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TOPOLOGY AND GEOMETRY OF SUBMANIFOLDS IMMERSED IN SPACE FORMS

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Abstract

Let M^m $(m > 3)$ be a compact submanifold immersed in a space form $N^n(c)$ with $c \geq 0$. In this paper, it is showed that if the square length s of the second fundamental form and the mean curvature H of M^m satisfy $s < \frac{m^2}{m-2}H^2 + 4c$, then for $p = 2, 3, \ldots, m-2$, there is no stable integral p-current in M^m, and the homology groups $H_p(M, Z) = 0$.

1. Introduction

Let M^m be a submanifold immersed in a Riemannian manifold N^n. Denote by $V(N, M)$ the normal bundle of M^m in N^n. For a smooth section $\nu \in C(V(N, M))$, the shape operator A_ν determined by ν satisfies

$$\langle A_\nu X, Y \rangle = \langle h(X, Y), \nu \rangle,$$

where $X, Y \in C(TM)$ and h is the second fundamental form of the submanifold M^m.

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Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_x(N, M)$ and $A_\lambda = A_{\nu_\lambda}$. Let s be the square length of the second fundamental form, and H be the mean curvature vector field of the submanifold M^m. Then

$$s = \sum_\lambda \mathrm{tr} A_\lambda^2, \quad H = \frac{1}{m} \sum_\lambda (\mathrm{tr} A_\lambda) \nu_\lambda.$$

For each fixed index λ, choose an orthonormal basis $\{E_a\}$ in $T_x M$ such that

$$A_\lambda E_a = k_{\lambda_a} E_a, \quad a = 1, 2, \ldots, m.$$

Then

$$\mathrm{tr} A_\lambda = \sum_a k_{\lambda_a}, \quad \mathrm{tr} A_\lambda^2 = \sum_a k_{\lambda_a}^2.$$

And thus

$$\frac{1}{m} \mathrm{tr} A_\lambda^2 = \frac{1}{m} \sum_a k_{\lambda_a}^2 \geq \left(\frac{1}{m} \sum_a k_{\lambda_a} \right)^2 = \left(\frac{1}{m} \mathrm{tr} A_\lambda \right)^2.$$

So, for any submanifold $M^m \hookrightarrow N^n$, s and H always satisfy the inequality: $s \geq mH^2$.

Relationship between s and H influences the geometric and topological structure of submanifolds. As an extension of the well-known gap theorem in minimal submanifolds, Okumura [3] proved that

Theorem O. *Let M^m be a compact connected submanifold immersed in a space form $N^n(c)$ with $c \geq 0$ and satisfy the following condition*

(C) *the connection of the normal bundle is flat and the mean curvature vector field H is parallel with respect to the connection of the normal bundle.*

If

$$s < \frac{m^2}{m-1} H^2 \text{ on } M^m, \tag{1}$$

then M^m is totally umbilical.

Cancelling condition (C), the first author [6] proved that

Theorem Z. *Let* M^m *be an oriented, connected submanifold immersed in a simply connected space form* $N^n(c)$, $m = \dim M \geq 4$. *If one of the following is satisfied*:

C_1: M^m *is compact*, $c \geq 0$, *and* $s < \frac{m^2}{m-1} H^2$ *on* M^m;

C_2: M^m *is complete*, $c > 0$, *and* $s \leq \frac{m^2}{m-1} H^2$ *on* M^m,

then M^m *is homeomorphic to a sphere.*

Shiohama and Xu [4] showed a more generalized result:

Theorem SX. *Let* M^m *be an oriented complete submanifold in a space form* $N^n(c)$ *with* $c \geq 0$. *If*

$$\sup\left\{s - \left[mc + \frac{m^3}{2(m-1)} H^2 - \frac{m(m-2)}{2(m-1)} \sqrt{m^2 H^4 + 4(m-1)cH^2}\right]\right\} < 0, \quad (2)$$

then M^m *is homeomorphic to a sphere.*

Note that the inequality (2) will be reduced to $s < \frac{m^2}{m-1} H^2$ if the ambient space $N^n(c)$ is the Euclidean space E^n.

The above conclusions indicate that for any immersed $M^m \to N^n(c)$, the inequality $s \geq mH^2$ is always hold. If $s < \frac{m^2}{m-1} H^2$ on M^m, then under the conditions of Okumura, M^m is totally umbilical. And when the condition (C) of Okumura is deleted, the M^m, topologically, is a sphere.

