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AN EXTENSION OF THE THOM-PORTEOUS FORMULA TO A CERTAIN CLASS OF COHERENT SHEAVES

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Abstract

The goal is a theorem which allows computations analogous to the Thom-Porteous formula for a morphism $\sigma : E \to F$ of coherent sheaves, which are not vector bundles, over a scheme X. In particular if $Y \subset X$ is the subset where either E or F is not a vector bundle, then the goal is to find a class supported on the set $D_k(\sigma) = \{x \in X - Y : \operatorname{rank}(\sigma(x)) \leq k\} \cup Y$.

S. Diaz has one method for accomplishing this goal: find a blow up $p : \widetilde{X} \to X$ such that the double dual of the pullbacks of E and F, namely $(p^*E)^{**}$ and $(p^*F)^{**}$, are vector bundles over $\widetilde{X}$. Hence over $\widetilde{X}$ there is a morphism of vector bundles $(p^*\sigma)^{**} : (p^*E)^{**} \to (p^*F)^{**}$. For an appropriate choice of k, apply the Thom-Porteous formula to compute the fundamental class of $D_k((p^*\sigma)^{**})$. Then $p_*[|D_k((p^*\sigma)^{**})|]$ is a class supported on $D_k(\sigma)$ in X. To derive a formula from this construction it suffices to express the Chern classes of $(p^*E)^{**}$ and

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$(p^*F)^{**}$ in terms of known information about E and F. A formula for these Chern classes is derived for E and F belonging to a certain class of coherent sheaves.

1. Introduction

Given a morphism σ of vector bundles E and F of rank e and f, respectively, over a purely n-dimensional Cohen-Macaulay scheme X, a nonnegative integer $k \leq \min\{e, f\}$, and a degeneracy locus

$$D_k(\sigma) = \{x \in X : \operatorname{rank}(\sigma(x)) \leq k\}$$

of codimension $(e-k)(f-k)$, the Thom-Porteous formula [2] gives the fundamental class of the degeneracy locus in the Chow group of X in terms of the Chern classes of E and F as follows:

$$[|D_k(\sigma)|] = \Delta_{f-k}^{(e-k)}(c(F-E)) \cap [X],$$

where $c(F-E)$ denotes the formal quotient

$$c(F-E) = \frac{c(F)}{c(E)} = \frac{1 + c_1(F)t + c_2(F)t^2 + \cdots}{1 + c_1(E)t + c_2(E)t^2 + \cdots},$$

and given a formal sum $c = c_0 + c_1 + c_2 + \cdots$, $\Delta_q^{(p)}(c)$ denotes

$$\Delta_q^{(p)}(c) = \det\begin{bmatrix} c_q & c_{q+1} & c_{q+2} & \cdots & c_{q+p-1} \\ c_{q-1} & c_q & c_{q+1} & \cdots & c_{q+p-2} \\ \vdots & & & \ddots & \\ c_{q-p+1} & c_{q-p+2} & c_{q-p+3} & \cdots & c_q \end{bmatrix}.$$

Example 1. Let $\mathbb{P}^2$ be projective space over a field K and let $\sigma : \mathcal{O}_{\mathbb{P}^2}^2 \to \mathcal{O}_{\mathbb{P}^2}^3(1)$ be given by the matrix

$$[\sigma] = \begin{bmatrix} x & z \\ y & x \\ 0 & y \end{bmatrix}.$$

Taking two by two minors gives $D_1(\sigma) = \{(0, 0, 1)\}$. By the Thom-Porteous formula the fundamental class of $D_1(\sigma)$ is given as follows, where H is the rational equivalence class of a hyperplane in $\mathbb{P}^2$:

$$[|D_1(\sigma)|] = \Delta_2^{(1)}\left(\frac{c(\mathcal{O}^3_{\mathbb{P}^2}(1))}{c(\mathcal{O}^2_{\mathbb{P}^2})}\right) \cap [\mathbb{P}^2]$$

