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A PROBLEM OF STOCHASTIC GEOMETRY ON A CIRCLE

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Abstract

Let us consider, in the Euclidean plane $\mathbf{E}_2$ a fixed convex body $\mathbf{K}_0$ and a system $\{\mathbf{K}_1, ..., \mathbf{K}_m\}$ of n-dimensional convex bodies. Assume that the sets $\mathbf{K}_i$ $(i = 1, ..., m)$ have random positions, being stochastically independent and uniformly distributed in a limited domain of $\mathbf{E}_2$, and denote by $\mathcal{S}_m$ the area of the convex body $\mathcal{K}_m = \mathbf{K}_0 \bigcap (\mathbf{K}_1 \cap \mathbf{K}_2 \cap \cdots \cap \mathbf{K}_m)$. The aim of this paper is the study of the random variable $\mathcal{S}_m$.

1. Introduction

Let us consider $\mathbf{E}_2$ be the Euclidean two dimensional space of coordinates x_1, x_2 in which operates the group of Euclidean motion G_3.

The elementary Kinematic measure in the plane is

$$d\mathbf{K} = dP \wedge dO_1,$$

where $dP = dx_1 \wedge dx_2$ and dO_1 is the infinitesimal area element of the 1-dimensional unit circle $\mathbf{S}_1$ (see [1]). In this space we consider a fixed convex body $\mathbf{K}_0$ and a system $\{\mathbf{K}_1, ..., \mathbf{K}_m\}$ of m convex sets, which are placed at random with uniform distribution on a limited domain of $\mathbf{E}_2$.

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We suppose that the $\mathbf{K}_i$ are stochastically independent and that they meet $\mathbf{K}_0$. Since $\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_m$ are convex sets, $\mathcal{K}_m := \mathbf{K}_0 \cap (\mathbf{K}_1 \cap \mathbf{K}_2 \cap \cdots \cap \mathbf{K}_m)$ is a convex set; its area $\mathcal{S}_m := \mu(\mathcal{K}_m)$ is a random variable for which we want to determine the mean value $\mathbb{E}(\mathcal{S}_m)$, the second moment $\mathbb{E}(\mathcal{S}_m^2)$ and the variance $\sigma^2(\mathcal{S}_m)$. If we assume that the convex sets $\mathbf{K}_i$ are congruent to a convex set $\mathbf{K}$, of area S, we obtain the result of Santaló in [2] and Stoka in [3]. As application we consider a system of random circles $\mathbf{\Sigma}_i$ $(i = 1, \ldots, 4)$ of constant radius R_i and a fixed circle of radius R_0.

2. Main Results

We assume that the convex set $\mathbf{K}_i$ $(i = 0, 1, \ldots, m)$ has area S_i and boundary $\partial\mathbf{K}_i$ of length L_i. We compute the following integral

$$I_1 := \int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing\}} \mathcal{S}_m d\mathbf{K}_1 \cdots d\mathbf{K}_m.$$

We put

$$J_1 := \int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing,\, P \in \mathbf{K}_0 \cap \mathcal{K}_m\}} dP d\mathbf{K}_1 \cdots d\mathbf{K}_m.$$

With this positions we have

$$J_1 = \int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing\}} d\mathbf{K}_1 \cdots d\mathbf{K}_m \int_{\{P \in \mathbf{K}_0 \cap \mathcal{K}_m\}} dP = I_1.$$

Taking into account that

$$\int_{\{P \in \mathbf{K}_i\}} d\mathbf{K}_i = 2\pi S_i,$$

and by the fact that the convex sets $\mathbf{K}_i$ are stochastically independent, we obtain that

$$J_1 = 2^m \pi^m S_0 \prod_{i=1}^{m} S_i.$$

Hence

$$\int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing\}} \mathcal{S}_m d\mathbf{K}_1 \cdots d\mathbf{K}_m = 2^m \pi^m S_0 \prod_{i=1}^{m} S_i.$$

But it is easy to see, using a classical result of Santaló and the independence of the convex sets $\mathbf{K}_i$, that

$$\int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing\}} d\mathbf{K}_1 \cdots d\mathbf{K}_m = \prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_0 L_i).$$

