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STATISTICAL INFERENCE BASED ON FUZZY RANDOM VECTORS

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Abstract

In this paper, we discuss on fuzzy random vector and its probability and introduce the definition of independent fuzzy random vectors and likelihood function. We present the sufficient statistic and the maximum likelihood estimator of mean vector for multivariate normal distribution. Further, we prove Neyman-Pearson lemma for testing of mean vector.

1. Introduction

Fuzzy random variables were introduced in the literature by Kwakernaak [6] and slightly modified by Kruse and Meyer [5]. The concept of fuzzy random variable was presented by these authors as a model to describe a fuzzy report of an existing numerical variable associated with a random experiment, that is, as a fuzzy perception of a classical random variable (which Kwakernaak, Kruse and Meyer refer to

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as the original random variable). Puri and Ralescu [8] have presented fuzzy random variables as a model to deal with an existing qualification process associated with a random experiment. In this paper we will develop and make use of the concept of fuzzy random variables as intended by Puri and Ralescu [8] and add some new definitions for the vectorial case, and we will apply them for statistical inference based on fuzzy data.

It is the intention of our study to apply the results to the data communication system and production system. For example, we draw n data (vectorial) to select one from the production line system and the ith section of the selection is interrupted for a moment, hence the ith data (vectorial) becomes vague. Therefore, we need to deal with n size of random sample with one vague vector and have to make some statistical decisions about them.

2. Statistical Inference Based on Fuzzy Data

Let (Ω, A, P) be a probability space and $F(\mathbb{R})$ be the set of all piecewise continuous functions (fuzzy sets of $\mathbb{R}$) or all discrete functions $\widetilde{X}(\omega) : R \to [0, 1]$ (subject to certain measurable conditions).

Definition 2.1 (Puri and Ralescu). A *fuzzy random variable* (F.R.V.) is a function $\widetilde{X} : \Omega \to F(\mathbb{R})$, such that

$$\{(\omega, x) : x \in X_\alpha(\omega)\} \in \mathcal{A} \times \mathcal{B}; \quad \forall \alpha \in [0, 1],$$

where $\mathcal{B}$ denotes the Borel set of $\mathbb{R}$ and $X_\alpha : \Omega \to P(\mathbb{R})$, such that

$$\widetilde{X}_\alpha(\omega) = \{x \in \mathbb{R} \mid \widetilde{X}(\omega)(x) \geq \alpha\}.$$

For a F.R.V. $\widetilde{X}$ and $\omega \in \Omega$, let $\widetilde{X}(\omega)$ be a fuzzy set with the membership function $\widetilde{X}(\omega)(x)$.

Example 2.1 (Puri and Ralescu). Suppose we toss a coin one time, we have $\Omega = \{H, T\}$. If it comes up H, we win about 50 dollars, and if it comes up T, we win about 10 dollars, then we may write

$$\widetilde{X}(H)(x) = [1 - (x - 50)^2]^+, \quad \widetilde{X}(T)(x) = [1 - (x - 10)^2]^+,$$

where $[a]^+ = \max(a, 0)$.

Definition 2.2 (Zadeh). Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then a fuzzy event in Ω is a fuzzy set whose membership function $\widetilde{A}\,(\widetilde{A} : \Omega \to [0, 1])$ is Borel-measurable. The probability of a *fuzzy event* $\widetilde{A}$ is defined by Zadeh with the Lebesgue-Stieltjes integral as

$$P(\widetilde{A}) = \int_{\Omega} \widetilde{A}(\omega) dP.$$

Definition 2.3. By the definition of a F.R.V. $\widetilde{X}$, the membership function $\widetilde{X}(\omega)(x)$ is a *discrete function* or *continuous function*. $\widetilde{X}(\omega)(x)$ is a measurable function and $\widetilde{X}(\omega)(X)$ is a classical random variable, the expectation of $\widetilde{X}(\omega)(X)$ is

$$\int_{-\infty}^{\infty} \widetilde{X}(\omega)(x) f(x) dx,$$

where $f(x)$ is the p.d.f. of r.v. X, such that support X is the reference set of fuzzy set $\widetilde{X}(\omega)$, therefore we have

$$E(\widetilde{X}(\omega)(X)) = P(\widetilde{X}(\omega)).$$

Definition 2.4. We say $\underline{\widetilde{X}} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_p)'$ is a *fuzzy random vector*, if all $\widetilde{X}_i\ (i = 1, 2, \ldots, p)$ be a F.R.V. For a fuzzy random vector $\underline{\widetilde{X}}$ and $\omega \in \Omega$, let $\underline{\widetilde{X}}(\omega)$ be a vector of fuzzy sets with the membership function

$$\underline{\widetilde{X}}(\omega)(\underline{x}) = (\widetilde{X}_1(\omega)(x_1), \widetilde{X}_2(\omega)(x_2), \ldots, \widetilde{X}_p(\omega)(x_p))'.$$

Definition 2.5. Let g be p.d.f. (p.m.f.) of random vector $\underline{X}$, for a fuzzy random vector $\underline{\widetilde{X}}$ and $\omega \in \Omega$, $\widetilde{P}(\underline{\widetilde{X}}(\omega))$ is called the *probability measure* of vector of fuzzy sets $\underline{\widetilde{X}}(\omega)$, defined as

