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SOME RESULTS ON GENERALIZED SOFT SUBGROUPS

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Abstract

This paper is devoted to give some properties of generalized soft subgroups with respect to a variety of groups ν, which generalizes the work of Blackburn and Héthelyi [1] and Héthelyi [3] with respect to the abelian variety. It is shown that if A is a ν-soft subgroup of index greater than p, then the A-invariant subgroups of $V(G)V^*(N_G(A))$ containing $V^*(N_G(A))$ form a chain and also shown that if the p-group G has a uniserially embedded subgroup P of order p, then either G has a cyclic subgroup of index p or is of maximal class.

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1. Introduction and Preliminary Results

Let F_∞ be a free group freely generated by a countable set $\{x_1, x_2, ...\}$. Let ν be a variety of groups defined by a subset V of F_∞. We assume that the reader is familiar with the notions of the verbal subgroup, $V(G)$, and the marginal subgroup, $V^*(G)$, associated with a variety of groups ν and a given group G. See [2, 8, 9] for more information on variety of groups.

We define a series of a group G with respect to a given variety ν as follows:

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$$G = V_0(G) \supseteq V_1(G) = V(G) \supseteq V_2(G) \supseteq \cdots \supseteq V_n(G) \supseteq \cdots,$$

where $V_n(G) = [V_{n-1}(G)V^*G]$ for $n > 0$, which is called the *lower* ν-*marginal series* of G with respect to the variety ν. The corresponding *upper* ν-*marginal series* of G is defined as follows:

$$1 = V_0^*(G) \subseteq V_1^*(G) = V^*(G) \subseteq \cdots \subseteq V_n^*(G) \subseteq \cdots,$$

where, $V_n^*(G)$ will be defined by

$$\frac{V_n^*(G)}{V_{n-1}^*(G)} = V^*\left(\frac{G}{V_{n-1}^*(G)}\right), \quad n > 0.$$

By induction on i, one may check that $V_i^*\left(\frac{G}{V_j^*(G)}\right) = \frac{V_{i+j}^*(G)}{V_j^*(G)}$, for all $j \geq 0$. Clearly $V_c(G) = 1$ if and only if $V_c^*(G) = G$, for all $c \geq 0$. (See also [5, 6, 7]).

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Let G be a finite p-group, where p is a prime number. As in [3] a proper subgroup H of G is called ν-*soft* if H is a maximal abelian subgroup of G and is of index p in its normalizer with respect to a variety of groups ν. The main properties of soft subgroups are given in [3], [4] and [1]. A soft subgroup H of G is always uniserially embedded in G, that is, the subgroups of G containing H form a chain.

A p-group G is called a ν-*CF-group* with respect to variety ν, if the index of any term of the lower ν-marginal series of G beyond $V(G)$ in its predecessor is at most p. We show that if G has a soft subgroup different from A, then G is a ν-*CF*-group. Our result, in a way, are similar to the works of N. Blackburn and L. Héthelyi in abelian variety. (See also [1]).

Theorem 1.1. *Let* ν *be a variety of groups defined by the set of laws* V. *Suppose that* A *is a maximal normal abelian subgroup of the non-abelian* p-*group* G *and that* G/A *is cyclic. Suppose that* G *has soft subgroup* B *distinct from* A. *Then*

(i) $G = AB$.

(ii) $d(G/V^*(G)) = 2$, *and if* $|G : A| = p^{\alpha}$, $G/V(G)V^*(G)$ *is of type* (p^{α}, p).

(iii) *G is a* ν-*CF-group*.

Proof. (i) If B is normal in G, then $|G : B| = p$, since B is soft, and A is not contained in B, since A, B are self-centralising and distinct: hence $G = AB$. If B is not normal in G, let M be the unique maximal subgroup of G containing B. Let $R = V(G)V^*(N)$, where $N = N_G(B)$. By [4] Theorem 2, $|G : R| = p^2$ and $M \geq R$; further, if $x \in M \setminus R$, then x is conjugate to an element y of $B \setminus V^*(N)$ and $C_G(y) = B$. Hence if $x \in M \setminus R$, then $x \notin A$, since A, B are self-centralising and distinct. Hence $A \cap M \leq R$. By [4, Corollary 6], G/R is non-cyclic, so $A \not\leq R$. Hence $A \not\leq M$. Since B is soft and AB is a subgroup containing B but not in M, $AB = G$.