In this paper we shall further relax restrictions on s and H and prove the following:

Main Theorem. *Let* M^m *be an oriented compact submanifold immersed in a simply connected space form* $N^n(C)$ *with* $c \geq 0$, $m = \dim M > 3$. *If* $s < \frac{m^2}{m-2} H^2 + 4c$ *on* M^m, *then for* $p = 2, 3, \ldots$,

$m - 2,$ *there are no stable integral p-currents in* M^m *and*

$$H_2(M, Z) = H_3(M, Z) = \cdots = H_{m-2}(M, Z) = 0.$$

Example. As a submanifold, $M^m = S^1 \times S^{m-1}$ can be immersed into E^{m+2}. We can get that

$$s = m, \quad H^2 = \frac{1}{m^2}[(m-1)^2 + 1],$$

and thus

$$\frac{m^2}{m-1} H^2 < s < \frac{m^2}{m-2} H^2 \text{ on } M^m.$$

The first author [5] proved that for $0 < p < m_1 + m_2$, $p \neq m_1$ and $p \neq m_2$, there is no stable integral p-current in $S^{m_1} \times S^{m_2}$ and

$$H_p(S^{m_1} \times S^{m_2}, Z) = 0.$$

This conclusion tells us that when $m > 3$,

$$H_p(S^1 \times S^{m-1}, Z) = 0, \quad p = 2, 3, \ldots, m-2.$$

2. Proof of Main Theorem

For a given integer $p \in (0, m)$, let V be a p-dimensional subspace in $T_x M$ and $\{e_i\}$ be an orthonormal basis of V. Define a selfadjoint linear map $Q^A : V \to V$ associated with the immersion $M^m \to N^n(c)$ by

$$Q^A X = \sum_{\lambda} \left[2\left(\sum_i \langle A_\lambda^2 X, e_i \rangle e_i - B_\lambda^2 X \right) - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda) B_\lambda X \right], \tag{3}$$

where $X \in C(TM)$ and B_λ is a map on V associated with A_λ defined by

$$B_\lambda X = \sum_i \langle A_\lambda X, e_i \rangle e_i.$$

Q^A is independent of the choice of bases of $V_x(N, M)$ and V. Its trace is given by

$$\mathrm{tr}Q^A = \sum_i \langle Q^A e_i, e_i\rangle = \sum_\lambda \left[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda\right], \quad (4)$$

where $\{e_\alpha\}$ is an orthonormal basis of $V^\perp$ which is the orthogonal complement of V in $T_x M$.

Because $N^n(c)$ is a simply connected space form, it can be considered as a totally umbilical hypersurface of E^{n+1} [1, p.41]. The first author [5] proved the following:

Lemma. *Let M^m be a compact submanifold immersed in a totally umbilical hypersurface N^n with the sectional curvature $c \geq 0$ of E^{n+1}, and p be a given integer, $p \in (0, m)$. If for any $x \in M$ and any p-subspace V of $T_x M$,*

$$\mathrm{tr}Q^A < p(m-p)c, \quad (5)$$

then there is no stable integral p-current in M^m and $H_p(M, Z) = H_{m-p}(M, Z) = 0$.

Now we calculate $\mathrm{tr}Q^A$. Note that $\{e_i, e_\alpha\}$ is an orthonormal basis of $T_x M$, we have

$$\mathrm{tr}A_\lambda = \sum_i \langle A_\lambda e_i, e_i\rangle + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle, \quad (6)$$

$$\mathrm{tr}B_\lambda = \sum_i \langle A_\lambda e_i, e_i\rangle, \quad (7)$$

$$\begin{aligned}\mathrm{tr}A_\lambda^2 &= \sum_i \langle A_\lambda^2 e_i, e_i\rangle + \sum_\alpha \langle A_\lambda^2 e_\alpha, e_\alpha\rangle \\ &= \sum_i \langle A_\lambda e_i, A_\lambda e_i\rangle + \sum_\alpha \langle A_\lambda e_\alpha, A_\lambda e_\alpha\rangle. \quad (8)\end{aligned}$$

Because

$$A_\lambda e_i = \sum_j \langle A_\lambda e_i, e_j\rangle e_j + \sum_\alpha \langle A_\lambda e_i, e_\alpha\rangle e_\alpha,$$

$$\sum_i \langle A_\lambda e_i, A_\lambda e_i \rangle = \sum_{i,j} \langle A_\lambda e_i, e_j \rangle^2 + \sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 \tag{9}$$

and

$$\sum_\alpha \langle A_\lambda e_\alpha, A_\lambda e_\alpha \rangle = \sum_{\alpha,i} \langle A_\lambda e_\alpha, e_i \rangle^2 + \sum_{\alpha,\beta} \langle A_\lambda e_\alpha, e_\beta \rangle^2. \tag{10}$$

