_{j,l}(x) = \sqrt{2}\sum_k p_{k-2l}\phi_{j+1,k}(x), \tag{2.5}$$

$$\psi_{j,l}(x) = \sqrt{2}\sum_k q_{k-2l}^i \psi_{j+1,k}(x), \quad i = 1, 2. \tag{2.6}$$

Defined below are certain functions:

$$\begin{cases} \Phi = \phi(x)\phi(y), & \Psi^1 = \phi(x)\psi^1(y), & \Psi^2 = \phi(x)\psi^2(y), \\ \Psi^3 = \psi^1(x)\phi(y), & \Psi^4 = \psi^1(x)\psi^1(y), & \Psi^5 = \psi^1(x)\psi^2(y), \\ \Psi^6 = \psi^2(x)\phi(y), & \Psi^7 = \psi^2(x)\psi^1(y), & \Psi^8 = \psi^2(x)\psi^2(y), \end{cases}$$

and

$$\begin{cases} \Phi_{j,k,m} = \phi_{j,k}(x)\phi_{j,m}(y), & \Psi^1_{j,k,m} = \phi_{j,k}(x)\psi^1_{j,m}(y), \quad \Psi^2_{j,k,m} = \phi_{j,k}(x)\psi^2_{j,m}(y), \\ \Psi^3_{j,k,m} = \psi^1_{j,k}(x)\phi_{j,m}(y), & \Psi^4_{j,k,m} = \psi^1_{j,k}(x)\psi^1_{j,m}(y), \quad \Psi^5_{j,k,m} = \psi^1_{j,k}(x)\psi^2_{j,m}(y), \\ \Psi^6_{j,k,m} = \psi^2_{j,k}(x)\phi_{j,m}(y), & \Psi^7_{j,k,m} = \psi^2_{j,k}(x)\psi^1_{j,m}(y), \quad \Psi^8_{j,k,m} = \psi^2_{j,k}(x)\psi^2_{j,m}(y). \end{cases}$$

The algorithms for decomposition and reconstruction are elucidated by the following theorem.

Theorem 2.1 (Theorem of decomposition and reconstruction)**.** *Let* $\Psi = \{\psi_1, \psi_2\}$ *be a mother wavelet and* ϕ *be a refinable function which generates an MRA. Suppose that*

$$V_j = \overline{span\{\phi_{j,k},\ k \in Z\}},$$

$$W_j^i = \overline{span\{\psi^i_{j,k},\ k \in Z\}}, \quad i = 1, 2, \quad V_j^{(2)} = V_j \otimes V_j,$$

$$W_j^{(2)} = V_j \otimes W_j^1 + V_j \otimes W_j^2 + W_j^1 \otimes V_j + W_j^1 \otimes W_j^1$$

$$+ W_j^1 \otimes W_j^2 + W_j^2 \otimes V_j + W_j^2 \otimes W_j^1 + W_j^2 \otimes W_j^2.$$

Then

(1) $V_j^{(2)} = \overline{span\{\Phi_{j,k,m},\ k, m \in Z\}}$;

(2)

$$W_j^{(2)} = \overline{span\{\Psi^1_{j,k,m}, \Psi^2_{j,k,m}\Psi^3_{j,k,m}, \Psi^4_{j,k,m}\Psi^5_{j,k,m}, \Psi^6_{j,k,m}, \Psi^7_{j,k,m}, k, m \in Z\}};$$

(3) $V_{j+1}^{(2)} = V_j^{(2)} + W_j^{(2)}$;

(4) $\bigcup_j W_j^{(2)}$ *is dense in* $L^2(R)$;

(5) *Algorithm of decomposition*

$$\begin{cases} C_{j,k,m} = \sum_l \sum_n p_{l-2k} p_{n-2m} C_{j+1,l,n} \\ D^1_{j,k,m} = \sum_l \sum_n p_{l-2k} q^1_{n-2m} D^1_{j+1,l,n} \\ D^2_{j,k,m} = \sum_l \sum_n p_{l-2k} q^2_{n-2m} D^2_{j+1,l,n} \\ D^3_{j,k,m} = \sum_l \sum_n q^1_{l-2k} p_{n-2m} D^3_{j+1,l,n} \\ D^4_{j,k,m} = \sum_l \sum_n q^1_{l-2k} q^1_{n-2m} D^4_{j+1,l,n} \\ D^5_{j,k,m} = \sum_l \sum_n q^1_{l-2k} q^2_{n-2m} D^5_{j+1,l,n} \\ D^6_{j,k,m} = \sum_l \sum_n q^2_{l-2k} p_{n-2m} D^6_{j+1,l,n} \\ D^7_{j,k,m} = \sum_l \sum_n q^2_{l-2k} q^1_{n-2m} D^7_{j+1,l,n} \\ D^8_{j,k,m} = \sum_l \sum_n q^2_{l-2k} q^2_{n-2m} D^8_{j+1,l,n}. \end{cases} \tag{2.7}$$

