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WAVELET LINEAR DENSITY ESTIMATION FOR NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract

This note considers the wavelet based linear density estimator for the probability density function considered in Prakasa Rao [11]. The results obtained for associated sequences by Prakasa Rao [11] are extended to the case of negatively dependent sequences.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables. A finite family of

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random variables $\{X_1, X_2, \dots, X_N\}$ is said to be *negatively dependent* (ND) if

$$P\left\{\bigcap_{j=1}^{n}(X_j \leq x_j)\right\} \leq \prod_{j=1}^{n} P\{X_j \leq x_j\}$$

and

$$P\left\{\bigcap_{j=1}^{n}(X_j > x_j)\right\} \leq \prod_{j=1}^{n} P\{X_j > x_j\}.$$

An infinite family of random variables is said to be *ND* if every finite subfamily is ND.

The following lemma was proved in Bozorgnia et al. [1]. We use it in obtaining the main result in the next section.

Lemma 1.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of ND random variables and* $\{f_n, n \geq 1\}$ *be a sequence of Borel functions all of which are monotone increasing* (*or all are monotone decreasing*). *Then* $\{f(X_n), n \geq 1\}$ *is a sequence of ND random variables.*

Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables with a common one-dimensional marginal probability density function f. Prakasa Rao [11] proposed a wavelet based linear estimator of f in case the sequence of random variables given above is associated. Recently, such results have also been extended to the negatively associated sequences by Doosti et al. [6]. This allows one to obtain upper bounds on the L_p losses for the resulting estimator as shown in Prakasa Rao [11]. The purpose of this note is to extend these results for estimating the density of ND of random variables.

Some preliminaries of the linear wavelet estimator of a probability density function is given in Section 2 and Section 3 provides the bounds on the L_p-losses for the proposed estimator.

2. Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables on the probability space $(\Omega, \aleph, P)$. We suppose that X_i has a bounded and compactly supported marginal density $f(\cdot)$, with respect to the Lebesgue measure, which does not depend on i. We are interested in estimating this density from n observations X_i, $i = 1, \ldots, n$. The motivation behind wavelet based linear estimator of the density comes from a formal expansion (see Daubechies [2, 3]) for any function $f \in \mathbf{L}_2(\mathbf{R})$,

$$f = \sum_{k \in Z} \alpha_{j_0, k} \phi_{j_0, k} + \sum_{j \geq j_0} \sum_{k \in Z} \delta_{j, k} \psi_{j, k}$$

$$= P_{j_0} f + \sum_{j \geq j_0} D_j f,$$

where the functions

$$\phi_{j_0, k}(x) = 2^{j_0/2} \phi(2^{j_0} x - k)$$

and

$$\psi_{j, k}(x) = 2^{j/2} \psi(2^j x - k)$$

constitute an (inhomogeneous) orthonormal basis of $\mathbf{L}_2(\mathbf{R})$. Here $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet, respectively. Wavelet coefficients are given by the integrals

$$\alpha_{j_0, k} = \int f(x) \phi_{j_0, k}(x) dx,$$

$$\delta_{j, k} = \int f(x) \psi_{j, k}(x) dx.$$

We suppose that both ϕ and $\psi \in \mathbf{C}^{r+1}$, $r \in \mathbf{N}$, have compact supports included in $[-\delta, \delta]$. Note that, by Corollary 5.5.2 in Daubechies [2], ψ is orthogonal to polynomials of degree $\leq r$, i.e.,

$$\int \psi(x) x^l dx = 0, \quad \forall l = 0, 1, \ldots, r.$$

We suppose that f belongs to the Besov class (see Meyer [10], Section VI. 10), $F_{s, p, q} = \{f \in B_{p, q}^s, \| f \|_{B_{p, q}^s} \leq M\}$ for some $0 < s < r + 1$, $p \geq 1$ and $q \geq 1$, where

$$\| f \|_{B^s_{p,q}} = \| P_{j_0} f \|_p + \left(\sum_{j \geq j_0} (\| D_j f \|_p 2^{js})^q \right)^{1/q},$$

with $\| g \|_p = \left(\int | g |^p \right)^{(1/p)}$.

We may also say $f \in B^s_{p,q}$ if and only if

$$\| \alpha_{j_0,\cdot} \|_{l_p(Z)} < \infty, \text{ and } \left(\sum_{j \geq j_0} (\| \delta_{j,\cdot} \|_{l_p(Z)} 2^{j(s+1/2-1/p)})^q \right)^{1/q} < \infty, \tag{2.1}$$

where $\| \gamma_{j,\cdot} \|_{l_p(Z)} = \left(\sum_{k \in Z} \gamma^p_{j,k} \right)^{1/p}$. We consider Besov spaces essentially because of their executional expressive power (see Triebel [13] and the discussion in Donoho et al. [5]). We construct the density estimator

$$\hat{f} = \sum_{k \in K_{j_0}} \hat{\alpha}_{j_0,k} \phi_{j_0,k}, \text{ with } \hat{\alpha}_{j_0,k} = \frac{1}{n} \sum_{i=1}^{n} \phi_{j_0,k}(X_i), \tag{2.2}$$

where K_{j_0} is the set of k such that $\operatorname{supp}(f) \cap \operatorname{supp}(\phi_{j_0,k}) \neq \emptyset$.