(1) if g is continuous, then

$$P(\underline{\tilde{X}}(\omega)) = \tilde{P}(\tilde{X}_1(\omega), \tilde{X}_2(\omega), \ldots, \tilde{X}_p(\omega))$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\tilde{X}_1(\omega)(x_1) \wedge \tilde{X}_2(\omega)(x_2)$$

$$\wedge \cdots \wedge \tilde{X}_p(\omega)(x_p))g(x_1, x_2, \ldots, x_p)dx_1dx_2 \ldots dx_p,$$

(2) if g is discrete, then

$$P(\underline{\tilde{X}}(\omega)) = \tilde{P}(\tilde{X}_1(\omega), \tilde{X}_2(\omega), \ldots, \tilde{X}_p(\omega))$$

$$= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_p}(\tilde{X}_1(\omega)(x_1) \wedge \tilde{X}_2(\omega)(x_2)$$

$$\wedge \cdots \wedge \tilde{X}_p(\omega)(x_p))g(x_1, x_2, \ldots, x_p).$$

Example 2.2. Let $\underline{X}$ be a random vector with p.d.f. $N_3(\underline{\mu}, \Sigma)$. Then we have

$$P(\underline{\tilde{X}}(\omega)) = \tilde{P}(\tilde{X}_1(\omega), \tilde{X}_2(\omega), \ldots, \tilde{X}_p(\omega))$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\tilde{X}_1(\omega)(x_1) \wedge \tilde{X}_2(\omega)(x_2) \wedge \tilde{X}_3(\omega)(x_3))$$

$$\times |\, 2\pi\Sigma\,|^{\frac{-1}{2}} e^{\left(\frac{-1}{2}\right)(\underline{x}-\underline{\mu})'\,\Sigma^{-1}(\underline{x}-\underline{\mu})} dx_1dx_2dx_3.$$

Definition 2.6. Let $\underline{X}$ be a random vector with p.d.f. $f(\underline{x};\, \theta)$ $(\theta \in \Theta \subseteq R^s)$, and $\underline{\tilde{X}}_1$ be a fuzzy random vector associated with p.d.f. $f(\underline{x};\, \theta)$, and $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ be random vectors with p.d.f. $f_1(\underline{x};\, \theta)$; $f_2(\underline{x};\, \theta), \ldots, f_n(\underline{x};\, \theta)$, respectively. For all $\omega \in \Omega$, and $\underline{A}_j = (A_{j1}, A_{j2}, \ldots, A_{jp})$; $(j = 2, \ldots, n)$, where $(A_{jk}(k = 1, 2, \ldots, p)$ are Borel sets of $\mathbb{R}$), if

$$P_\theta(\underline{\tilde{X}}_1(\omega), \underline{X}_2 \in \underline{A}_2, \ldots, \underline{X}_n \in \underline{A}_n) = P_\theta(\underline{\tilde{X}}_1(\omega))\prod_{j=2}^{n} P_\theta(\underline{X}_j \in \underline{A}_j),$$

then $\underline{\tilde{X}}_1(\omega), \underline{X}_2, \ldots, \underline{X}_n$ are independent.

Definition 2.7. Let $\underline{X}_1, \underline{X}_2, ..., \underline{X}_n$ be i.i.d. with p.d.f. $f(\underline{x}; \theta)$, and $\underline{\widetilde{X}}_1$ be a fuzzy random vector associated with p.d.f. $f(\underline{x}; \theta)$. For fixed $\omega \in \Omega$, $\underline{\widetilde{X}}_1(\omega), \underline{X}_2, ..., \underline{X}_n$ are called the *n size of random sample* with one vague vector, if $\underline{\widetilde{X}}_1(\omega), \underline{X}_2, ..., \underline{X}_n$ are independent.

Definition 2.8. Let $\underline{X}_1, \underline{X}_2, ..., \underline{X}_{i-1}, \underline{\widetilde{X}}_i(\omega), \underline{X}_{i+1}, ..., \underline{X}_n$ be random sample of size n with one vague vector $\underline{\widetilde{X}}_i(\omega)$ associated with p.d.f. $f(\underline{x}; \theta)$, $L(\theta, \underline{x}_1, \underline{x}_2, ..., \underline{x}_{i-1}, \underline{\widetilde{X}}_i(\omega), \underline{x}_{i+1}, ..., \underline{x}_n)$ is called the *likelihood function* of $\underline{X}_1 = \underline{x}_1$, $\underline{X}_2 = \underline{x}_2, ..., \underline{X}_{i-1} = \underline{x}_{i-1}$, $\underline{\widetilde{X}}_i(\omega)$, $\underline{X}_{i+1} = \underline{x}_{i+1}, ...,$ $\underline{X}_n = \underline{x}_n$, defined as follows:

(1) if $f(\underline{x}; \theta)$ is discrete, then

$$L(\theta; \underline{x}_1, \underline{x}_2, ..., \underline{x}_{i-1}, \underline{\widetilde{X}}_i(\omega), \underline{x}_{i+1}, ..., \underline{x}_n)$$

$$= \sum_{x_i} \underline{\widetilde{X}}_i(\omega)(\underline{x}_i) f(\underline{x}_i; \theta) \prod_{j=1, \neq i}^{n} f(\underline{x}_j; \theta),$$

(2) if $f(\underline{x}; \theta)$ is continuous, then

$$L(\theta; \underline{x}_1, \underline{x}_2, ..., \underline{x}_{i-1}, \underline{\widetilde{X}}_i(\omega), \underline{x}_{i+1}, ..., \underline{x}_n)$$

$$= \int_{-\infty}^{\infty} \underline{\widetilde{X}}_i(\omega)(\underline{x}_i) f(\underline{x}_i; \theta) \prod_{j=1, \neq i}^{n} f(\underline{x}_j; \theta).$$

