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A NEW NONLINEAR EVOLUTION EQUATION HIERARCHY AND ITS BINARY SYMMETRIC CONSTRAINED FLOWS

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Abstract

Based on a subalgebra of loop algebra $\widetilde{A}_2$, we design a new 3×3 isospectral problem. By making use of Tu's scheme, a nonlinear equation hierarchy is obtained. Moreover, it is shown that the equation hierarchy is integrable in the sense of Liouville and possesses tri-Hamiltonian structure. Then we construct its binary symmetric constrained flows, which are reduced to Hamiltonian systems in the end.

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1. Introduction

Finding new integrable Hamiltonian systems is an important and interesting task. Integrable Hamiltonian systems contain infinite-dimensional systems and finite-dimension systems. Tu's trace identity [1-3] and nonlinearization technique [4, 7] are effective approaches to produce them, respectively. This paper tries to construct a 3×3 isospectral problem, and uses Tu's scheme to obtain an nonlinear evolution equation hierarchy. As its reduction cases, the generalized mKdV [5] hierarchy is presented. Under the Bargmann or Neumann constraints between the potentials and eigenvalues, and the eigenvalue problem is nonlinearization as a finite-dimensional integrable system. Recently this approach was developed to treat higher-order constraints. By establishing binary symmetric constraints, three constrained flows of the hierarchy are finally presented.

2. A Higher Order Loop Algebra and a Corresponding Integrable System

Consider the loop algebra $\widetilde{A}_2$ as follows:

$$e_1(i, n) = \begin{pmatrix} 0 & -\lambda^{3n+i} & 0 \\ -\lambda^{3n+i} & 0 & \lambda^{3n+i} \\ 0 & \lambda^{3n+i} & 0 \end{pmatrix}, \quad e_2(i, n) = \begin{pmatrix} 0 & -\lambda^{3n+i} & 0 \\ \lambda^{3n+i} & 0 & -\lambda^{3n+i} \\ 0 & \lambda^{3n+i} & 0 \end{pmatrix},$$

$$e_3(i, n) = \begin{pmatrix} -\lambda^{3n+i} & 0 & \lambda^{3n+i} \\ 0 & 2\lambda^{3n+i} & 0 \\ \lambda^{3n+i} & 0 & -\lambda^{3n+i} \end{pmatrix}, \quad e_4(i, n) = \begin{pmatrix} \lambda^{3n+i} & 0 & \lambda^{3n+i} \\ 0 & 2\lambda^{3n+i} & 0 \\ \lambda^{3n+i} & 0 & \lambda^{3n+i} \end{pmatrix},$$

$$[e_1(i, m), e_2(j, n)] = \begin{cases} 2e_3(i+j, m+n), & i+j < 3, \\ 2e_3(i+j-3, m+n+1), & i+j \geq 3, \end{cases}$$

$$[e_1(i, m), e_3(j, n)] = \begin{cases} 4e_2(i+j, m+n), & i+j < 3, \\ 4e_2(i+j-3, m+n+1), & i+j \geq 3, \end{cases}$$

$$[e_1(i, m), e_4(j, n)] = \begin{cases} 2e_2(i+j, m+n), \ i+j < 3, \\ 2e_2(i+j-3, m+n+1), \ i+j \geq 3, \end{cases}$$

$$[e_2(i, m), e_3(j, n)] = \begin{cases} 4e_1(i+j, m+n), \ i+j < 3, \\ 4e_1(i+j-3, m+n+1), \ i+j \geq 3, \end{cases}$$

$$[e_2(i, m), e_4(j, n)] = \begin{cases} 2e_1(i+j, m+n), \ i+j < 3, \\ 2e_1(i+j-3, m+n+1), \ i+j \geq 3, \end{cases}$$

$$[e_3(i, m), e_4(j, n)] = 0, \ \ deg(e_j(i, n)) = 3n + i, \ \ i = 0, 1, 2, \ \ j = 1, 2, 3, 4. \tag{1}$$