$$= \Delta_2^{(1)}\left(\frac{(1 + Ht)^3}{1}\right)$$

$$= \Delta_2^{(1)}(1 + 3Ht + 3Pt^2)$$

$$= \det[3P]$$

$$= 3P.$$

In this simple example it is possible to verify $3P$ is the fundamental class of $D_1(\sigma)$. $D_1(\sigma)$ is a closed subscheme of $\mathbb{P}^2$ supported at the point $(0, 0, 1)$ so $[|D_1(\sigma)|]$ is nP, where n is the length of the local ring of $D_1(\sigma)$ at $(0, 0, 1)$. The local ring is $\dfrac{K[x, y]}{(x^2 - y, xy, y^2)}$ which is a 3-dimensional vector space over K (with basis $\{1, x, x^2\}$). Hence the fundamental class is $3P$.

The Thom-Porteous formula applies only to morphisms of vector bundles; however many interesting subschemes can only be described as the degeneracy locus of a morphism of coherent sheaves. Given a morphism $\sigma : E \to F$ of coherent sheaves over a scheme X, the goal is a formula that allows analogous computations. In fact Harris and Morrison [4] ask, "Is there a Porteous type formula for maps of torsion-free coherent sheaves?" This paper gives such a formula for morphisms of coherent sheaves which meet certain conditions. In particular if $Y \subset X$ is the subset where either E or F is not a vector bundle, the goal is to find a class supported on the set

$$D_k(\sigma) \equiv \{x \in X - Y : \operatorname{rank}(\sigma(x)) \leq k\} \cup Y.$$

Diaz [1] has one method for accomplishing this goal: find a blow up $p : \widetilde{X} \to X$ such that the double dual of the pullbacks of E and F, namely $(p^*E)^{**}$ and $(p^*F)^{**}$, are vector bundles over $\widetilde{X}$. Hence over $\widetilde{X}$ there is a morphism of vector bundles

$$(p^*\sigma)^{**} : (p^*E)^{**} \to (p^*F)^{**}.$$

For an appropriate choice of k, apply the Thom-Porteous formula to compute the fundamental class of $D_k((p^*\sigma)^{**})$. Then $p_*[|D_k((p^*\sigma)^{**})|]$ is a class supported on $D_k(\sigma)$ in X. To derive a formula from this construction it suffices to express the Chern classes of $(p^*E)^{**}$ and $(p^*F)^{**}$ in terms of known information about E and F. An expression for these Chern classes is provided for E and F belonging to a certain class of coherent sheaves. Section 2 details the result (proofs are given in Section 4), and Section 3 applies the result to a simple example.

2. Extension of the Thom-Porteous Formula

Definition 1. Let I be a coherent sheaf of ideals on a nonsingular, quasi-projective scheme X of dimension n over a field K. Call I *homogeneous of degree* $(d_1, \ldots, d_k)$ *with respect to local parameters at a set of distinct closed points* $\{x_1, \ldots, x_k\} \subset X$ if there is some choice of *local coordinates* $u_1, \ldots, u_n$ defined on neighborhoods U_i of x_i such that $I(U_i)$ has a set of generators each of which is a degree $d_i > 0$ homogeneous polynomial in $u_1, \ldots, u_n$ with coefficients in K. (The dependence of the set of local coordinates $u_1, \ldots, u_n$ on i has been suppressed.)

Definition 2. Let $\{U_i\}_{i \in \Lambda}$ be an open cover of a scheme X and D be an effective Cartier divisor on X such that $|D| \subset \bigcup_{i \in \Lambda'} U_i$ and $|D| \cap U_i = \varnothing$ for all $i \notin \Lambda'$, where $\Lambda' \subset \Lambda$. Write local equations for D as u_i on U_i for $i \in \Lambda'$ and 1 on U_i for $i \notin \Lambda'$. Let F_1 and F_2 be vector bundles of rank f_1 and f_2, respectively on X. Suppose F_1 splits on $U = \bigcup_{i \in \Lambda'} U_i$ and write $F_1|_U = L_1 \oplus \cdots \oplus L_{f_1}$. Let $\phi : F_1 \to F_2$ be a morphism of rank r on

$X - |D|$ dropping rank by $k \geq 1$ on $|D|$. Fix (locally) free bases for F_1 and F_2, respecting the splitting $F_1|_U = L_1 \oplus \cdots \oplus L_{f_1}$, so ϕ has a matrix representation $[\phi]$. Define the *jth column vanishing* M_j *of* $[\phi]$ *on* $|D|$ to be the greatest positive integer r such that for every i, each entry of the jth column of $[\phi(U_i)]$ is in the ideal $(u_i)^r$ in $\mathcal{O}_X(U_i)$. Let the *total column vanishing* M be the sum $M = M_1 + \cdots + M_{f_1}$.