Example 2.3. Let $\underline{\widetilde{X}}_1, \underline{X}_2, ..., \underline{X}_n$ be random sample of size n with one vague vector $\underline{\widetilde{X}}_1(\omega)$ associated with $N_p(\underline{\mu}, I)$, where $\underline{\mu}$ is an unknown parameter and suppose that the membership function $\widetilde{X}_{1i}(\omega)$ $= e^{\frac{-1}{2}(x_{1i} - m_i(\omega))^2}$, where $m_i(\omega)\,(i = 1, 2, ..., p)$ is a known real number (only dependent on ω), if we apply Zadeh's idea, i.e., $(\widetilde{X}_{11}(\omega)(x_{11}) \wedge$

$\widetilde{X}_{12}(\omega)(x_{12}) \wedge \cdots \wedge \widetilde{X}_{1p}(\omega)(x_{1p})) = \prod_{i=1}^{p} \widetilde{X}_{1i}(\omega)(x_{1i}),$ then

$$L(\underline{\mu};\ \underline{\widetilde{X}}_1(\omega),\ \underline{x}_2,\ \ldots,\ \underline{x}_n)$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(\underline{x}_1 - \underline{m}(\omega))'(\underline{x}_1 - \underline{m}(\omega))} \frac{1}{|2\pi I|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_1 - \underline{\mu})'(\underline{x}_1 - \underline{\mu})} d\underline{x}_1$$

$$\times \prod_{j=2}^{n} \frac{1}{|2\pi I|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_j - \underline{\mu})'(\underline{x}_j - \underline{\mu})}$$

$$= \int_{-\infty}^{\infty} \frac{1}{|2\pi I|^{\frac{1}{2}}} e^{\left(\underline{x}_1 - \frac{\underline{\mu} + \underline{m}(\omega)}{2}\right)'\left(\underline{x}_1 - \frac{\underline{\mu} + \underline{m}(\omega)}{2}\right) - \frac{-1}{4}(\underline{\mu} - \underline{m}(\omega))'(\underline{\mu} - \underline{m}(\omega))}$$

$$\times \prod_{j=2}^{n} \frac{1}{|2\pi I|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_j - \underline{\mu})'(\underline{x}_j - \underline{\mu})} d\underline{x}_1$$

$$= 2^{\frac{-(n+1)p}{2}} \pi^{\frac{-(n-1)p}{2}} e^{\frac{-1}{4}(\underline{\mu} - \underline{m}(\omega))'(\underline{\mu} - \underline{m}(\omega))} e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j - \underline{\mu})'(\underline{x}_j - \underline{\mu})},$$

where $\underline{m}(\omega) = (m_1(\omega),\ m_2(\omega),\ \ldots,\ m_p(\omega))'$.

Definition 2.9. For fixed $\omega \in \Omega$, let $\underline{\widetilde{X}}_1(\omega),\ \underline{X}_2,\ \ldots,\ \underline{X}_n$ be the n size of random vector with one vague vector associated with p.d.f. $f(\underline{x};\ \theta)$, and $\underline{T} = t(\underline{X}_2,\ \ldots,\ \underline{X}_n)$ be a statistic. We say $\underline{T}$ is *sufficient* for θ, if

$$L(\theta;\ \underline{\widetilde{X}}_1(\omega),\ \underline{x}_2,\ \ldots,\ \underline{x}_n) = g(\underline{t};\ \theta)h(\underline{x}_2,\ \ldots,\ \underline{x}_n),$$

where g depends only on $\underline{t}$ and θ, and h is independent of θ.

Example 2.4. In Example 2.3 we demonstrated that $\underline{\overline{X}} = (\overline{X}_1,\ \overline{X}_2,\ \overline{X}_3,\ \ldots,\ \overline{X}_p)'$ is sufficient for $\underline{\mu}$, we have

$$L(\underline{\mu};\ \underline{\tilde{X}}_1(\omega),\ \underline{x}_2,\ \dots,\ \underline{x}_n)$$

$$= 2^{\frac{-(n+1)p}{2}}\pi^{\frac{-(n-1)p}{2}}e^{\frac{-1}{4}(\underline{\mu}-\underline{m}(\omega))'(\underline{\mu}-\underline{m}(\omega))}e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j-\underline{\mu})'(\underline{x}_j-\underline{\mu})}$$

$$= \underbrace{e^{\frac{-1}{4}(\underline{\mu}-\underline{m}(\omega))'(\underline{\mu}-\underline{m}(\omega))}e^{\frac{-(n-1)}{2}(\underline{\bar{x}}-\underline{\mu})'(\underline{\bar{x}}-\underline{\mu})}}_{g}\underbrace{2^{\frac{-(n+1)p}{2}}\pi^{\frac{-(n-1)p}{2}}e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j-\underline{\bar{x}})'(\underline{x}_j-\underline{\bar{x}})}}_{h},$$

where $\underline{\bar{X}} = (\underline{\bar{X}}_1, \underline{\bar{X}}_2, \underline{\bar{X}}_3, \dots, \underline{\bar{X}}_p)'$ and $\bar{X}_i = \frac{1}{n-1}\sum_{k=2}^{n} X_{ki}$ $(i = 1, 2, \dots, p)$.