In terms of (1), an isospectral problem is in the following:

$$\phi_x = U\phi, \ \ \lambda_t = 0, \ \ \phi = (\phi_1, \phi_2, \phi_3)^T, \ \ U = \begin{pmatrix} \lambda & -2q & \lambda \\ -2r & 2\lambda & 2r \\ \lambda & 2q & \lambda \end{pmatrix}$$

$$= e_4(1, 0) + (q+r)e_1(0, 0) + (q-r)e_2(0, 0). \tag{2}$$

Let

$$V = \begin{pmatrix} -c(0) - \lambda c(1) - \lambda^2 c(2) & V_{12} & c(0) + \lambda c(1) + \lambda^2 c(2) \\ V_{21} & 2c(0) + 2\lambda c(1) + 2\lambda^2 c(2) & V_{23} \\ c(0) + \lambda c(1) + \lambda^2 c(2) & V_{32} & -c(0) - \lambda c(1) - \lambda^2 c(2) \end{pmatrix},$$

where

$$V_{12} = -a(0) - \lambda a(1) - \lambda^2 a(2) - b(0) - \lambda b(1) - \lambda^2 b(2),$$

$$V_{21} = -a(0) - \lambda a(1) - \lambda^2 a(2) + b(0) + \lambda b(1) + \lambda^2 b(2),$$

$$V_{23} = a(0) + \lambda a(1) + \lambda^2 a(2) - b(0) - \lambda b(1) - \lambda^2 b(2),$$

$$V_{32} = a(0) + \lambda a(1) + \lambda^2 a(2) + b(0) + \lambda b(1) + \lambda^2 b(2),$$

$$a(0) = \sum_{m \geq 0} a(0, m)\lambda^{-3m}, \ \ a(1) = \sum_{m \geq 0} a(1, m)\lambda^{-3m}, \ \dots .$$

Solving the stationary zero-curvature equation

$$V_x = [U, V] \tag{3}$$

yields

$$
\begin{aligned}
& a_x(0, m) = -2b(2, m+1) + 4(q-r)c(0, m), \\
& a_x(1, m) = -2b(0, m) + 4(q-r)c(1, m), \\
& a_x(2, m) = -2b(1, m) + 4(q-r)c(2, m), \\
& b_x(0, m) = -2a(2, m+1) + 4(q+r)c(0, m), \\
& b_x(1, m) = -2a(0, m) + 4(q+r)c(1, m), \\
& b_x(2, m) = -2a(1, m) + 4(q+r)c(2, m), \\
& c_x(0, m) = 2(q+r)b(0, m) - 2(q-r)a(0, m), \\
& c_x(1, m) = 2(q+r)b(1, m) - 2(q-r)a(1, m), \\
& c_x(2, m) = 2(q+r)b(2, m) - 2(q-r)a(2, m), \ c(0, 0) = \beta = const, \\
& b(0, 0) = a(0, 0) = a(1, 0) = b(1, 0) = c(1, 0) = c(2, 0) = b(2, 0) = a(2, 0) = 0, \\
& a(2, 1) = 2\beta q + 2\beta r, \ b(2, 1) = 2\beta q - 2\beta r, \ c(2, 1) = 0, \\
& a(1, 1) = -\beta q_x + \beta r_x, \ b(1, 1) = -\beta q_x - \beta r_x, \ c(1, 1) = -4\beta qr, \\
& a(0, 1) = \frac{1}{2}\beta q_{xx} + \frac{1}{2}\beta r_{xx} - 8\beta q^2 r - 8\beta q r^2, \\
& b(0, 1) = \frac{1}{2}\beta q_{xx} - \frac{1}{2}\beta r_{xx} - 8\beta q^2 r + 8\beta q r^2, \ c(0, 1) = -2\beta q r_x + 2\beta r q_x.
\end{aligned} \tag{4}
$$

Note

$$
\begin{aligned}
V_+^{(n)} &= \sum_{m=0}^{n} \sum_{i=0}^{2} (a(i, m)e_1(i, n-m) + b(i, m)e_2(i, n-m) \\
&\qquad + c(i, m)e_3(i, n-m)), \\
V_-^{(n)} &= \lambda^{3n} V - V_+^{(n)},
\end{aligned} \tag{5}
$$

then we have $-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]$. It is easy to verify that the terms of the left-hand side in (5) are of degree ≥ 0, while the terms of the right-hand side in (5) are of degree ≤ 0. Therefore, the terms of both sides in (5) are of degree 0. It follows that

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2b(2, n+1)e_1(0, 0) + 2a(2, n+1)e_2(0, 0).$$