(6) *Algorithm of reconstruction*

$$\begin{aligned} C_{j+1,l,n} = \sum_{k,m} & p_{l-2k} p_{n-2m} C_{j,k,m} + p_{l-2k} q^1_{n-2m} D^1_{j,k,m} \\ & + p_{l-2k} q^2_{n-2m} D^2_{j,k,m} + q^1_{l-2k} p_{n-2m} D^3_{j,k,m} + q^1_{l-2k} q^1_{n-2m} D^4_{j,k,m} \\ & + q^1_{l-2k} q^2_{n-2m} D^5_{j,k,m} + q^2_{l-2k} p_{n-2m} D^6_{j,k,m} + q^2_{l-2k} q^1_{n-2m} D^7_{j,k,m} \\ & + q^2_{l-2k} q^2_{n-2m} D^8_{j,k,m}. \end{aligned} \tag{2.8}$$

Proof. By the definitions of $V_j^{(2)}$, $W_j^{(2)}$, statements (1) and (2) hold. The work in [3] leads to $V^1_{j+1} = V^1_j + W^1_j + W^2_j$, from which we observe that (3) is true. Let $f \in V^{(2)}_{j+1}$. Then $f = f_j + e_j$, $f_j \in V_j^{(2)}$, $e_j \in W_j^{(2)}$. Thus

$$f = \sum_{(k,m)} \left\{ C_{j,k,m} \Phi_{j,k,m} + \sum_{i=1}^{2} D^i_{j,k,m} \Psi^i_{j,k,m} \right\}.$$

From equations (2.5), (2.6), it follows that

$$\begin{aligned} C_{j,k,m} &= \langle f_{j+1}, \Phi_{j,k,m} \rangle = \langle f_{j+1}, \phi_{j,k}(x) \phi_{j,m}(y) \rangle \\ &= 2 \left\langle f_{j+1}, \left(\sum_l p_{l-2k} \phi_{j+1,l} \right) \left(\sum_n p_{n-2m} \phi_{j+1,n} \right) \right\rangle \end{aligned}$$

$$= 2\sum_{l}\sum_{n} p_{l-2k}p_{n-2m}C_{j+1,l,n}.$$

One can prove similarly that the other statements in (2.7) hold. By [3], it can be obtained that

$$\phi_{j+1,l}(x) = \sqrt{2}\sum_{k}(p_{l-2k}\phi_{j,k}(x) + q^{1}_{l-2k}(x)\psi^{1}_{j,k}(x) + q^{2}_{l-2k}(x)\psi^{2}_{j,k}(x)),$$

$$\phi_{j+1,l}(y) = \sqrt{2}\sum_{k}(p_{l-2k}\phi_{j,k}(y) + q^{1}_{l-2k}(x)\psi^{1}_{j,k}(y) + q^{2}_{l-2k}(y)\psi^{1}_{j,k}(y)).$$

By further induction,

$$\begin{aligned}\Phi_{j+1,l,n} = 2\sum_{k,m}\{&p_{l-2k}p_{n-2m}\Phi_{j,k,m} + p_{l-2k}q^{1}_{n-2m}\Psi^{1}_{j,k,m} + p_{l-2k}q^{2}_{n-2m}\Psi^{2}_{j,k,m}\\&+ q^{1}_{l-2k}p_{n-2m}\Psi^{3}_{j,k,m} + q^{1}_{l-2k}q^{1}_{n-2m}\Psi^{4}_{j,k,m} + q^{1}_{l-2k}q^{2}_{n-2m}\Psi^{5}_{j,k,m}\\&+ q^{2}_{l-2k}p_{n-2m}\Psi^{6}_{j,k,m} + q^{2}_{l-2k}q^{1}_{n-2m}\Psi^{7}_{j,k,m} + q^{2}_{l-2k}q^{2}_{n-2m}\Psi^{8}_{j,k,m}\}\end{aligned}$$