Definition 3. A coherent sheaf E over a nonsingular, integral, quasi-projective scheme X of dimension $n \geq 2$ over a field K is *nice* at a finite set of distinct closed points $\{x_1, \ldots, x_k\} \subset X$ if it satisfies the following conditions:

- E has a locally-free resolution

$$0 \to E_2 \xrightarrow{\phi} E_1 \to E \to 0.$$

- The first nonzero Fitting ideal I of E is supported on the set $\{x_1, \ldots, x_k\}$.

- I is degree d_i homogeneous with respect to local parameters at x_i for $i = 1, \ldots, k$.

- If $p : \widetilde{X} \to X$ is the blow up of X along I and $e_i = p^{-1}(x_i)$, then the total column vanishing of $[p^*\phi]$ on e_i is $M_i = d_i$ for $i = 1, \ldots, k$.

The following theorem gives a formula for the Chern class of the double dual of the pullback of a nice sheaf.

Theorem 1. *Let X be a nonsingular, integral, quasi-projective scheme of dimension $n \geq 2$ over a field K, and let E be a coherent sheaf over X which is nice at a finite set of distinct closed points $\{x_1, \ldots, x_k\}$. Let $p : \widetilde{X} \to X$ be the blow up of X along the first nonzero Fitting ideal of E, and let L_i be the invertible sheaf associated to $p^{-1}(x_i)$ for $i = 1, \ldots, k$. Then*

$$c_t((p^*E)^{**}) = \frac{c_t(p^*E_1)}{c_t(p^*E_2) \cdot (1 + c_1(L_1)t)^{M_1} \cdots (1 + c_1(L_m)t)^{M_k}}.$$

Definition 4. Given a morphism $\sigma : E \to F$ of coherent sheaves over a scheme X, suppose $Y \subset X$ is the subset where either E or F is not a vector bundle. Let e and f be the ranks of E and F, respectively over $X - Y$, and choose a nonnegative integer $k \leq \min\{e, f\}$. Then the *kth degeneracy locus* $D_k(\sigma)$ *of* σ is

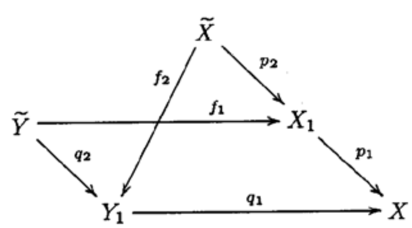
$$D_k(\sigma) = \{x \in X - Y : \operatorname{rank}(\sigma(x)) \leq k\} \cup Y.$$

Definition 5. Given a morphism $\sigma : E \to F$ of coherent sheaves over a scheme X, let the first nonzero Fitting ideals of E and F be I_E and I_F, respectively. Let $q_1 : X_1 \to X$ be the blow up of X along I_E, and let $q_2 : \widetilde{X} \to X_1$ be the blow up of X_1 along $q_1^{-1} I_F \cdot \mathcal{O}_{X_1}$. Then $p = q_1 \circ q_2 : \widetilde{X} \to X$ is the *double blow up of X along the first nonzero Fitting ideals of E and F.*

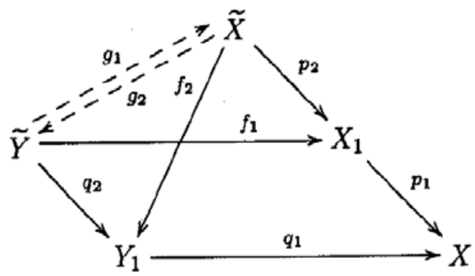
The following lemma shows the order in which the ideals I_E and I_F are blown up does not matter.