Finally we have the following result

Theorem 2.1. *The mean value of the random variable* $\mathcal{S}_m$ *is*

$$\mathbb{E}(\mathcal{S}_m) = \frac{2^m \pi^m S_0 \prod_{i=1}^{m} S_i}{\prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_0 L_i)}.$$

Remark. If we assume that the convex sets $\mathbf{K}_i$ are congruent to a convex set $\mathbf{K}$ of area S and boundary $\partial\mathbf{K}$ of length L, it is known that [2]:

$$\mathbb{E}(S_m) = \frac{(2\pi S)^m S_0}{(2\pi(S_0 + S) + L_0 L)^m}.$$

Corollary 2.2. *The probability that a fixed point* P *in* $\mathbf{K}_0$ *belongs to* $\mathcal{K}_m$ *is given by*

$$p = \frac{2^m \pi^m \prod_{i=1}^{m} S_i}{\prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_0 L_i)}.$$

Proof. Easy by the fact that this probability is exactly given by the expression

$$p = \frac{\int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing,\, P \in \mathbf{K}_0 \cap \mathcal{K}_m\}} dP d\mathbf{K}_1 \cdots d\mathbf{K}_m}{\int_{\{\mathbf{K}_i \cap \mathbf{K}_0 \neq \varnothing,\, P \in \mathbf{K}_0\}} dP d\mathbf{K}_1 \cdots d\mathbf{K}_m}.$$

Using the same arguments of Stoka in [3] we can compute the variance for the random variable $\mathcal{S}_m$, finding.

Theorem 2.3. *The second moment of the random variable* $\mathcal{S}_m$ *is*

$$\mathbb{E}(\mathcal{S}_m^2) = \frac{2\int_{\mathbf{G} \cap \mathbf{K}_0 \neq \varnothing} \Phi(\mathbf{K}_i, m; \lambda) d\mathbf{G}}{\prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_i L_0)},$$

where

$$\Phi(\mathbf{K}_i, m; \lambda) = \int_0^{\lambda} \int_0^{v} \prod_{i=1}^{m} \mu_i(\mathbf{K}_i, u) u du dv,$$

and where $d\mathbf{G}$ *is the density for sets of lines in the plane* $\mathbf{E}_2$, λ *is the length of the chord determined by the convex body* $\mathbf{K}_0$ *on the line* $\mathbf{G}$ *and* $\mu_i(\mathbf{K}_i, u)$ *is the measure of all the line segments of length* u *entirely contained in the convex body* $\mathbf{K}_i$.

Remark 2.4. The variance of $\mathcal{S}_m$ is

$$\sigma^2(\mathcal{S}_m) = \frac{1}{\prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_i L_0)}$$

$$\times \left(2\int_{\mathbf{G} \cap \mathbf{K}_0 \neq \varnothing} \Phi(\mathbf{K}_i, m; \lambda) d\mathbf{G} - \frac{2^{2m} \pi^{2m} S_0^2 \prod_{i=1}^{m} S_i^2}{\prod_{i=1}^{m} (2\pi(S_0 + S_i) + L_0 L_i)} \right).$$

3. Applications

Let us consider, in the Euclidean plane $\mathbf{E}_2$ a fixed circle $\mathbf{\Sigma}_0$ of radius R_0 and a system $\mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_4$ of circles of constant radius R_i. Assume that the sets $\mathbf{\Sigma}_i$ $(i = 1, \ldots, 4)$ have random positions, being stochastically independent and uniformly distributed in a limited domain of $\mathbf{E}_2$. We denote by $\mathcal{S}_4$ the area of the convex body $\mathbf{\Sigma}_4 = \mathbf{\Sigma}_0 \cap (\mathbf{\Sigma}_1 \cap \mathbf{\Sigma}_2 \cap \cdots \cap \mathbf{\Sigma}_4)$.