(ii) Since G/A is cyclic, there exists $b \in B$ such that $G = A\langle b\rangle$. And $|A : A \cap M| = |AM : M| = p$, there exists $a \in A$ such that $A = (A \cap M)\langle a\rangle$. Thus $G = (A \cap M)\langle a, b\rangle$. But

$$V(G) \leq A \cap M \leq R = V(G)V^*(N).$$

So $A \cap M = V(G)(A \cap M \cap V^*(N)) = V(G)V^*(G)$ (even if $B \lhd G$). Thus $|A : V(G)V^*(G)| = p$ and $G = V(G)V^*(G)\langle a, b\rangle$. Hence $G/V(G)V^*(G)$ is of type (p^{α}, p) and $d(G/V^*(G)) = 2$.

(iii) Let $c = [a, b]$. Since $a^p \in V(G)V^*(G)$, $[a^p, b] \in V_2(G)$. Now

$$[a^p, b] = a^{-p}(a^p)^p = (a^{-1}a^p)^p = [a, b]^p,$$

so $[a, b]^p \in V_2(G)$. Since $V(G) = \langle [a, b], V_2(G)\rangle$, $|V(G) : V_2(G)| = p$. It follows by an easy induction that $|V_i(G) : V_{i-1}(G)| = p$ for $V_i(G) \neq 1$: if $i \geq 3$ and $V_{i-1} = \langle u, V_i(G)\rangle$ with $u^p \in V_i(G)$, then $V_i(G) = \langle [a, u], [b, u],$

$V_{i+1}(G)\rangle$, and $[a, u] = 1$, since $u \in V(G) \leq A$. And $[b, u]^p \in V_{i+1}(G)$, since $[b, u]^p = (u^p)^{-p} u^p = [b, u^p]$. Thus G is ν-*CF*-group.

Corollary 1.2. *Suppose that A is a maximal normal abelian subgroup of the non-abelian p-group G such that* G/A *is cyclic. If there is a soft subgroup B of G contained in the maximal subgroup of G containing A, then* $|G : A| = p$.

Proof. For $AB = G$ cannot hold, so $B = A$.

We now consider the case $|G : A| = p$.

Theorem 1.3. *Let G is a p-group of class greater than* 2 *and A is an abelian subgroup of G of index p. Then the following are equivalent.*

(i) *Every maximal abelian subgroup of G is* ν*-soft.*

(ii) *G has a* ν*-soft subgroups distinct from A.*

(iii) $G/V^*(G)$ *is a p-group of maximal class.*

(iv) $|V_2^*(G) : V^*(G)| = p$.

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Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) By Theorem 1.1, G is a ν-*CF*-group and $G/V(G)V^*(G)$ is of type (p, p). If $\overline{G} = G/V^*(G)$, $\overline{G}/V(\overline{G})$ is of type (p, p) and $\overline{G}$ is a ν-*CF*-group, so $\overline{G}$ is of maximal class.

(iii) ⇒ (iv) is trivial.

(iv) ⇒ (i) Let $|V_2^*(G) : V^*(G)| = p$ and B is a maximal abelian subgroup of G. Suppose that $N = N_G(B)$, where $A \neq B$. Since $G = AB$, $N = (N \cap A)B$ and

$$[N \cap A, G] = [N \cap A, AB] = [N \cap A, B] \leq A \cap B = V^*(G),$$

so $N \cap A \leq V_2^*(G)$. Thus $|N : B| \leq |V_2^*(G)B : B| \leq |V_2^*(G) : V_1^*(G)| = p$ and $|N : B| = p$, hence B is ν-soft.