from which it follows that

$$\begin{aligned}C_{j+1,l,n} &= \langle\Phi_{j+1,l,n}, f\rangle\\&= 2\sum_{k,m}\{p_{l-2k}p_{n-2m}C_{j,k,m} + p_{l-2k}q^{1}_{n-2m}D^{1}_{j,k,m}\\&\quad+ p_{l-2k}q^{2}_{n-2m}D^{2}_{j,k,m} + q^{1}_{l-2k}p_{n-2m}D^{3}_{j,k,m} + q^{1}_{l-2k}q^{1}_{n-2m}D^{4}_{j,k,m}\\&\quad+ q^{1}_{l-2k}q^{2}_{n-2m}D^{5}_{j,k,m} + q^{2}_{l-2k}p_{n-2m}D^{6}_{j,k,m}\\&\quad+ q^{2}_{l-2k}q^{1}_{n-2m}D^{7}_{j,k,m} + q^{2}_{l-2k}q^{2}_{n-2m}D^{8}_{j,k,m}\}.\end{aligned}$$

The proof is thus complete.

3. Symmetric Extension Transforms Theorem

Perfect reconstruction of image after wavelet frame transform is of fundamental importance for image compression. The methods of symmetric extension transforms are described by the following theorem.

Theorem 3.1 (Symmetric extension transforms based on wavelet frames theory)**.** *Suppose that the length of the datum* $\{x_0, x_1, ..., x_{M-1}\}$ *is*

M. Divide symmetric extension transforms into three ways as follows:

(1) *If the lengths of high-pass filters and low-pass filter are odd*, *and they are* (*anti*)*symmetric about* 0, *then the following*:

Symmetric extension operator in time-domain

$$\begin{cases} x_n = x_{-n} & n < 0, \\ x_n = x_{2M-n-2} & n > M-1. \end{cases} \tag{3.1}$$

Symmetric extension operator of the low-frequency in frequency-domain

$$\begin{cases} S_l = S_{-l} & l < 0 \text{ on the left}, \\ S_l = S_{M-l-1} & l > \dfrac{M}{2} - 1 \text{ on the right}. \end{cases} \tag{3.2}$$

Symmetric extension operator of the high-frequency in frequency-domain

$$\begin{cases} d_l = d_{-l} & l < 0 \text{ on the left}, \\ d_l = d_{M-l-1} & l > \dfrac{M}{2} - 1 \text{ on the right}. \end{cases} \tag{3.3}$$

(2) *If the lengths of high-pass filters and that of low-pass filter are odd*, *and if the low-pass filter is* (*anti*)*symmetric about* 0 *while high-pass filters are* (*anti*)*symmetric about* 1, *then the following*:

Symmetric extension operator in time-domain

$$\begin{cases} x_n = x_{-n} & n < 0, \\ x_n = x_{2M-n-2} & n > M-1. \end{cases} \tag{3.4}$$

Symmetric extension operator of the low-frequency in frequency-domain

$$\begin{cases} S_l = S_{-l} & l < 0 \text{ on the left}, \\ S_l = S_{M-l-1,} & l > \dfrac{M}{2} - 1 \text{ on the right}. \end{cases} \tag{3.5}$$

Symmetric extension operator of the high-frequency in frequency-domain

$$\begin{cases} d_l = d_{-l-1} & l < 0 \text{ on the left}, \\ d_l = d_{M-l-2} & l > \dfrac{M}{2} - 1 \text{ on the right}. \end{cases} \tag{3.6}$$

This conclusion holds in case when every 2 *or* 1 *of the* 3 *centers of symmetry of FIR filters are/is* 1 *and the other one/two is/are* 0.