Substituting (9) and (10) into (8), we get

$$\begin{aligned} \mathrm{tr}A_\lambda^2 &= \sum_i \langle A_\lambda e_i, e_i \rangle^2 + \sum_{i\neq j} \langle A_\lambda e_i, e_j \rangle^2 + 2\sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 \\ &\quad + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle^2 + \sum_{\alpha\neq\beta} \langle A_\lambda e_\alpha, e_\beta \rangle^2. \end{aligned} \tag{11}$$

By (6) and (7) we have

$$\begin{aligned} (\mathrm{tr}A_\lambda)^2 &= \left(\sum_i \langle A_\lambda e_i, e_i \rangle\right)^2 + \left(\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle\right)^2 \\ &\quad + 2\sum_{i,\alpha} \langle A_\lambda e_i, e_i \rangle \langle A_\lambda e_\alpha, e_\alpha \rangle \\ &= \left(\sum_i \langle A_\lambda e_i, e_i \rangle\right)^2 + \left(\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle\right)^2 \\ &\quad + 2(\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda. \end{aligned} \tag{12}$$

(11) and (12) gives

$$\begin{aligned} \mathrm{tr}A_\lambda^2 - \frac{1}{2}(\mathrm{tr}A_\lambda)^2 &= 2\sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda \\ &\quad + \sum_i \langle A_\lambda e_i, e_i \rangle^2 + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle^2 \\ &\quad + \sum_{i\neq j} \langle A_\lambda e_i, e_j \rangle^2 + \sum_{\alpha\neq\beta} \langle A_\lambda e_\alpha, e_\beta \rangle^2 \\ &\quad - \frac{1}{2}\left(\sum_i \langle A_\lambda e_i, e_i \rangle\right)^2 - \frac{1}{2}\left(\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle\right)^2. \end{aligned}$$

Therefore

$$2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda$$

$$= \mathrm{tr}A_\lambda^2 - \frac{1}{2}(\mathrm{tr}A_\lambda)^2 + \frac{1}{2}\left(\sum_i \langle A_\lambda e_i, e_i\rangle\right)^2 + \frac{1}{2}\left(\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle\right)^2$$

$$- \sum_i \langle A_\lambda e_i, e_i\rangle^2 - \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle^2$$

$$- \sum_{i\neq j} \langle A_\lambda e_i, e_j\rangle^2 - \sum_{\alpha\neq\beta} \langle A_\lambda e_\alpha, e_\beta\rangle^2. \tag{13}$$

If follows from the Schwarz inequality that

$$\left(\sum_i \langle A_\lambda e_i, e_i\rangle\right)^2 \leq p\sum_i \langle A_\lambda e_i, e_i\rangle^2,$$

$$\left(\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle\right)^2 \leq (m-p)\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle^2.$$

Substituting these into (13), we obtain

$$2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda$$

$$\leq \mathrm{tr}A_\lambda^2 - \frac{1}{2}(\mathrm{tr}A_\lambda)^2 + \left(\frac{p}{2} - 1\right)\sum_i \langle A_\lambda e_i, e_i\rangle^2$$

$$+ \left(\frac{m-p}{2} - 1\right)\sum_\alpha \langle A_\lambda e_\alpha, e_\alpha\rangle^2. \tag{14}$$

Assume $\frac{p}{2} - 1 \geq 0$ and $\frac{m-p}{2} - 1 \geq 0$, that is, $2 \leq p \leq m-2$. By (11) and (14) we get

$$2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 - (\mathrm{tr}A_\lambda - \mathrm{tr}B_\lambda)\mathrm{tr}B_\lambda$$

$$\leq \mathrm{tr}A_\lambda^2 - \frac{1}{2}(\mathrm{tr}A_\lambda)^2 + \max\left\{\left(\frac{p}{2} - 1\right), \frac{m-p}{2} - 1\right\}\mathrm{tr}A_\lambda^2$$

$$= \frac{1}{2}[\max\{p,\ m - p\}\mathrm{tr}A_\lambda^2 - (\mathrm{tr}A_\lambda)^2].$$

Substituting the above inequality into (4), we obtain

$$\mathrm{tr}Q^A \leq \frac{1}{2}[\max\{p,\ m - p\}s - m^2H^2].$$

Hence, if $s < \frac{m^2}{m-2}H^2 + 4c$ on M^m, $m = \dim M > 3$, then for $p = 2, 3, \ldots, m - 2$,

$$s < \frac{m^2}{\max\{p,\ m - p\}}H^2 + 2\min\{p,\ m - p\}c,$$

and thus $\mathrm{tr}Q^A < p(m - p)c$. The proof is completed.

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