Theorem 1.4. *Let the p-group G has a maximal normal abelian subgoup A for which* G/A *is cyclic. If* $|V^*(G) \cap V(G)| = p$, *then G has a* ν*-soft subgroup distinct from A.*

Proof. Let $G = A\langle b\rangle$ and $B = C_G(b)$. Now $C_G(b) \cap A = V^*(G)$, and since $G = A\langle b\rangle$, $B = V^*(G)\langle b\rangle$. Hence B is abelian and self centralising. Suppose that $N = N_G(B)$. If $x \in N$, $[x, b] \in B \cap V(G) = V^*(G) \cap V(G)$, since $B \cap A = V^*(G)$. Hence there is a mapping ξ of N/B into $V^*(G) \cap V(G)$ given by $(xB)\Xi = [x, b]$ $(x \in N)$. And ξ is injective, so $|N : B| \leq |V^*(G) \cap V(G)| = p$. Hence $|N : B| = p$ and B is soft.

Let ν be the variety of abelian groups. In this case it follows from above theorem that if G is a ν-*CF*-group (= *CF*-group) having the usual subgroup A, then G has a soft subgroup B distinct from A. But despite the equivalence of (i) and (iii) in Theorem 1.3, it is not the case that a group G for which $G/V^*(G)$ is a ν-*CF*-group with two generators necessarily has a ν-soft subgroup. (See [1] Theorem 1.3).

2. The Main Results

A subgroup H of a p-group G is n-uniserial with respect to a variety of groups ν, if for each $i = 1, \ldots, n$, there is no unique subgroup K_i such that $H \leq K_i$ and $|K_i : H| = p^i$. In case the subgroups of G containing H form a chain we say that H is ν-*uniserially embedded* in G. In this section we give some important results of ν-soft subgroups which is a vast generalization of soft subgroups in abelian variety.

Theorem 2.1. *Let* ν *be a variety of groups defined by the set of laws V. Let A be a* ν*-soft subgroup of index greater than P. Suppose that* $N_1 = N_G(A)$, $R = V(G)V^*(N_1)$. *Then the A-invariant subgroups of R containing* $V^*(N_1)$ *form a chain.*

Proof. The subgroup of G containing A form a chain

$$A = N_0 < N_1 < N_2 < \cdots < N_{k-1} = M,$$

where $|G : M| = p$ and $N_i = N_G(N_{i-1})$ $(i = 1, \ldots, k-1)$. Thus $R \leq M$ and

$$R \cap N_0 \leq R \cap N_1 \leq \cdots \leq R \cap N_{k-1} = R \leq M$$

is a sequence of A-invariant subgroups of R containing $V^*(N_1)$. Let X be an A-invariant subgroup of R containing $V^*(N_1)$. Since $A \leq AX \leq G$, $AX = N_i$ for some i. Thus $X \leq R \cap N_i$ and $|X| \cdot |A \cap X| = |N_i|$. Since $A \cap X \geq V^*(N_1)$,

$$|A : A \cap X| \leq |A : V^*(N_1)| = p,$$

(by [3, Lemma 1]). Hence $|N_i| \leq p|X|$. But since $N_i \not\leq R$, $|R \cap N_i| < |N_i|$, so $|X| \geq |R \cap N_i|$. Hence $X = R \cap N_i$. Thus the A-invariant subgroups of R containing $V^*(N_1)$ form a chain

$$V^*(N_1) = R \cap N_0 < R \cap N_1 < \cdots < R \cap N_{k-1} = R.$$

Theorem 2.2. *Let* v *be a variety of groups defined by the set of laws V. Let G be a non-abelian p-group and for every* $x \in G \setminus V^*(G)$, $C_G(x)$ *is abelian. Then either G has an abelian subgroup of index p or the exponent of* $G \setminus V^*(G)$ *is p.*

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Proof. Let A be a maximal normal abelian subgroup of G. Suppose that $s \in G \setminus A$. Let $H = \langle A, s \rangle$, so H is non-abelian and $V^*(H) < A$. Thus $H/V^*(H)$ has a normal subgroup $Y/V^*(H)$ of order p with $Y \leq A$. If $Y = \langle V^*(H), a \rangle$, $a^p \in V^*(H)$, so $a^p = (a^p)^s = (a^s)^p = (a \cdot [a, s])^p = a^p \cdot [a, s]^p$ and $[a, s]^p = 1$. Also $[a, s] \in V^*(H)$, so $[a, s^p] = [a, s]^p = 1$. Hence $C_G(s^p)$ contains $\langle a, s \rangle$; as this is non-abelian, $s^p \in V^*(G)$.