Lemma 1. *Let I_E and I_F be coherent sheaves of ideals on a noetherian scheme X. Suppose $p_1 : X_1 \to X$ is the blow up of X along I_E, and $p_2 : \widetilde{X} \to X_1$ is the blow up of X_1 along $p_1^{-1} I_F \cdot \mathcal{O}_{X_1}$. Suppose further $q_1 : Y_1 \to X$ is the blow up of X along I_F and $q_2 : \widetilde{Y} \to Y_1$ is the blow up of Y_1 along $q_1^{-1} I_E \cdot \mathcal{O}_{Y_1}$. Then $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic schemes.*

Proof. [5, Corollary II.7.15] gives unique morphisms $f_1 : \widetilde{Y} \to X_1$ and $f_2 : \widetilde{X} \to Y_1$ so the following diagram commutes:



Then $f_1^{-1}(p_1^{-1}I_F \cdot \mathcal{O}_{X_1}) \cdot \mathcal{O}_{\widetilde{Y}} = q_2^{-1}(q_1^{-1}F \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{\widetilde{Y}}$ is an invertible sheaf of ideals on $\widetilde{Y}$. Hence [5, Proposition II.7.14] gives a unique morphism $g_1 : \widetilde{Y} \to \widetilde{X}$ factoring f_1. Similarly there is a unique morphism $g_2 : \widetilde{X} \to \widetilde{Y}$ factoring f_2:



The sheaf of ideals $q_2^{-1}(q_1^{-1}I_E \cdot Y_1) \cdot \widetilde{Y}$ is invertible so $h = g_2 \circ g_1$ is the unique scheme morphism from $\widetilde{Y}$ to $\widetilde{Y}$ factoring q_2. However the identity morphism also has this property so $g_2 \circ g_1 = 1$. Similarly $g_1 \circ g_2 = 1$.

Theorem 2 (Extension of Thom-Porteous formula). *Suppose* $\sigma : E \to F$ *is a morphism of coherent sheaves over a nonsingular, integral, quasi-projective scheme X of dimension* $n \geq 2$ *over a field K. Let E and F be nice at a finite set of distinct closed points* $\{x_1, ..., x_l\} \subset X$. *Let* $p : \widetilde{X} \to X$ *be the double blow up of X along the first nonzero Fitting ideals of E and F. Let e, f, and k be as in Definition* 4. *Then a class supported on* $D_k(\sigma)$ *is given by the expression*

$$p_*[| D_k((p^*\sigma))^{**} |] = p_*(\Delta_{f-k}^{(e-k)}(c((p^*F)^{**} - (p^*E)^{**})) \cap [\widetilde{X}]),$$

where the Chern classes of $(p^*E)^{**}$ *and* $(p^*F)^{**}$ *are given by Theorem* 1.

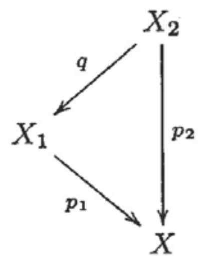
Proof. Apply the Thom-Porteous formula to the morphism

$$(p^*\sigma)^{**} : (p^*E)^{**} \to (p^*F)^{**}$$

of vector bundles over $\widetilde{X}$.

The following lemma shows the class given by Theorem 2 is unique in a certain sense.

Lemma 2. *Let X be a variety and* $p_1 : X_1 \to X$ *be a blow up. Let* $p_2 : X_2 \to X$ *be a blow up, and* $q : X_2 \to X_1$ *be a morphism making the following diagram commute*:



Given vector bundles E and F on X_1, $E' = q^*E$ *and* $F' = q^*F$ *are vector bundles on* X_2. *Let* $e = \text{rank}(E)$, $f = \text{rank}(F)$, $k \leq \min\{e, f\}$, *and for the determinantal expressions of the Thom-Porteous formula write*

$$\Delta_1 := \Delta_{f-k}^{(e-k)}(c(F - E)) \cap [X_1],$$

and

$$\Delta_2 := \Delta_{f-k}^{(e-k)}(c(F' - E')) \cap [X_2].$$

Then $q_*(\Delta_2) = \Delta_1$.