Definition 2.10. Let $\underline{\tilde{X}}_1, \underline{X}_2, \dots, \underline{X}_n$ be the n size of random sample with one vague vector $\underline{\tilde{X}}_1(\omega)$ associated with p.d.f. $f(\underline{x};\ \theta)$, if there exists $\hat{\theta}$ such that

$$L(\hat{\theta};\ \underline{\tilde{X}}_1(\omega),\ \underline{x}_2,\ \dots,\ \underline{x}_n) = \max\{L(\theta;\ \underline{\tilde{X}}_1(\omega),\ \underline{x}_2,\ \dots,\ \underline{x}_n) : \theta \in \Theta\},$$

then the estimator $\hat{\theta}$ is called a *maximum likelihood estimate* of θ.

Example 2.5. Let $\underline{\tilde{X}}_1, \underline{X}_2, \dots, \underline{X}_n$ be the n size of random sample with one vague vector $\underline{\tilde{X}}_1(\omega)$ associated with p.d.f. $N_p(\underline{\mu}, I_p)$, where $\underline{\mu}$ is an unknown vectorial parameter, it is desired $\hat{\underline{\mu}}$, we have

$$L(\underline{\mu};\ \underline{\tilde{X}}_1(\omega),\ \underline{x}_2,\ \dots,\ \underline{x}_n) = 2^{\frac{-(n+1)p}{2}}\pi^{\frac{-(n-1)p}{2}}e^{\frac{-1}{4}(\underline{\mu}-\underline{m}(\omega))'(\underline{\mu}-\underline{m}(\omega))}$$

$$\times e^{\frac{-(n-1)}{2}(\underline{\bar{x}}-\underline{\mu})'(\underline{\bar{x}}-\underline{\mu})}e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j-\underline{\bar{x}})'(\underline{x}_j-\underline{\bar{x}})},$$

hence

$$l(\underline{\mu}) = \ln L(\underline{\mu};\ \underline{\tilde{X}}_1(\omega),\ \underline{x}_2,\ \dots,\ \underline{x}_n)$$

$$= -\frac{(n+1)}{2}p\ln 2 - \frac{(n-1)}{2}p\ln\pi - \frac{(n-1)}{2}(\underline{\bar{x}}-\underline{\mu})'(\underline{\bar{x}}-\underline{\mu})$$

$$-\frac{1}{4}(\underline{\mu}-\underline{m}(\omega))'(\underline{\mu}-\underline{m}(\omega)) - \frac{1}{2}\sum_{j=2}^{n}(\underline{x}_j-\underline{\bar{x}})'(\underline{x}_j-\underline{\bar{x}}),$$

$$\Rightarrow \frac{\partial l(\underline{\mu})}{\partial \underline{\mu}} = (n-1)(\underline{\overline{X}} - \underline{\mu}) - \frac{1}{2}(\underline{\mu} - \underline{m}(\omega)) = 0,$$

$$\Rightarrow \quad \hat{\underline{\mu}} = \frac{2(n-1)\underline{\overline{X}} + \underline{m}(\omega)}{(2n-1)}.$$

The distribution of maximum likelihood estimator $\hat{\underline{\mu}}\left(\frac{2(n-1)\underline{\overline{X}} + \underline{m}(\omega)}{(2n-1)}\right)$ is

$$N_p\left(\frac{2(n-1)\underline{\mu} + \underline{m}(\omega)}{(2n-1)}, \frac{4(n-1)^2}{(2n-1)^2} I_p\right)$$

for case $p = 1$, see [3].

Example 2.6. Let $\underline{\widetilde{X}}_1, \underline{X}_2, \ldots, \underline{X}_n$ be the n size of random sample with one vague vector $\underline{\widetilde{X}}_1(\omega)$ associated with p.d.f. $N_p(\underline{\mu}, \Sigma_0)$, where $\underline{\mu}$ is an unknown vectorial parameter, it is desired $\hat{\underline{\mu}}$, we have

$$L(\underline{\mu}, \Sigma_0; \underline{\widetilde{X}}_1(\omega), \underline{x}_2, \ldots, \underline{x}_n)$$

$$= \int_{R^p} \underline{\widetilde{X}}_1(\omega_0)(\underline{x}_1) \frac{1}{|2\pi\Sigma_0|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_1 - \underline{\mu})' \Sigma_0^{-1} (\underline{x}_1 - \underline{\mu})} d\underline{x}_1$$

$$\times \prod_{j=2}^{n} \frac{1}{|2\pi\Sigma_0|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_j - \underline{\mu})' \Sigma^{-1} (\underline{x}_j - \underline{\mu})},$$

if we suppose that $\underline{\widetilde{X}}_1(\omega)(\underline{x}_1) = e^{\frac{-1}{2}(\underline{x}_1 - \underline{m}(\omega))'(\underline{x}_1 - \underline{m}(\omega))}$, then

$$L(\underline{\mu}, \Sigma_0; \underline{\widetilde{X}}_1(\omega), \underline{x}_2, \ldots, \underline{x}_n)$$

$$= \int_{R^p} e^{\frac{-1}{2}(\underline{x}_1 - \underline{m}(\omega))'(\underline{x}_1 - \underline{m}(\omega))} \frac{1}{|2\pi\Sigma_0|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_1 - \underline{\mu})' \Sigma_0^{-1} (\underline{x}_1 - \underline{\mu})} d\underline{x}_1$$

$$\times \prod_{j=2}^{n} \frac{1}{|2\pi\Sigma_0|^{\frac{1}{2}}} e^{\frac{-1}{2}(\underline{x}_j - \underline{\mu})' \Sigma_0^{-1} (\underline{x}_j - \underline{\mu})}.$$