Thus $s^p \in V^*(G)$ for all $s \in G \setminus A$, in particular G/A is of exponent p. If $|G/A| > p$, choose $x \in G \setminus A$ with $xA \in V^*(G/A)$ and $y \in G$, $y \notin A\langle x \rangle$. Then $(x^i y^j a)^p \in V^*(G)$ for all $a \in A$ and $(i, j) \not\equiv (0, 0)(p)$. Hence if ξ, η are the automorphisms of the abelian group $A/V^*(G)$ given by

$$\overline{a}\xi = \overline{a}^x, \quad \overline{a}\eta = \overline{a}^y \quad (\overline{a} \in A/V^*(G)),$$

then $\overline{a}^{x^i y^j} = \overline{a}\xi^i\eta^j$, so

$$(x^i y^j)^p \{\overline{a}((\xi^i\eta^j)^{p-1} + \cdots + \xi^i\eta^j + 1)\} = 1$$

for all $\overline{a} \in A/V^*(G)$. Hence

$$((\xi^i\eta^j)^{p-1} + \cdots + \xi^i\eta^j + 1) = 0$$

for all $(i, j) \not\equiv (0, 0)(p)$. But then

$$0 = \left(\sum_{i=0}^{p-1} \xi^i\right) \cdot \left(\sum_{j=0}^{p-1} \eta^j\right) = \sum_{j=0}^{p-1} \eta^j + \sum_{j=0}^{p-1}\left(\sum_{i=1}^{p-1} (\xi\eta^j)^i\right) = 0 + \sum_{j=0}^{p-1} (-1).$$

Thus $p \cdot 1 = 0$ and $A/V^*(G)$ is elementary abelian. Thus the exponent of $G/V^*(G)$ is p.

ν-soft subgroups are uniserially embedded, but this is also possible for subgroups P of order p, although these are never soft. In the following we investigate this situation.

Theorem 2.3. *Let* ν *be a variety of groups defined by the set of laws* V. *Suppose that the p-group* G *has a uniserially embedded subgroup* P *of order* p. *Then either* G *has a cyclic subgroup of index* p *or is of maximal class* (*coclass* 1).

Proof. We proceed by induction. It is trivial if $P \lhd G$, since G/P has only one maximal subgroup and is therefore cyclic. So we suppose this is not the case, then the class k of G is at least 2. Let N be a subgroup of order p contained in $V_k(G)$, where

$$G = V_0(G) \supseteq V_1(G) = V(G) \supseteq V_2(G) \supseteq \cdots \supseteq V_k(G) \supseteq 1$$

is the lower ν-marginal series of G with respect to the variety ν. Thus $N \lhd G$ and $P \neq N$, and PN/N is uniserially embedded in G/N. By the inductive hypothesis, either G/N has a cyclic subgroup of index p or is of maximal class.

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Suppose first that G/N is of maximal class. If $N = V_k(G)$, then G is of maximal class. Otherwise $|V_k(G) : N| = p$ and $V_k(G)$ is marginal and elementary abelian of order p^2. Since P is not normal, $P \not\leq V_k(G)$; but P normalizes all the $p+1$ subgroups of $V_k(G)$ of order p and is contained in at least $p+1$ subgroups of order p^2, contrary to the hypothesis.

Now suppose that G/N has a cyclic subgroup M/N of index p. If M is cyclic there is nothing to prove, so we suppose M is abelian of type (p^r, p). If $r = 1$, then $|G| = p^3$ and G is of maximal class. If $r \geq 2$, then M has a characteristic subgroup K of order p such that M/K is not cyclic. Hence $K \neq N$ and $K \lhd G$. Then PK and PN are subgroups of order p^2 containing P, so $PK = PN = L$, say. Thus $L = KN \leq V^*(G)$ and $P \lhd G$.

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