Proof. Since $\Delta_{f-k}^{(e-k)}(c(F' - E')) \cap [X_2]$ is a polynomial in the Chern classes of q^*E and q^*F, it follows from [2, Theorem 3.2(c)] that

$$q_*(\Delta_2) = \Delta_{f-k}^{(e-k)}(c(F - E)) \cap q_*[X_2].$$

Let $l = [R(X_2) : R(X_1)]$, where $R(\cdot)$ denotes the field of rational functions. By [2, Section 1.4], $q_*(\Delta_2) = l\Delta_1$. Since q is a birational morphism, $l = 1$.

3. Example

Define morphisms $\alpha : \mathcal{O}^2_{\mathbb{P}^2}(1) \to \mathcal{O}^5_{\mathbb{P}^2}(1)$ and $i : \mathcal{O}^2_{\mathbb{P}^2} \to \mathcal{O}^5_{\mathbb{P}^2}(1)$ by the matrices

$$[\alpha] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

and

$$[i] = \begin{bmatrix} x_2 & x_1 & 0 & x_3 & 0 \\ 0 & 0 & x_2 & x_1 & x_1 \end{bmatrix}^T.$$

Let $E \cong \mathcal{O}^2_{\mathbb{P}^2}(1)$ and F be sheaves defined by the following locally free resolutions:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathcal{O}^2_{\mathbb{P}^2}(1) & \xrightarrow{\pi_1} & E & \longrightarrow & 0 \\ & & & & \downarrow \alpha & & \downarrow \sigma & & \\ 0 & \longrightarrow & \mathcal{O}^2_{\mathbb{P}^2} & \xrightarrow{i} & \mathcal{O}^5_{\mathbb{P}^2}(1) & \xrightarrow{\pi_2} & F & \longrightarrow & 0. \end{array}$$

The morphism $\sigma : E \to F$ is induced by $\pi_2\alpha$ since $\pi_2\alpha$ vanishes on $\ker(\pi_1) = 0$. E and F are torsion-free coherent sheaves with locally free locus $Y = \mathbb{P}^2 - \{(0, 0, 1)\}$. $D_1(\sigma) = \{(0, 1, 0), (0, 0, 1)\}$.

Let $p : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ along the first nonzero Fitting ideal of F. Let e, H, p^*H, and P be the rational equivalence classes of the exceptional divisor, a hyperplane in $\mathbb{P}^2$, the pullback of H, and a point, respectively. Then

$$\begin{aligned} c_t((p^*F)^{**}) &= \frac{c_t(\mathcal{O}^5_{\widetilde{\mathbb{P}^2}}(1))}{c_t(\mathcal{O}_{\widetilde{\mathbb{P}^2}}) \cdot (1 + et)} \\ &= \frac{(1 + p^*Ht)^5}{1 + et} \\ &= (1 + 5p^*Ht + 10Pt^2)(1 - et - Pt^2) \\ &= 1 + 5p^*Ht - et + 9Pt^2. \end{aligned}$$

The Thom-Porteous formula gives the fundamental class of $D_1((p^*\sigma)^{**})$ as follows:

$$[|D_1((p^*\sigma)^{**})|] = \Delta_2^{(1)}(c((p^*F)^{**} - (p^*E)^{**})) \cap [\widetilde{\mathbb{P}^2}]$$

$$= \Delta_2^{(1)}(1 + 5p^*Ht - et + 9Pt^2)/((1 + p^*Ht)^2)$$

$$= \Delta_2^{(1)}(1 + 5p^*Ht - et + 9Pt^2)(1 - 2p^*Ht + 3Pt^2)$$

$$= \Delta_2^{(1)}(1 + 3p^*Ht - et + 2Pt^2)$$

$$= \det[2P]$$

$$= 2P.$$

To obtain a class in the Chow group $A_0(\mathbb{P}^2)$ supported on $D_1(\sigma)$ apply the group homomorphism p_*

$$p_*([|D_1((p^*\sigma)^{**})|]) = 2P.$$

On $\widetilde{\mathbb{P}^2}$ the double dual map drops rank at two points. The Thom-Porteous formula counts each of these point once [2, Lemma 12.1]. Thus each of the points (0, 1, 0) and (0, 0, 1) in $\mathbb{P}^2$ is counted exactly once (each is counted at least once by [2, Section 1.4].