Taking $V^{(n)} = V_+^{(n)}$, then the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{6}$$

admits the Lax integrable system

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -b(2, n+1) - a(2, n+1) \\ -b(2, n+1) + a(2, n+1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} 4a(2, n+1) - 4b(2, n+1) \\ 4a(2, n+1) + 4b(2, n+1) \end{pmatrix}$$

$$= J_1 \begin{pmatrix} 4a(2, n+1) - 4b(2, n+1) \\ 4a(2, n+1) + 4b(2, n+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a_x(0, n) + \frac{1}{2}b_x(0, n) - 4qc(0, n) \\ \frac{1}{2}a_x(0, n) - \frac{1}{2}b_x(0, n) + 4rc(0, n) \end{pmatrix}$$

$$= \begin{pmatrix} 2q\partial^{-1}q & \frac{1}{8}\partial - 2q\partial^{-1}r \\ \frac{1}{8}\partial - 2r\partial^{-1}q & 2r\partial^{-1}r \end{pmatrix} \begin{pmatrix} 4a(0, n) - 4b(0, n) \\ 4a(0, n) + 4b(0, n) \end{pmatrix}$$

$$= J_2 \begin{pmatrix} 4a(0, n) - 4b(0, n) \\ 4a(0, n) + 4b(0, n) \end{pmatrix}$$

$$= \begin{pmatrix} q\partial^{-1}q\partial - \partial q\partial^{-1}q & -\frac{1}{16}\partial^2 + q\partial^{-1}r\partial + \partial q\partial^{-1}r \\ \frac{1}{16}\partial^2 - r\partial^{-1}q\partial - \partial r\partial^{-1}q & -r\partial^{-1}r\partial + \partial r\partial^{-1}r \end{pmatrix} \begin{pmatrix} 4a(1, n) - 4b(1, n) \\ 4a(1, n) + 4b(1, n) \end{pmatrix}$$

$$= J_3 \begin{pmatrix} 4a(1, n) - 4b(1, n) \\ 4a(1, n) + 4b(1, n) \end{pmatrix}. \tag{7}$$

A direct calculation reads

$$\left\langle V, \frac{\partial U}{\partial q}\right\rangle = 4[a(0) + \lambda a(1) + \lambda^2 a(2) - b(0) - \lambda b(1) - \lambda^2 b(2)],$$

$$\left\langle V, \frac{\partial U}{\partial r}\right\rangle = 4[a(0) + \lambda a(1) + \lambda^2 a(2) + b(0) + \lambda b(1) + \lambda^2 b(2)],$$

$$\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = 4c(0) + 4\lambda c(1) + 4c(2)\lambda^2.$$

Substituting the above formula into trace identity leads to

$$\frac{\delta}{\delta u}\left(\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle\right) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix}\left\langle V, \frac{\partial U}{\partial q}\right\rangle \\ \left\langle V, \frac{\partial U}{\partial r}\right\rangle\end{pmatrix}. \tag{8}$$

Comparing the coefficients of λ^{-3n-2}, λ^{-3n-1} and λ^{-3n} in (8) gives rise to

$$\frac{\delta}{\delta u}4c(1, n+1) = (-3n - 1 + \gamma)\begin{pmatrix}4a(2, n+1) - 4b(2, n+1) \\ 4a(2, n+1) + 4b(2, n+1)\end{pmatrix} \tag{9}$$

$$\frac{\delta}{\delta u}4c(2, n+1) = (-3n + \gamma)\begin{pmatrix}4a(0, n) - 4b(0, n) \\ 4a(0, n) + 4b(0, n)\end{pmatrix} \tag{10}$$

$$\frac{\delta}{\delta u}4c(0, n) = (-3n + 1 + \gamma)\begin{pmatrix}4a(1, n) - 4b(1, n) \\ 4a(1, n) + 4b(1, n)\end{pmatrix}. \tag{11}$$