Therefore,

$$L(\underline{\mu}, \Sigma_0; \underline{\widetilde{X}}_1(\omega), \underline{x}_2, ..., \underline{x}_n)$$

$$= |\Sigma_0 + I|^{\frac{-1}{2}} e^{\frac{1}{2}(\underline{m}(\omega)-\underline{\mu})'((I+\Sigma_0^{-1})^{-1}-I)(\underline{m}(\omega)-\underline{\mu})}$$

$$\times \frac{1}{|2\pi\Sigma_0|^{\frac{(n-1)}{2}}} e^{\frac{-(n-1)}{2} tr\, \Sigma_0^{-1} S - \frac{(n-1)}{2} tr\, \Sigma_0^{-1}(\underline{\bar{x}} - \underline{\mu})(\underline{\bar{x}} - \underline{\mu})'},$$

and

$$l(\underline{\mu}) = \ln L(\underline{\mu}, \Sigma_0; \underline{\widetilde{X}}_1(\omega), \underline{x}_2, ..., \underline{x}_n)$$

$$= \frac{-1}{2} \ln|\Sigma_0 + I| + \frac{1}{2}(\underline{m}(\omega) - \underline{\mu})'((I + \Sigma_0^{-1})^{-1} - I)(\underline{m}(\omega) - \underline{\mu})$$

$$- \frac{(n-1)}{2} \ln|2\pi| + \frac{(n-1)}{2} \ln|\Sigma_0^{-1}| - \frac{-(n-1)}{2} tr\, \Sigma_0^{-1} S$$

$$- \frac{(n-1)}{2} tr\, \Sigma_0^{-1}(\underline{\bar{x}} - \underline{\mu})(\underline{\bar{x}} - \underline{\mu})',$$

hence

$$\frac{\partial l(\underline{\mu})}{\partial \underline{\mu}} = -((I + \Sigma_0^{-1})^{-1} - I)(\underline{m}(\omega) - \underline{\mu}) + (n-1)\Sigma_0^{-1}(\underline{\bar{x}} - \underline{\mu}),$$

and then

$$\frac{\partial l(\underline{\mu})}{\partial \underline{\mu}} = 0 \Rightarrow \hat{\underline{\mu}} = [((I + \Sigma_0^{-1})^{-1} - I) - (n-1)\Sigma_0^{-1}]^{-1}$$

$$\times [((I + \Sigma_0^{-1})^{-1} - I)\underline{m}(\omega) - (n-1)\Sigma_0^{-1}\underline{\bar{X}}].$$

Definition 2.11. Let $\underline{\widetilde{X}}_1, \underline{X}_2, \underline{X}_n$ be the n size of random sample with one vague vector $\underline{\widetilde{X}}_1(\omega)$ associated with p.d.f. $f(\underline{x}; \theta)$. For fixed $\omega \in \Omega$, the expectation and the variance of the function $\underline{T} = t(\underline{\widetilde{X}}_1(\omega)(\underline{X}_1), \underline{X}_2, ..., \underline{X}_n)$ are defined as

$$E(\underline{T}') = \int_{R^p} \cdots \int_{R^p} \underline{t}' \prod_{i=1}^{n} f(\underline{x}_i;\ \theta) d\underline{x}_i$$

and

$$Var(\underline{T}) = Cov(\underline{T},\ \underline{T}) = E[(\underline{T} - E(\underline{T}))(\underline{T} - E(\underline{T}))'] = E(\underline{T}\,\underline{T}') - E(\underline{T})E'(\underline{T}).$$

Example 2.7. Let $\underline{\widetilde{X}}_1,\ \underline{X}_2,\ ...,\ \underline{X}_n$ be the n size of random sample with one vague vector $\underline{\widetilde{X}}_1(\omega)$ associated with p.d.f. $N_3(\underline{\mu},\ I)$ and

$$\underline{T} = (\widetilde{X}_{11}(\omega)(X_{11}) + X_{22},\quad \widetilde{X}_{12}(\omega)(X_{12}),\ X_{53} + X_{24})',$$

where $\widetilde{X}_{1j}(\omega)(x_{1j}) = e^{\frac{-1}{2}(x_{1j} - m_j(\omega))^2}$. Then we have

$$E(\underline{T}') = \int_{R^3} \cdots \int_{R^3} \underline{t}' \prod_{i=1}^{5} f(\underline{x}_i,\ \theta) d\underline{x}_i$$

$$= \left(\int_{R^3} \cdots \int_{R^3} (\widetilde{X}_{11}(\omega)(x_{11}) + x_{22}) \frac{1}{|\,2\pi I\,|^{\frac{1}{2}}} e^{\frac{-1}{2}\sum_{i=1}^{5} (\underline{x}_i - \underline{\mu})'(\underline{x}_i - \underline{\mu})}, \right.$$

$$\int_{R^3} \cdots \int_{R^3} \widetilde{X}_{12}(\omega)(x_{12}) \frac{1}{|\,2\pi I\,|^{\frac{1}{2}}} e^{\frac{-1}{2}\sum_{i=1}^{5} (\underline{x}_i - \underline{\mu})'(\underline{x}_i - \underline{\mu})},$$

$$\left. (x_{53} + x_{21}) \frac{1}{|\,2\pi I\,|^{\frac{1}{2}}} e^{\frac{-1}{2}\sum_{i=1}^{5} (\underline{x}_i - \underline{\mu})'(\underline{x}_i - \underline{\mu})} \right)$$

$$= \left(\frac{1}{\sqrt{2}} e^{\frac{-1}{4}(\mu_1 - m_1(\omega))^2} + \mu_2,\ \frac{1}{\sqrt{2}} e^{\frac{-1}{4}(\mu_2 - m_2(\omega))^2},\ \mu_3 + \mu_1 \right)$$

and

$$E(\underline{T}) = \left(\frac{1}{\sqrt{2}} e^{\frac{-1}{4}(\mu_1 - m_1(\omega))^2} + \mu_2,\ \frac{1}{\sqrt{2}} e^{\frac{-1}{4}(\mu_2 - m_2(\omega))^2},\ \mu_3 + \mu_1 \right)'.$$