4. Details

Lemma 3 (Existence of partial tensor). *Let $\{U_i\}_{i\in\Lambda}$ be an open cover of a scheme X over a field K, and let E be a rank r vector bundle splitting on $U = \bigcup_{i\in\Lambda'} U_i$, where $\Lambda' \subset \Lambda$. Suppose E splits on U as $E|_U = L_1 \oplus \cdots \oplus L_r$ for line bundles $L_1, \ldots, L_r$ on U. Let L be a line bundle with associated Cartier divisor D supported on a closed set $|D| \subset U$ such that $|D| \cap U_i = \varnothing$ for all $i \notin \Lambda'$. Let $M_1, \ldots, M_q$ be a finite sequence of positive integers with $q \le r$. Then there is a vector bundle G on X, call it the partial tensor of E with L by the sequence $M_1, \ldots, M_q$, such that*

(1) $G|_U \cong (L_1 \otimes L^{M_1}) \oplus \cdots \oplus L_q \otimes L^{M_q} \oplus L_{q+1} \oplus \cdots \oplus L_r$, *and*

(2) $G|_{U_i} = E|_{U_i}$, *for all* $i \notin \Lambda'$.

If $g_{i,j} : U_i \cap U_j \to GL(k, m)$ *give the transition data for* E *and* D *has local equations* $u_i \in \mathcal{O}_X(U_i)$ *with respect to the open cover* $\{U_i\}_{i \in \Lambda}$, *then* G *has transition data* $\tilde{g}_{i,j}$ *given by*

$$[\tilde{g}_{i,j}] = \begin{bmatrix} u_i^{M_1} & & & & & 0 \\ & \ddots & & & & \\ & & u_i^{M_q} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix} \cdot [g_{i,j}]$$

$$\cdot \begin{bmatrix} u_j^{-M_1} & & & & & 0 \\ & \ddots & & & & \\ & & u_j^{-M_q} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix}.$$

There is an injective vector bundle morphism $f : E \to G$ *such that for all* $i \in \Lambda$

$$[f(U_i)] = \begin{bmatrix} u_i^{M_1} & & & & & 0 \\ & \ddots & & & & \\ & & u_i^{M_q} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix},$$

and G *has Chern polynomial* $c_t(G) = c_t(E)(1 + c_1(L)t)^M$, *where* $M = M_1 + \cdots + M_q$.

Proof. It suffices to assume $q = 1$ and $M_1 = 1$. Shrink the U_i if necessary so that L and L_j for $j = 1, \ldots, r$ are trivial on every U_i for $i \in \Lambda'$. Let $h_{i,j} : U_i \cap U_j \to K^*$ and $g_{i,j} : U_i \cap U_j \to GL(r, K)$ be the transition functions for L and E, respectively. For $i \in \Lambda'$, say $L(U_i) = \langle e_0 \rangle$ and $E(U_i) = \langle e_1, \ldots, e_r \rangle$. Then define

$$G(U_i) = \langle e_0 \otimes e_1, e_2, \ldots, e_r \rangle,$$

and for $i \notin \Lambda'$ define $G(U_i) = E(U_i)$. One needs to specify the transition functions $\widetilde{g}_{i,j}$ for G. Consider $i, j \in \Lambda'$. There is a function $a_0 : U_i \cap U_j \to K^*$ such that $[h_{i,j}] = [a_0]$. Since E splits on U, there are functions $a_l : U_i \cap U_j \to K^*$ for $l = 1, \ldots, r$ such that

$$[g_{i,j}] = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{bmatrix}.$$

For $i, j \in \Lambda'$ define

$$[\widetilde{g}_{i,j}] = \begin{bmatrix} a_0 a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_r \end{bmatrix}.$$

Next consider $i \in \Lambda'$ and $j \notin \Lambda'$. Here $[h_{i,j}] = [a_0]$ and there are functions $a_{s,t} : U_i \cap U_j \to K^*$, such that

$$[g_{i,j}] = \begin{bmatrix} a_{1,1} & \cdots & a_{1,r} \\ & \ddots & \\ a_{r,1} & \cdots & a_{r,r} \end{bmatrix}.$$