Inserting the initial values in (4) into (9), (10) and (11) gives $\gamma = 0$. Thus, the relations (9), (10) and (11) can determine the following three Hamiltonian functions:

$$\begin{cases}\dfrac{\delta H(2, 3n+2)}{\delta u} = \begin{pmatrix}4a(2, n+1) - 4b(2, n+1) \\ 4a(2, n+1) + 4b(2, n+1)\end{pmatrix}, \\ H(2, 3n+2) = -\dfrac{4c(1, n+1)}{3n+1},\end{cases} \tag{12}$$

$$\begin{cases}\dfrac{\delta H(3, 3n+1)}{\delta u} = \begin{pmatrix}4a(0, n) - 4b(0, n) \\ 4a(0, n) + 4b(0, n)\end{pmatrix}, \\ H(3, 3n+1) = -\dfrac{4c(2, n+1)}{3n},\end{cases} \tag{13}$$

$$\begin{cases} \dfrac{\delta H(1,\, 3n+3)}{\delta u} = \begin{pmatrix} 4a(1,\, n) - 4b(1,\, n) \\ 4a(1,\, n) + 4b(1,\, n) \end{pmatrix}, \\ H(1,\, 3n+3) = -\dfrac{4c(0,\, n)}{3n-1}. \end{cases} \tag{14}$$

Therefore, we have

$$u_t = J_1 \frac{\delta H(2,\, 3n+2)}{\delta u} = J_2 \frac{\delta H(3,\, 3n+1)}{\delta u} = J_3 \frac{\delta H(1,\, 3n+3)}{\delta u}. \tag{15}$$

In terms of (4), a recurrence operator L is given by

$$\begin{pmatrix} 4a(1,\, n+1) - 4b(1,\, n+1) \\ 4a(1,\, n+1) + 4b(1,\, n+1) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\partial - 8r\partial^{-1}q & 8r\partial^{-1}r \\ -8q\partial^{-1}q & -\frac{1}{2}\partial + 8q\partial^{-1}r \end{pmatrix} \begin{pmatrix} 4a(2,\, n+1) - 4b(2,\, n+1) \\ 4a(2,\, n+1) + 4b(2,\, n+1) \end{pmatrix}$$

$$= L \begin{pmatrix} 4a(2,\, n+1) - 4b(2,\, n+1) \\ 4a(2,\, n+1) + 4b(2,\, n+1) \end{pmatrix}$$

which satisfies the following:

$$J_1 L = L^* J_1 = J_2,\; J_2 L = L^* J_2 = J_3$$

which implies (15) is Liouville integrable. We can prove arbitrary combination of $\{J_1,\, J_2,\, J_3\}$ is still a symplectic operator by a direct calculation. Here we omit to prove it. When $n = 1$ in (15), a nonlinear evolution equation is presented as

$$\begin{cases} q_t = \beta\left(-\frac{1}{2} q_{xxx} + 16q^2 r_x + 8qrq_x\right), \\ r_t = \beta\left(-\frac{1}{2} r_{xxx} + 16q_x r^2 + 8qrr_x\right). \end{cases} \tag{16}$$

Especially, taking $q = \pm r$ in (16) gives the generalized mKdV equation

$$q_t = \beta\left(-\frac{1}{2} q_{xxx} + 24q^2 q_x\right). \tag{17}$$

When $H = qr,\ qr = q_x r_x$, we obtain

$$H_t = -\frac{1}{2}H_{xxx} + \frac{3}{2}H_x + 24HH_x.$$

3. Binary Constrained Flows of the Hierarchy

Consider the adjoint spectral problem of (2) as follows

$$\varphi_x = U^*\varphi = \begin{pmatrix} -\lambda & 2r & -\lambda \\ 2q & -2\lambda & -2q \\ -\lambda & -2r & -\lambda \end{pmatrix}\varphi,\ \lambda_t = 0,\ \varphi = (\varphi_1, \varphi_2, \varphi_3)^T. \tag{18}$$

A direct calculation from (2) and (18) yields

$$\frac{\delta\lambda}{\delta u} = \begin{pmatrix} \dfrac{\delta\lambda}{\delta q} \\ \dfrac{\delta\lambda}{\delta r} \end{pmatrix} = \begin{pmatrix} -2\phi_2\varphi_1 + 2\phi_2\varphi_3 \\ -2\phi_1\varphi_2 + 2\phi_3\varphi_2 \end{pmatrix}.$$

Let us consider the nonlinearization problem of the Lax pairs and the adjoint Lax pairs of the spectral problem of (2).