Theorem 2.1. *Let* $\underline{X}$ *be a random vector with p.d.f.* $f(\underline{x};\, \theta)$ *and* $\underline{\widetilde{X}}_1$, $\underline{X}_2, \ldots, \underline{X}_n$ *be the n size of random sample with one vague vector* $\underline{\widetilde{X}}_1(\omega)$ *associated with p.d.f.* $f(\underline{x};\, \theta)$. *Then, for any unbiased estimator* $\underline{T} = t(\underline{\widetilde{X}}_1(\omega)(\underline{X}_1), \underline{X}_2, \ldots, \underline{X}_n)$ *of* θ,

$$Var(\underline{T}) \geq F^{-1},$$

where

$$F = Var(\underline{W}) = E(\underline{W}\underline{W}') = -E\left(\frac{\partial^2 \log \prod_{i=1}^{n} f(\underline{x}_i;\, \theta)}{\partial\theta\partial\theta'}\right),$$

and

$$\underline{W} = \frac{\partial}{\partial\theta} \log \prod_{i=1}^{n} f(\underline{x}_i;\, \theta).$$

Proof. We have

$$\theta = E(\underline{t}') = \int_{R^P} \cdots \int_{R^P} \underline{t}' \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n,$$

therefore,

$$1 = \int_{R^P} \cdots \int_{R^P} \underline{t}' \frac{\partial}{\partial\theta} \log \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n$$

$$= \int_{R^P} \cdots \int_{R^P} \frac{\partial}{\partial\theta} \log \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) \underline{t}' \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n$$

$$= \int_{R^P} \cdots \int_{R^P} \underline{w}\,\underline{t}' \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n$$

$$= Cov(\underline{W}, \underline{T}),$$

because $\int_{R^P} \cdots \int_{R^P} \prod_{i=1}^{n} f(\underline{x}_i;\, \theta) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n = 1$. We know that

$$\max\left(\frac{(\underline{a}'\underline{x})^2}{(\underline{x}'B\underline{x})};\ \underline{x} \in R^P\right) = \underline{a}'\underline{B^{-1}a}, \left(\text{at } x = \frac{B^{-1}\underline{a}}{(\underline{a}'B^{-1}\underline{a})^{\frac{-1}{2}}}\right) \text{ (see [8])},$$

therefore

$$\max([Corr(\underline{a}'\underline{T},\ \underline{c}'\underline{W})]^2 : \underline{c} \in R^P) = \frac{\underline{a}'F^{-1}\underline{a}}{\underline{a}'Var(T)\underline{a}} \leq 1; \quad \forall \underline{a} \in R^P,$$

and it results that $Var(T) \geq F^{-1}$.

The proof of the theorem is complete.

Theorem 2.2 (Neyman-Pearson)**.** *Let* $\underline{X} : \Omega \to R^P$ *be a random vector with p.d.f.* $f(\underline{x};\ \theta)\,(\theta \in \Theta = \{\theta_0,\ \theta_1\} \subseteq R^s)$. *Let* $\underline{\widetilde{X}}_1(\omega),\ \underline{X}_2,\ \ldots,\ \underline{X}_n$ *be the n size of random sample with one vague vector associated with p.d.f.* $f(\underline{x};\ \theta)$. *Consider testing of the null hypothesis* $H_0 : \theta = \theta_0$ *against the alternative hypothesis* $H_1 : \theta = \theta_1$. *Let*

$$S = \left\{W \subseteq E^{n-1} : \int_{-\infty}^{\infty}\int_W \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0)\,d\underline{x}_n \ldots d\underline{x}_1 \geq \alpha\right\},$$

where $E^{n-1} = \{(\underline{x}_2,\ \ldots,\ \underline{x}_n)' : \underline{x}_j = (x_{j1},\ \ldots,\ x_{jp})' - \infty < x_{jl} < \infty\ (l = 1,\ \ldots,\ p,\ j = 2,\ \ldots,\ n)\}$.

W is called the critical region of the level α *of significance,* $\alpha \in\ ]0, 1[$. *Let* $W_0 \in S$ *satisfy*:

(i) *if* $(\underline{x}_2,\ \ldots,\ \underline{x}_n)' \in W_0$, *then*

$$\int_{-\infty}^{\infty} \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1)\,d\underline{x}_1 > k \int_{-\infty}^{\infty} \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0)\,d\underline{x}_1,$$

(ii) *if* $(\underline{x}_2,\ \ldots,\ \underline{x}_n)' \in W_0^c$, *then*

$$\int_{-\infty}^{\infty} \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1) d\underline{x}_1 < k \int_{-\infty}^{\infty} \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_1,$$

where $k > 0$, *then* W_0 *is a best critical region of the level* α *of significance for testing* $H_0 : \theta = \theta_0$ *against* $H_1 : \theta = \theta_1$.