(3) *If the lengths of high-pass filters and that of low-pass filter are even and they are* (*anti*)*symmetric about* $\frac{1}{2}$, *then the following*:

Symmetric extension operator in time-domain

$$\begin{cases} x_n = x_{-n-1} & n < 0, \\ x_n = x_{2M-n-1} & n > M-1. \end{cases} \tag{3.7}$$

Symmetric extension operator of the low-frequency in frequency-domain

$$\begin{cases} S_{-l} = S_{l-1} & \text{on the left}, \\ S_{\frac{M}{2}-1+l} = S_{\frac{M}{2}-l} & \text{on the right}. \end{cases} \tag{3.8}$$

Symmetric extension operator of the high-frequency in frequency-domain

$$\begin{cases} d_{-l} = d_{l-1} & \text{on the left}, \\ d_{\frac{M}{2}-1+l} = d_{\frac{M}{2}-l} & \text{on the right}. \end{cases} \tag{3.9}$$

Then the algorithms (2.7) *and* (2.8) *can realize perfect reconstruction of image after wavelet frame transform*.

Proof. Among the three statements, we take the second and the third as selections, since the other one can be proved in a similar way.

Suppose that h, g, f are filters, then from [3], it follows that

$$\begin{cases} S_l = \sum_n x_n h_{n-2l} = \sum_n x_{n+2l} h_n, \\ d_l^1 = \sum_n x_n g_{n-2l} = \sum_n x_{n+2l} g_n, \\ d_l^2 = \sum_n x_n f_{n-2l} = \sum_n x_{n+2l} f_n. \end{cases} \tag{3.10}$$

Together with $h_{-n} = h_n$, $g_n = g_{2-n}$, $f_n = f_{2-n}$, it follows that

$$S_{-l} = \sum_n x_{n-2l} h_n = \sum_n x_{-n+2l} h_n = \sum_n x_{-n+2l} h_{-n} = \sum_n x_{n+2l} h_n = S_l,$$

$$S_{\frac{M}{2}-1+l} = \sum_n x_{n+M-2+2l} h_n = \sum_n x_{M-n-2l} h_{-n} = \sum_n x_{M+n-2l} h_n = S_{\frac{M}{2}-l},$$

$$d^1_{-l} = \sum_n x_{n-2l} g_n = \sum_n x_{-n+2l} g_n = \sum_n x_{-n+2l} g_{2-n} = \sum_n x_{n+2l-2} g_n = d^1_{l-1},$$

$$d^1_{\frac{M}{2}-1+l} = \sum_n x_{n+M-2+2l} g_n = \sum_n x_{M-n-2l} g_{2-n} = \sum_n x_{M+n-2l-2} g_n = d^1_{\frac{M}{2}-1-l}.$$

Similarly, $d^2_{-l} = d^2_{l-1}$, $d^2_{\frac{M}{2}-1+l} = d^2_{\frac{M}{2}-1-l}$ hold.

Associate $h_n = h_{1-n}$, $g_n = g_{1-n}$, $f_n = f_{1-n}$ with (3.10), we can deduce that

$$S_{-l} = \sum_n x_{n-2l} h_n = \sum_n x_{-n+2l-1} h_n = \sum_n x_{-n+2l-1} h_{1-n}$$

$$= \sum_n x_{n+2l-2} h_n = S_{l-1},$$

$$S_{\frac{M}{2}-1+l} = \sum_n x_{n+M-2+2l} h_n = \sum_n x_{M-n-2l+1} h_{1-n} = \sum_n x_{M+n-2l} h_n = S_{\frac{M}{2}-l},$$

$$d^1_{-l} = \sum_n x_{n-2l} g_n = \sum_n x_{-n+2l-1} g_n = \sum_n x_{-n+2l-1} g_{1-n}$$