Theorem 3.1. *The mean value of the random variable* $\mathcal{S}_4$ *is*

$$\mathbb{E}(\mathcal{S}_4) = \pi R_0^2 \prod_{i=1}^{4} \frac{R_i^2}{(R_0 + R_i)^2}.$$

Now, fixing $s \in \mathbf{N}$ and a convex body $\mathbf{K}_0$, we put

$$J_s(\mathbf{K}_0) := \int_{\{\mathbf{G} \cap \mathbf{K}_0 \neq \varnothing\}} \lambda^s d\mathbf{G},$$

where λ is the length of the chord obtained as intersection between a random line $\mathbf{G}$ and $\mathbf{K}_0$. If $\mathbf{K}_0 = \mathbf{\Sigma}_0$ we can compute $J_s(\mathbf{\Sigma}_0)$ using the following formula due to Stoka [3],

$$J_s(\mathbf{\Sigma}_0) = 2^s \pi B\left(\frac{1}{2}, \frac{s+2}{2}\right) R_0^{1+s},$$

where $B\left(\frac{1}{2}, \frac{s+2}{2}\right)$ is the Bessel function of parameters $\frac{1}{2}$ and $\frac{s+2}{2}$.

If we denote by $\mathcal{N}$ the set of all segments, of length u, that lie completely in $\mathbf{\Sigma}_i$, the measure $\mu(\mathcal{N})$ is computed by means of the elementary Kinematic measure. Hence

$$\mu(\mathcal{N}) = \frac{\pi^2}{4}(2R_i - u)^2,$$

then

$$\prod_{i=1}^{4} \mu_i(\mathbf{\Sigma}_i, u) = \frac{\pi^8}{4^4} \prod_{i=1}^{4} (2R_i - u)^2.$$

Using the previous results we obtain, by direct calculations, that

$$\mathbb{E}(\mathcal{S}_4^2) = \frac{\int_{\{\mathbf{G} \cap \mathbf{\Sigma}_0 \neq \varnothing\}} \Phi(\mathbf{\Sigma}_i, 4; \lambda) d\mathbf{G}}{2^3 \pi^8 \prod_{i=1}^{4} (R_0 + R_i)^2},$$

such that

$$\int_{\{\mathbf{G} \cap \mathbf{\Sigma}_0 \neq \varnothing\}} \Phi(\mathbf{\Sigma}_i, 4; \lambda) d\mathbf{G}$$

$$= 2^{11}\pi B\left(\frac{1}{2}, \frac{13}{2}\right)\delta_0^{12}\mu_{11} + 2^{10}\pi B\left(\frac{1}{2}, 6\right)\delta_0^{11}\mu_{10} + 2^{9}\pi B\left(\frac{1}{2}, \frac{11}{2}\right)\delta_0^{10}\mu_9$$

$$+ 2^{8}\pi B\left(\frac{1}{2}, \frac{10}{2}\right)\delta_0^{9}\mu_8 + 2^{7}\pi B\left(\frac{1}{2}, \frac{9}{2}\right)\delta_0^{8}\mu_7 + 2^{6}\pi B\left(\frac{1}{2}, 4\right)\delta_0^{7}\mu_6$$

$$+ 2^{5}\pi B\left(\frac{1}{2}, \frac{7}{2}\right)\delta_0^{6}\mu_5 + 2^{4}\pi B\left(\frac{1}{2}, 3\right)\delta_0^{5}\mu_4 + 2^{3}\pi B\left(\frac{1}{2}, \frac{5}{2}\right)\delta_0^{4}\mu_3,$$

where

$$\mu_{11} := \frac{\pi^8}{28160},$$

$$\mu_{10} := -\frac{\pi^8}{5760}(R_1 + R_2 + R_3 + R_4),$$

$$\mu_9 = \left[\frac{\pi^8 R_1^2}{4608} + R_1\left(\frac{\pi^8 R_2^2}{1152} + \frac{\pi^8 R_3}{1152} + \frac{\pi^8 R_4}{1152}\right) + \frac{\pi^8 R_2^2}{4608}\right.$$

$$\left. + R_2\left(\frac{\pi^8 R_3}{1152} + \frac{\pi^8 R_4}{1152}\right) + \frac{\pi^8 R_3^2}{4608} + \frac{\pi^8 R_3 R_4}{1152} + \frac{\pi^8 R_4^2}{4608}\right],$$