The fact that ϕ has a compact support implies that K_{j_0} is finite and $\operatorname{card} K_{j_0} = O(2^{j_0})$. Wavelet density estimators aroused much interest in the recent literature, see Donoho et al. [4] and Doukhan and Leon [7]. In the case of independent samples the properties of the linear estimator (2.2) have been studied for a variety of error measures and density classes (see Kerkyacharian and Picard [8], Leblanc [9] and Tribouley [14]). It was shown, for example, that these estimators are minimax with respect to L_p-risk for densities belonging to Besov space $B^s_{p,q}$. When the error of estimation is measure in $L_{p'}$-norm, with $p' \geq p$, the linear wavelet estimators are not optimal anymore, although they are still minimax in the class of linear estimators (see Donoho et al. [4] and Kerkyacharian and Picard [8]).

3. Main Results

In the following theorems we take density to have compact support on [0, 1]. The scale function may typically be taken to be a compactly supported density on [0, 1], in the following theorem we take it to be monotone, such as the linear density or uniform density on [0, 1]. Theorem 3.1 gives bounds on $E_f\|\hat{f} - f\|_{p'}^2$ for $P' > \max(2, p)$.

Theorem 3.1. *Let* $f \in F_{s,p,q}$ *with* $s \geq 1/p$, $p \geq 1$, *and* $q \geq 1$. *Then for* $p' \geq \max(2, p)$, *there exists a constant C such that*

$$\mathbf{E}\|\hat{f} - f\|_{p'}^2 \leq Cn^{-\frac{2s'}{1+2s'}},$$

where $s' = s + 1/p' - 1/p$ *and* $2^{j_0} = n^{\frac{1}{1+2s'}}$.

Proof. First, we decompose $\mathbf{E}\|\hat{f} - f\|_{p'}^2$ into a bias term and stochastic term

$$\mathbf{E}\|\hat{f} - f\|_{p'}^2 \leq 2(\|f - P_{j_0}f\|_{p'}^2 + \mathbf{E}\|\hat{f} - P_{j_0}f\|_{p'}^2) = 2(T_1 + T_2). \tag{3.1}$$

Now, we want to find upper bounds for T_1 and T_2. From Leblanc [9, p. 83]

$$T_1 \leq K2^{-2s'j_0}. \tag{3.2}$$

Next, we have

$$T_2 = \mathbf{E}\|\hat{f} - P_{j_0}f\|_{p'}^2 = \mathbf{E}\left\|\sum_{k \in K_{j_0}} (\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})\phi_{j_0,k}(x)\right\|_{p'}^2.$$

Now the use of Lemma 1 in Leblanc [9, p. 82] (using Meyer [10]) gives

$$T_2 \leq C\mathbf{E}\left\{\sum_{k \in K_{j_0}} \|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}\|_{l_{p'}}^2\right\} 2^{2j_0(1/2 - 1/p')}.$$

Further, by using Jensen's inequality the above equation implies

$$T_2 \leq C2^{2j_0(1/2-1/p')}\left\{\sum_{k\in K_{j_0}} \mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\right\}^{2/p'}. \tag{3.3}$$

To complete the proof, it is sufficient to estimate $\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}$. We know that

$$\hat{\alpha}_{j_0,k} - \alpha_{j_0,k} = \frac{1}{n}\sum_{i=1}^{n}\{[\phi_{j_0,k}(X_i) - \alpha_{j_0,k}]\}.$$

Denote $\xi_i = [\phi_{j_0,k}(X_i) - \alpha_{j_0,k}]$. Because of ND property (Lemma 1.1) and monotonicity of scale function, we know $\{\xi_i, n \geq 1\}$ remains a sequence of ND random variables. Moreover $\|\xi_i\|_\infty \leq K2^{j_0/2}\|\phi\|_\infty$, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 \leq \|f\|_\infty$ and $|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}| = \frac{1}{n}\left|\sum_{i=1}^{n}\xi_i\right|$.

Now, we need the following theorem from Rivaz et al. [12].