For $i \in \Lambda'$ and $j \notin \Lambda'$ define

$$[\widetilde{g}_{i,j}] = \begin{bmatrix} a_0 a_{1,1} & \cdots & a_0 a_{1,r} \\ a_{2,1} & \cdots & a_{2,r} \\ & \ddots & \\ a_{r,1} & \cdots & a_{r,r} \end{bmatrix}.$$

For $i \notin \Lambda'$ and $j \in \Lambda'$, define $\widetilde{g}_{i,j} = \widetilde{g}_{j,i}^{-1}$ (see below for existence of the inverse). Finally for $i, j \notin \Lambda'$ define $\widetilde{g}_{i,j} = g_{i,j}$.

Now check the $\widetilde{g}_{i,j}$ satisfy the following axioms:

(1) $\widetilde{g}_{i,j} : U_i \cap U_j \to GL(r, K)$,

(2) $\widetilde{g}_{i,i} = 1$,

(3) $\widetilde{g}_{i,k} = \widetilde{g}_{i,j} \cdot \widetilde{g}_{j,k}$, and

(4) $\widetilde{g}_{i,j}^{-1} = \widetilde{g}_{j,i}$.

It is not hard to see (1) and (2) hold for $\widetilde{g}_{i,j}$ since they hold for $g_{i,j}$ and $h_{i,j}$. For axiom (4), first take the case $i, j \in \Lambda'$. Here since $h_{j,i} = h_{i,j}^{-1}$ and $g_{j,i} = g_{i,j}^{-1}$,

$$[\widetilde{g}_{i,j}^{-1}] = \begin{bmatrix} a_0 a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_r \end{bmatrix}^{-1} = \begin{bmatrix} a_0^{-1} a_1^{-1} & & & 0 \\ & a_2^{-1} & & \\ & & \ddots & \\ 0 & & & a_r^{-1} \end{bmatrix} = [\widetilde{g}_{j,i}].$$

Next consider $i \in \Lambda'$ and $j \notin \Lambda'$. Then $|\widetilde{g}_{i,j}| = a_0 |g_{i,j}|$ and $\widetilde{g}_{i,j}^{-1} = \widetilde{g}_{j,i}$ by definition. Finally consider $i, j \notin \Lambda'$. Here $\widetilde{g}_{i,j}^{-1} = g_{i,j}^{-1} = g_{j,i} = \widetilde{g}_{j,i}$. This suffices to check axiom (4). Lastly consider axiom (3). First take the case $i, j, k \in \Lambda'$. Here the matrices are diagonal so axiom (3) holds for $\widetilde{g}_{i,j}$ since it holds for $g_{i,j}$ and $h_{i,j}$. If $i, j, k \notin \Lambda'$, then axiom (3) holds since $\widetilde{g}_{i,j} = g_{i,j}$. Suppose $i, j \in \Lambda'$ and $k \notin \Lambda'$. Then

$$[\widetilde{g}_{i,j}] \cdot [\widetilde{g}_{j,k}] = \begin{bmatrix} a_0 a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_r \end{bmatrix} \cdot \begin{bmatrix} b_0 b_{1,1} & \cdots & b_0 b_{1,r} \\ b_{2,1} & \cdots & b_{2,r} \\ & \ddots & \\ b_{r,1} & \cdots & b_{r,r} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 b_0 a_1 b_{1,1} & \cdots & a_0 b_0 a_1 b_{1,r} \\ a_2 b_{2,1} & \cdots & a_2 b_{2,r} \\ & \ddots & \\ a_r b_{r,1} & \cdots & a_r b_{r,r} \end{bmatrix}.$$

Note that

$$[g_{i,k}] = [g_{i,j}] \cdot [g_{j,k}] = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_r \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,r} \\ b_{2,1} & \cdots & b_{2,r} \\ & \ddots & \\ b_{r,1} & \cdots & b_{r,r} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 b_{1,1} & \cdots & a_1 b_{1,r} \\ a_2 b_{2,1} & \cdots & a_2 b_{2,r} \\ & \ddots & \\ a_r b_{r,1} & \cdots & a_r b_{r,r} \end{bmatrix}.$$