Proof. For any $W \in S$, we have

$$\int_{-\infty}^{\infty} \int_{W} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_n \ldots d\underline{x}_1$$

$$= \int_{-\infty}^{\infty} \int_{W_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_n \ldots d\underline{x}_1 = \alpha,$$

so that

$$\int_{-\infty}^{\infty} \int_{W_0 - WW_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_n \ldots d\underline{x}_1$$

$$= \int_{-\infty}^{\infty} \int_{W - WW_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n,$$

from (i) and (ii),

$$\int_{-\infty}^{\infty} \int_{W_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1) d\underline{x}_1 d\underline{x}_2 \ldots d\underline{x}_n$$

$$- \int_{-\infty}^{\infty} \int_{W} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1) d\underline{x}_n \ldots d\underline{x}_1$$

$$= \int_{-\infty}^{\infty} \int_{W_0 - WW_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1) d\underline{x}_n \ldots d\underline{x}_1$$

$$- \int_{-\infty}^{\infty} \int_{W - WW_0} \cdots \int \underline{\widetilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_1) d\underline{x}_n \ldots d\underline{x}_1$$

$$\geq k \int_{-\infty}^{\infty} \int_{W_0 - WW_0} \cdots \int \underline{\tilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_n \ldots d\underline{x}_1$$

$$- k \int_{-\infty}^{\infty} \int_{W - WW_0} \cdots \int \underline{\tilde{X}}_1(\omega)(\underline{x}) \prod_{i=1}^{n} f(\underline{x}_i;\ \theta_0) d\underline{x}_n \ldots d\underline{x}_1$$

$$= 0.$$

The proof of the theorem is completed.

Example 2.8. Let $\underline{\tilde{X}}_1(\omega)$, $\underline{X}_2, \ldots, \underline{X}_n$ be the n size of random sample with one vector vague associated with p.d.f. $N_p(\underline{\mu}, I)$, where $\underline{\mu} \in \{\underline{\mu}_0, \underline{\mu}_1\}$, and the membership

$$\underline{\tilde{X}}_1(\omega)(\underline{x}_1) = e^{\frac{-1}{2}(\underline{x}_1 - \underline{m}(\omega))'(\underline{x}_1 - \underline{m}(\omega))},$$

where $\underline{m}(\omega)$ is known and independent of $\underline{\mu}$. It is desired to test the simple hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$ against $H_1 : \underline{\mu} = \underline{\mu}_1$ under the level α of significance, we have

$$Reject\ H_0 \Leftrightarrow \frac{L(\underline{\mu}_1)}{L(\underline{\mu}_0)} > k$$

but

$$\frac{L(\underline{\mu}_1)}{L(\underline{\mu}_0)} = \frac{2^{\frac{-(n+1)p}{2}} \pi^{\frac{-(n-1)p}{2}} e^{\frac{-1}{4}(\underline{\mu}_1 - \underline{m}(\omega))'(\underline{\mu}_1 - \underline{m}(\omega))} e^{\frac{-(n-1)}{2}(\bar{\underline{x}} - \underline{\mu}_1)'(\bar{\underline{x}} - \underline{\mu}_1)} e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j - \bar{\underline{x}})'(\underline{x}_j - \bar{\underline{x}})}}{2^{\frac{-(n+1)p}{2}} \pi^{\frac{-(n-1)p}{2}} e^{\frac{-1}{4}(\underline{\mu}_0 - \underline{m}(\omega))'(\underline{\mu}_0 - \underline{m}(\omega))} e^{\frac{-(n-1)}{2}(\bar{\underline{x}} - \underline{\mu}_0)'(\bar{\underline{x}} - \underline{\mu}_0)} e^{\frac{-1}{2}\sum_{j=2}^{n}(\underline{x}_j - \bar{\underline{x}})'(\underline{x}_j - \bar{\underline{x}})}}$$

$$= e^{-\frac{n-1}{2}[(\bar{\underline{x}} - \underline{\mu}_1)'(\bar{\underline{x}} - \underline{\mu}_1) - (\bar{\underline{x}} - \underline{\mu}_0)'(\bar{\underline{x}} - \underline{\mu}_0)]}$$

$$\times e^{-\frac{1}{4}[(\underline{\mu}_1 - \underline{m}(\omega))'(\underline{\mu}_1 - \underline{m}(\omega)) - (\underline{\mu}_0 - \underline{m}(\omega))'(\underline{\mu}_0 - \underline{m}(\omega))]},$$

therefore,

$$\frac{L(\underline{\mu}_1)}{L(\underline{\mu}_0)} > k$$

$$\Rightarrow -\frac{n-1}{2}[2(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0) + (\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\mu}_0 - \underline{\mu}_1)] > k_1$$

$$\Rightarrow (\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0) < k_2.$$

It is known that under hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$, $(\underline{\bar{x}} - \underline{\mu}_0) \sim N_p(\underline{0}, I)$, therefore,

$$\frac{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0)}{\sqrt{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\mu}_0 - \underline{\mu}_1)}} \sim N(0, 1),$$

and

$$P((\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0) < k_2 \,|\, \underline{\mu} = \underline{\mu}_0) = \alpha$$

$$\Rightarrow P\left(\frac{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0)}{\sqrt{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\mu}_0 - \underline{\mu}_1)}} < \frac{k_2}{\sqrt{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\mu}_0 - \underline{\mu}_1)}}\right) = \alpha$$

$$\Rightarrow \textit{Reject } H_0 \Leftrightarrow \frac{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\bar{x}} - \underline{\mu}_0)}{\sqrt{(\underline{\mu}_0 - \underline{\mu}_1)'(\underline{\mu}_0 - \underline{\mu}_1)}} < z_\alpha.$$