$$= \sum_n x_{n+2l-2} g_n = d^1_{l-1},$$

$$d^1_{\frac{M}{2}-1+l} = \sum_n x_{n+M-2+2l} g_n = \sum_n x_{M-n-2l+1} g_n$$

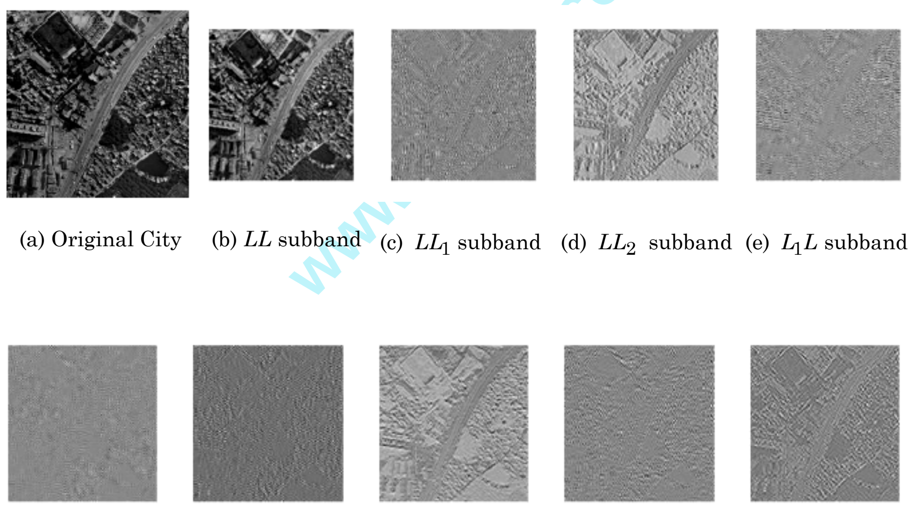
$$= \sum_n x_{M-n-2l+1} g_{1-n} = \sum_n x_{M+n-2l} g_n = d^1_{\frac{M}{2}-l}.$$

Similarly, $d^2_{-l} = d^2_{l-1}$, $d^2_{\frac{M}{2}-1+l} = d^2_{\frac{M}{2}-l}$ hold.

The proof is now complete.

4. Experiment

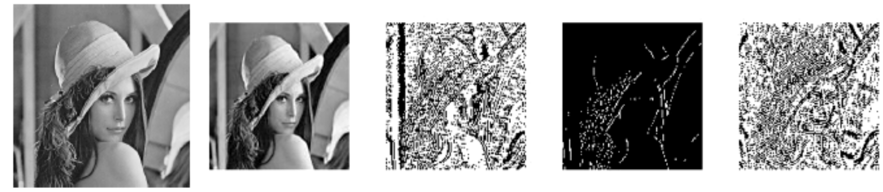
Nowadays, drastic development in the theory of wavelet frames has taken place. The major problem confronted by us is how to integrate theory with practice. By analyzing the similarities and differences between wavelet frames transform (WFT) and biorthogonal wavelet transform, it can easily be seen that time-frequency decompression of signals and the separation of high and low frequency signals can be achieved by WFT, when the perfect construction of image after WFT can be brought about. In the meanwhile, wavelet frames possess 2 or more mother functions and bear relatively high flexibility and greater capability to handle with high-frequency information. When wavelet frame possess 2 mother functions, suppose that low-pass filter is L, high-pass filters are L_1, L_2, then we get 9 subbands, LL, LL_1, LL_2, L_1L, L_1L_1, L_1L_2, L_2L, L_2L_1, L_2L_2. By the filters taken as an example in [12] whose symmetry satisfies Theorem 3.1, and by the algorithm of Theorem 2.1, we can obtain the following decomposition and reconstruction images of the city and that of Lena after WFT.



(a) Original City (b) LL subband (c) LL_1 subband (d) LL_2 subband (e) L_1L subband

(f) L_1L_1 subband (g) L_1L_2 subband (h) L_2L subband (k) L_2L_1 subband (l) L_2L_2 subband

Figure 1. Decomposition of image of the City.



(a) Original Lena (b) LL subband (c) LL_1 subband (d) LL_2 subband (e) L_1L subband

(f) L_1L_1 subband (g) L_1L_2 subband (h) L_2L subband (k) L_2L_1 subband (l) L_2L_2 subband

Figure 2. Decomposition of image of the Lena.

In order to prove perfect reconstruction of image after WFT by Theorems 2.1 and 3.1, the following definition is given

Definition 4.1.

$$MSE = \frac{\sum_{i,\, j=1}^{N} (\hat{f}(i,\, j) - f(i,\, j))^2}{N^2}, \tag{4.1}$$

where $f(i,\, j),\, i,\, j = 1, 2, \ldots, N$ represent original image, $\hat{f}(i,\, j)$, $i,\, j = 1, 2, \ldots, N$ represent reconstruction image.

By computing, MSE of Lena is 1.5×10^{-27}, MSE of City is 1.7×10^{-25}.

(a) Reconstruction of City (b) Reconstruction of Lena

Figure 3. Reconstruction of image.

As is observed, the reconstruction and the original images are completely equal, the above experimental results testify Theorems 2.1 and 3.1 in this paper.

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