$$\mu_8 := -\frac{\pi^8}{896}[R_1^2(R_2 + R_3 + R_4) + R_1(R_2^2 + 4R_2(R_3 + R_4)$$

$$+ R_3^2 + 4R_3R_4 + R_4^2) + R_2^2(R_3 + R_4)$$

$$+ R_2(R_3^2 + 4R_3R_4 + R_4^2) + R_3R_4(R_3 + R_4)],$$

$$\mu_7 = \left[R_1^2\left(\frac{\pi^8 R_2^2}{672} + R_2\left(\frac{\pi^8 R_3}{168} + \frac{\pi^8 R_4}{168}\right) + \frac{\pi^8 R_3^2}{672} + \frac{\pi^8 R_3 R_4}{168} + \frac{\pi^8 R_4^2}{672}\right)\right.$$

$$+ R_1\left(R_2^2\left(\frac{\pi^8 R_3}{168} + \frac{\pi^8 R_4}{168}\right) + R_2\left(\frac{\pi^8 R_3^2}{168} + \frac{\pi^8 R_3 R_4}{42} + \frac{\pi^8 R_4^2}{168}\right)\right.$$

$$\left. + \frac{\pi^8 R_3^2 R_4}{16} + \frac{\pi^8 R_3 R_4^2}{168}\right) + R_2^2\left(\frac{\pi^8 R_3^2}{672} + \frac{\pi^8 R_3 R_4}{168} + \frac{\pi^8 R_4^2}{672}\right)$$

$$\left. + R_2\left(\frac{\pi^8 R_3^2 R_4}{168} + \frac{\pi^8 R_3 R_4^2}{168}\right) + \frac{\pi^8 R_3^2 R_4^2}{672}\right],$$

$$\mu_6 = -\frac{\pi^8}{120}[R_1^2(R_2^2(R_3 + R_4) + R_2(R_3^2 + 4R_3R_4 + R_4^2)$$

$$+ R_3R_4(R_3 + R_4)) + R_1(R_2^2(R_3^2 + 4R_3R_4 + R_4^2)$$

$$+ 4R_2R_3R_4(R_3 + R_4) + R_3^2R_4^2) + R_2R_3R_4(R_2(R_3 + R_4) + R_3R_4)],$$

$$\mu_5 = \left[R_1^2\left(R_2^2\left(\frac{\pi^8R_3^2}{80} + \frac{\pi^8R_3R_4}{20} + \frac{\pi^8R_4^2}{80}\right)\right.\right.$$

$$+ R_2\left(\frac{\pi^8R_3^2R_4}{80} + \frac{\pi^8R_3R_4^2}{20}\right) + \left.\frac{\pi^8R_3^2R_4^2}{80}\right)$$

$$+ R_1\left(R_2^2\left(\frac{\pi^8R_3^2R_4}{20} + \frac{\pi^8R_3R_4^2}{20}\right) + \frac{\pi^8R_2R_3^2R_4^2}{20}\right) + \left.\frac{\pi^8R_2^2R_3^2R_4^2}{80}\right],$$

$$\mu_4 = \frac{\pi^8R_1R_2R_3R_4}{12}(R_1(R_2(R_3 + R_4) + R_3R_4) + R_2R_3R_4),$$

$$\mu_3 = \pi^8R_1^2R_2^2R_3^3R_4^2.$$

Finally, with the above notations, we give the following.

Theorem 3.2. *The variance of the random variable* $\mathcal{S}_4$ *is*

$$\sigma^2(\mathcal{S}_4) = \frac{1}{2^3\pi^8\prod_{i=1}^4(R_0 + R_i)^2}$$

$$\times\left(\int_{\mathbf{G}\cap\mathbf{\Sigma}_0\neq\varnothing}\Phi(\mathbf{\Sigma}_i,\ 4;\ \lambda)d\mathbf{G} - 2^3\pi^{10}R_0^4\prod_{i=1}^4\frac{R_i^2}{(R_0 + R_i)^2}\right).$$

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