Example 2.9. Let $\underline{\tilde{X}}_1(\omega), \underline{X}_2, \ldots, \underline{X}_n$ be the n size of random sample with one vector vague associated with p.d.f. $N_p(\underline{\mu}, I)$, where $\underline{\mu} \in \{\underline{\mu}_0, \underline{\mu}_1\}$, and the membership $\underline{\tilde{X}}_1(\omega)(\underline{x}_1) = e^{\frac{-1}{2}(\underline{x}_1 - \underline{m}(\omega))'(\underline{x}_1 - \underline{m}(\omega))}$, where $\underline{m}(\omega)$ is known and independent of $\underline{\mu}$. It is desired to test the simple hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$ against $H_1 : \underline{\mu} \neq \underline{\mu}_1$ under the level α of significance by the method G.L.R., we have

$$\frac{L(\underline{\mu}_0)}{L(\hat{\underline{\mu}})} = \exp\left\{-\frac{n-1}{2}(\underline{\bar{x}} - \underline{\mu}_0)'(\underline{\bar{x}} - \underline{\mu}_0) - \frac{1}{4}(\underline{\mu}_0 - \underline{m}(\omega))'(\underline{\mu}_0 - \underline{m}(\omega))\right.$$

$$+ -\frac{n-1}{2}\left(\underline{\bar{x}} - \frac{2(n-1)\underline{\bar{x}} + \underline{m}(\omega)}{2n-1}\right)'\left(\underline{\bar{x}} - \frac{2(n-1)\underline{\bar{x}} + \underline{m}(\omega)}{2n-1}\right)$$

$$\left. + \frac{1}{4}\left(\frac{2(n-1)\underline{\bar{x}} + \underline{m}(\omega)}{2n-1} - \underline{m}(\omega)\right)'\left(\frac{2(n-1)\underline{\bar{x}} + \underline{m}(\omega)}{2n-1} - \underline{m}(\omega)\right)\right\},$$

$$= \exp\left\{-\frac{1}{2(2n-1)}((n-1)^2(\underline{\bar{x}} - \underline{\mu}_0)'(\underline{\bar{x}} - \underline{\mu}_0)\right.$$

$$\left. - 2(n-1)(\underline{\bar{x}} - \underline{\mu}_0)'(\underline{\mu}_0 - \underline{m}(\omega)) + \frac{1}{2}(\underline{\mu}_0 - \underline{m}(\omega))'(\underline{\mu}_0 - \underline{m}(\omega)))\right\},$$

$$\textit{Reject } H_0 \Leftrightarrow 0 < \frac{L(\underline{\mu}_0)}{L(\hat{\underline{\mu}})} < k < 1$$

$$\Leftrightarrow (n-1)^2(\underline{\bar{x}} - \underline{\mu}_0)'(\underline{\bar{x}} - \underline{\mu}_0) - 2(n-1)(\underline{\bar{x}} - \underline{\mu}_0)'(\underline{\mu}_0 - \underline{m}(\omega))$$

$$+ (\underline{\mu}_0 - \underline{m}(\omega))'(\underline{\mu}_0 - \underline{m}(\omega)) > k'$$

$$\Leftrightarrow ((n-1)(\underline{\bar{x}} - \underline{\mu}_0) - (\underline{\mu}_0 - \underline{m}(\omega)))'((n-1)(\underline{\bar{x}} - \underline{\mu}_0)$$

$$- (\underline{\mu}_0 - \underline{m}(\omega))) > k'.$$

It is known that under hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$,

$$\underline{Y} = (n-1)(\underline{\bar{X}} - \underline{\mu}_0) - (\underline{\mu}_0 - \underline{m}(\omega)) \sim N_p(\underline{m}(\omega) - \underline{\mu}_0, (n-1)^2 I).$$

Therefore $\frac{\underline{Y}'\underline{Y}}{(n-1)^2} \sim \chi^2_{(p)}$ with the non-central parameter

$$\frac{(\underline{m}(\omega) - \underline{\mu}_0)'(\underline{m}(\omega) - \underline{\mu}_0)}{(n-1)^2}.$$

Hence

$$\textit{Reject } H_0 \Leftrightarrow \underline{Y}'\underline{Y} > \chi^2_{(p),1-\alpha}.$$

Conclusion

(1) The M.L.E. $\underline{\hat{\mu}}$ of $\underline{\mu}$ by random vector sample of size n with one vague vector $\underline{\tilde{X}}(\omega)$ from $N_p(\underline{\mu}, I(\text{or } \Sigma_0))$ is different from $\underline{\overline{X}} = (\overline{X}_1, \overline{X}_2, \overline{X}_3, \ldots, \overline{X}_p)' \left(\overline{X}_i = \frac{1}{n}\sum_{k=1}^{n} X_{ki}\ (i = 1, 2, \ldots, p)\right)$, which is the M.L.E. of $\underline{\mu}$ by n size of classical random vector sample from $N_p(\underline{\mu}, I(\text{or } \Sigma_0))$.

(2) For n statistical vector with one vague vector in crisp statistics, we usually use the $n-1$ stable vector to test the hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$ against $H_1 : \underline{\mu} = \underline{\mu}_1$. It is shown in Example 2.8 that we can use all the vectors (including the vague vector) to have the same result by our method.

(3) For n statistical vector with one vague vector in crisp statistics, we usually use the $n-1$ stable vector to test the hypothesis $H_0 : \underline{\mu} = \underline{\mu}_0$ against $H_1 : \underline{\mu} \neq \underline{\mu}_0$. It is shown in Example 2.9 that we can use all the vectors (including the vague vector) to have the same result by our method. But their M.L.E.s are different.

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