For N distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, we have

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U^*(u, \lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N, \tag{19}$$

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{tn} = V^{(n)}(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_{tn} = -V^{(n)T}(u, \lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N, \tag{20}$$

where

$$\lim_{|x|\to\infty} \phi_i = \lim_{|x|\to\infty} \varphi_i = 0, \quad i = 1, 2, 3.$$

Let $\phi_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{in})^T$, $\varphi_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in})^T$, $i = 1, 2, 3$ be solution of (19) and (20), $\langle\, ,\rangle$ denotes the standard inner product of R^N, $\wedge = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

We impose the following Bargmann constraint

$$J_1 \frac{\delta H(2, 3k+2)}{\delta u} = J_1 \alpha_k \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}, \tag{21}$$

where α_k is some non-zero constant.

When $k = 0$, (21) be conducted that

$$\frac{\delta H(2, 2)}{\delta u} = \begin{pmatrix} 16\beta r \\ 16\beta q \end{pmatrix} = \alpha_1 \begin{pmatrix} -2\Phi_2\Psi_1 + 2\Phi_2\Psi_3 \\ -2\Phi_1\Psi_2 + 2\Phi_3\Psi_2 \end{pmatrix}, \tag{22}$$

taking $\alpha_0 = 8\beta$, (21) becomes

$$\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} -\Phi_2\Psi_1 + \Phi_2\Psi_3 \\ -\Phi_1\Psi_2 + \Phi_3\Psi_2 \end{pmatrix}. \tag{23}$$

By substituting (23) into (19) and (20), we obtain the nonlinearlized Lax pairs

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u, \lambda_j)|_A \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U^*(u, \lambda_j)|_A \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$

$$j = 1, 2, \ldots, N,$$

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{tn} = V^{(n)}(u, \lambda_j)|_A \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_{tn} = -V^{(n)T}(u, \lambda_j)|_A \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$

$$j = 1, 2, \ldots, N, \tag{24}$$

where the subscript A means substitution of (23) into the expression and (24) can be expressed as the Hamiltonian regular form

$$\Phi_{ix} = \frac{\partial \widetilde{H}_1}{\partial \Psi_i}, \; \Psi_{ix} = -\frac{\partial \widetilde{H}_1}{\partial \Phi_i}, \quad i = 1, 2, 3 \tag{25}$$

with Hamiltonian function

$$\widetilde{H}_1 = \wedge(\langle \Phi_3, \Psi_1 \rangle + \langle \Phi_1, \Psi_1 \rangle + 2\langle \Phi_2, \Psi_2 \rangle + \langle \Phi_1, \Psi_3 \rangle + \langle \Phi_3, \Psi_3 \rangle)$$
$$- 2(-\langle \Phi_1, \Psi_2 \rangle + \langle \Phi_3, \Psi_2 \rangle)(\langle \Phi_2, \Psi_1 \rangle - \langle \Phi_2, \Psi_3 \rangle).$$

We impose the second Bargmann constraint

$$J_2 \frac{\delta H(3, 3n+1)}{\delta u} = J_2 \alpha_k \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}. \tag{26}$$

When $k = 1$, we get

$$\frac{\delta H(3, 4)}{\delta u} = \begin{pmatrix} 4\beta r_{xx} - 64\beta q r^2 \\ 4\beta q_{xx} - 64\beta q^2 r \end{pmatrix} = \alpha_1 \begin{pmatrix} -2\Phi_2\Psi_1 + 2\Phi_2\Psi_3 \\ -2\Phi_1\Psi_2 + 2\Phi_3\Psi_2 \end{pmatrix} \tag{27}$$

taking $\alpha_1 = -2\beta$ in (27), we get the following high binary symmetric constraints:

$$\begin{pmatrix} r_{xx} - 16qr^2 \\ q_{xx} - 16q^2 r \end{pmatrix} = \begin{pmatrix} \Phi_2\Psi_1 - \Phi_2\Psi_3 \\ \Phi_1\Psi_2 - \Phi_3\Psi_2 \end{pmatrix}. \tag{28}$$

By substituting (28) into (19) and (20), we obtain the following symmetric constraint flows:

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u, \lambda_j)|_B \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U^*(u, \lambda_j)|_B \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$
$$j = 1, 2, \ldots, N,$$
$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{tn} = V^{(n)}(u, \lambda_j)|_B \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_{tn} = -V^{(n)T}(u, \lambda_j)|_B \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$
$$j = 1, 2, \ldots, N, \tag{29}$$

where the subscript B means substitution of (28) into the expression. We use Jacobi-Ostrogradsky coordinates

$$q_1 = q + r,\ \ q_2 = q_x - r_x,\ \ p_1 = q_x + r_x,\ \ p_2 = q - r.$$

Noting

$$P_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{iN}, q_1, q_2)^T,\ \ Q_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{iN}, p_1, p_2)^T,$$

then (29) can be expressed as the Hamiltonian form

$$P_{ix} = \frac{\partial \widetilde{H}_2}{\partial Q_i},\ \ Q_{ix} = -\frac{\partial \widetilde{H}_2}{\partial P_i} \quad (i = 1, 2, 3) \tag{30}$$

with the Hamiltonian function

$$\begin{aligned}\widetilde{H}_2 &= \langle \wedge\Phi_1, \Psi_1\rangle + \langle \wedge\Phi_3, \Psi_1\rangle + 2\langle \wedge\Phi_2, \Psi_2\rangle + \langle \wedge\Phi_1, \Psi_3\rangle + \langle \wedge\Phi_3, \Psi_3\rangle \\ &\quad - (q_1 + p_2)\langle \Phi_2, \Psi_1\rangle + (p_2 - q_1)\langle \Phi_1, \Psi_2\rangle + (q_1 - p_2)\langle \Phi_3, \Psi_2\rangle \\ &\quad + (q_1 + p_2)\langle \Phi_2, \Psi_3\rangle + \frac{1}{2}p_1^2 - \frac{1}{2}q_2^2 + 2q_1^2 p_2^2 - p_2^4 - q_1^4.\end{aligned}$$

We impose the third Bargmann constraint

$$J_3 \frac{\delta H(1, 3n+3)}{\delta u} = J_3 \alpha_k \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}. \tag{31}$$

When $k = 1$, we get

$$\frac{\delta H(1, 6)}{\delta u} = \begin{pmatrix} 8\beta r_x \\ -8\beta q_x \end{pmatrix} = \alpha_1 \begin{pmatrix} -2\Phi_2\Psi_1 + 2\Phi_2\Psi_3 \\ -2\Phi_1\Psi_2 + 2\Phi_3\Psi_2 \end{pmatrix} \tag{32}$$

taking $\alpha_1 = -8\beta$ in (32), we get the following high binary symmetric constraints:

$$\begin{pmatrix} r_x \\ -q_x \end{pmatrix} = \begin{pmatrix} 2\Phi_2\Psi_1 - 2\Phi_2\Psi_3 \\ 2\Phi_1\Psi_2 - 2\Phi_3\Psi_2 \end{pmatrix}. \tag{33}$$

By substituting (33) into (19) and (20), we obtain the following symmetric constraint flow:

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u, \lambda_j)|_C \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix},\quad \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U^*(u, \lambda_j)|_C \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$

$$j = 1, 2, \ldots, N,$$

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_{tn} = V^{(n)}(u, \lambda_j)|_C \begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \quad \begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}_{tn} = -V^{(n)T}(u, \lambda_j)|_C \begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix},$$

$$j = 1, 2, \ldots, N, \tag{34}$$

where the subscript C means substitution of (33) into the expression. We use Jacobi-Ostrogradsky coordinates

$$q_1 = q, \quad p_1 = r.$$

Noting

$$P_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{iN}, q_1)^T, \quad Q_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{iN}, p_1)^T,$$

then (34) can be expressed as the Hamiltonian form

$$P_{ix} = \frac{\partial \widetilde{H}_3}{\partial Q_i}, \quad Q_{ix} = -\frac{\partial \widetilde{H}_3}{\partial P_i} \quad (i = 1, 2, 3) \tag{35}$$

with the Hamiltonian function

$$\widetilde{H}_3 = \langle \wedge\Phi_1, \Psi_1\rangle + \langle \wedge\Phi_3, \Psi_1\rangle + 2\langle \wedge\Phi_2, \Psi_2\rangle + \langle \wedge\Phi_1, \Psi_3\rangle + \langle \wedge\Phi_3, \Psi_3\rangle$$
$$- 2q_1\langle \Phi_2, \Psi_1\rangle - 2p_1\langle \Phi_1, \Psi_2\rangle + 2p_1\langle \Phi_3, \Psi_2\rangle + 2q_1\langle \Phi_2, \Psi_3\rangle.$$

By using the method in [8], a direct calculation, we find (25), (30) and (35) are the integrable Hamiltonian system in the sense of Liouville.

Remark. That loop algebra can be applied to other nonlinear evolution equations.

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