Theorem 3.2. *Let* $\xi_1, \ldots, \xi_n$ *be a sequence of ND identically distributed random variables such that* $\mathbf{E}(\xi_i) = 0$, $\|\xi_i\|_\infty < M$. *Then there exists* $C(p)$, *such that*

$$\mathbf{E}\left(\left|\sum_{i=1}^{n}\xi_i\right|^p\right) \leq C(p)\left\{M^{p-2}\sum_{i=1}^{n}\mathbf{E}(\xi_i^2) + \left(\sum_{i=1}^{n}\mathbf{E}(\xi_i^2)\right)^{p/2}\right\}, \quad p > 2.$$

Using the above theorem and the fact that $\mathrm{card}K_{j_0} = O(2^{j_0})$ we have

$$\left\{\sum_{k\in K_{j_0}}\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\right\}^{2/p'} \leq \left\{C2^{j_0}\frac{1}{n^{p'}}(n2^{j_0/2(p'-2)}c_1 + n^{p'/2}c_2)\right\}^{2/p'}$$

$$\leq K_1\left\{\frac{2^{j_0}}{n^{2(1-1/p')}} + \frac{2^{2j_0/p'}}{n}\right\}.$$

Now by substituting above inequality in (3.3), we get

$$T_2 \le K_1 2^{2j_0(1/2-1/p')}\left\{\frac{2^{j_0}}{n^{2(1-1/p')}} + \frac{2^{2j_0/p'}}{n}\right\}$$

$$= K_1\left\{\frac{2^{2j_0-2j_0/p'}}{n^{2-2/p'}} + \frac{2^{j_0}}{n}\right\}$$

$$= K_1\left\{\frac{2^{j_0}}{n}\left(\frac{2^{j_0}}{n}\right)^{1-2/p'} + \frac{2^{j_0}}{n}\right\},$$

since $n \ge 2^{j_0}$ and $1 - 2/p' \ge 0$ imply $\left(\frac{2^{j_0}}{n}\right)^{1-2/p'} \le 1.$ Hence

$$T_2 \le \frac{K_2 2^{j_0}}{n}. \tag{3.4}$$

By substituting (3.2), (3.4), and $2^{j_0} = \frac{1}{n^{1+2s'}}$ in (3.1) theorem is proved.

Remark. Suppose $1 < p' \le 2$. One can get upper bounds similar to those as Theorem 3.1 for the expected loss $\mathbf{E}\|\hat{f} - f\|_{p'}^{p'}$.

Observing that

$$\mathbf{E}\|\hat{f} - f\|_{p'}^{p'} \le 2^{p'-1}(\|f - P_{j_0}f\|_{p'}^{p'} + \mathbf{E}\|\hat{f} - P_{j_0}f\|_{p'}^{p'}), \tag{3.5}$$

$$\|f - P_{j_0}f\|_{p'}^{p'} \le C_1 2^{-p's'j_0}, \tag{3.6}$$

$$\mathbf{E}\|\hat{f} - P_{j_0}f\|_{p'}^{p'} \le C_2 2^{2j_0(p'/2-1)}\left\{\sum_{k\in K_{j_0}} \mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\right\}$$

$$\le C_2 2^{2j_0(p'/2-1)}\left\{\sum_{k\in K_{j_0}} \sqrt{\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{2p'}}\right\}$$

$$\le C_3 2^{2j_0(p'/2-1)}\left\{2^{j_0}\frac{1}{n^{p'}}(n2^{j_0(p'-2)} + n^{p'})\right\}. \tag{3.7}$$

for some positive constants C_1, C_2 and C_3.

References

[1] A. Bozorgnia, R. F. Patterson and R. L. Taylor, Limit theorems for dependent random variables, World Congress of Nonlinear Analysts'92, Vol. I-IV, pp. 1639-1650, de Gruyter, Berlin, 1996.

[2] I. Daubechies, Orthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988), 909-996.

[3] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conferences Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[4] D. L. Donoho, I. M. Johnstone, G. Kerkyacharian and D. Picard, Density estimation by wavelet thresholding, Ann. Statist. 24 (1996), 508-539.

[5] D. L. Donoho, I. M. Johnstone, G. Kerkyacharian and D. Picard, Wavelet shrinkage: asymptopia (with discussion), J. Roy. Statist. Soc. Ser. B 57(2) (1995), 301-369.

[6] H. Doosti, V. Fakoor, and Y. P. Chaubey, Wavelet linear density estimation for negatively associated sequences, J. Indian Statist. Assoc. (2006), to appear.

[7] P. Doukhan and J. R. León, Déviation quadratique d'estimateurs de densité par projections orthogonales, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 425-430.

[8] G. Kerkyacharian and D. Picard, Density estimation in Besov spaces, Statist. Probab. Lett. 13 (1992), 15-24.

[9] F. Leblanc, Wavelet linear density estimator for a discrete-time stochastic process: L_p -losses, Statist. Probab. Lett. 27 (1996), 71-84.

[10] Y. Meyer, Ondelettes et Opérateurs, Hermann, Paris, 1990.

[11] B. L. S. Prakasa Rao, Wavelet linear density estimation for associated sequences, J. Indian Statist. Assoc. 41 (2003), 369-379

[12] F. Rivaz, M. Amini, and A. Bozorgnia, Moment inequalities and applications for negative dependence random variables, Research Bulletin of Isfehan University (2006), to appear.

[13] H. Triebel, Theory of Function Spaces, II, Birkhäuser Verlag, Berlin, 1992.

[14] K. Tribouley, Practical estimation of multivariate densities using wavelet methods, Statist. Neerlandica 49 (1995), 41-62.