There is a function $h_{i,k} = c_0 : U_i \cap U_k \to K^*$, and by definition

$$[\widetilde{g}_{i,k}] = \begin{bmatrix} c_0 a_1 b_{1,1} & \cdots & c_0 a_1 b_{1,r} \\ a_2 b_{2,1} & \cdots & a_2 b_{2,r} \\ & \ddots & \\ a_r b_{r,1} & \cdots & a_r b_{r,r} \end{bmatrix}.$$

Now $h_{i,k} = h_{i,j} \cdot h_{j,k}$ implies $c_0 = a_0 b_0$ so axiom (3) holds in this case. Finally suppose $i \in \Lambda'$ and $j, k \notin \Lambda'$. Write $[g_{i,k}] = [c_{s,t}] = [g_{i,j}] \cdot [g_{j,k}]$ $= [a_{s,t}] \cdot [b_{s,t}]$. Then

$$[\widetilde{g}_{i,j}] \cdot [\widetilde{g}_{j,k}] = [\widetilde{g}_{i,j}] \cdot [g_{j,k}] = \begin{bmatrix} a_0 a_{1,1} & \cdots & a_0 a_{1,r} \\ a_{2,1} & \cdots & a_{2,r} \\ & \ddots & \\ a_{r,1} & \cdots & a_{r,r} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,r} \\ b_{2,1} & \cdots & b_{2,r} \\ & \ddots & \\ b_{r,1} & \cdots & b_{r,r} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 c_{1,1} & \cdots & a_0 c_{1,r} \\ c_{2,1} & \cdots & c_{2,r} \\ & \ddots & \\ c_{r,1} & \cdots & c_{r,r} \end{bmatrix}.$$

This is enough to check axiom (3) since $\widetilde{g}_{i,j} = \widetilde{g}_{i,k} \cdot \widetilde{g}_{k,j}$ together with the other axioms implies $\widetilde{g}_{i,k} = \widetilde{g}_{i,j} \cdot \widetilde{g}_{k,j}^{-1} = \widetilde{g}_{i,j} \cdot \widetilde{g}_{j,k}$ and $\widetilde{g}_{k,j} = \widetilde{g}_{i,k}^{-1} \cdot \widetilde{g}_{i,j} = \widetilde{g}_{k,i} \cdot \widetilde{g}_{i,j}$.

To see the transition data $\widetilde{g}_{i,j}$ have the form given in the statement of the lemma, note $a_0 = u_i/u_j$. For the morphism $f : E \to G$, fix an index i. For any open $V \subset U_i$ define

$$[f_i(V)] = \begin{bmatrix} u_i|_V & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

This defines a *local vector bundle morphism* $f_i : E|_{U_i} \to G|_{U_i}$ and $[\widetilde{g}_{i,j}] \cdot [f_j] = [f_i] \cdot [g_{i,j}]$. So there is a morphism $f : E \to G$ extending all the f_i. f is injective since each $f(U_i)$ is injective. Since f is an isomorphism on $X - |D|$, the cokernel of f is supported on $|D|$. Restrict attention to U, the open neighborhood, where E is trivial of rank r. Form the direct sum of the following exact sequence with $\mathcal{O}_X|_U^{r-1}$

$$0 \to \mathcal{O}_X|_U \overset{\iota}{\to} \mathcal{O}_X(D)|_U$$

to obtain $f|_U$

$$0 \to E \overset{f|_U}{\to} G|_U.$$

Since $\operatorname{coker}(f)(U_i)$ is the zero module whenever $U_i \cap |D| = \varnothing$, it follows that $\operatorname{coker}(f) \cong \operatorname{coker}(\iota) \cong \mathcal{O}_D(D)$. Applying the Whitney sum formula to the short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0,$$

and

$$0 \to E \overset{f}{\to} G \to \operatorname{coker}(f) \to 0$$

shows $\operatorname{coker}(f)$ has Chern polynomial $1 + c_1(L)t$ and G has Chern polynomial $c_t(E)(1 + c_1(L)t)$.

Lemma 4. *Let* $\{U_i\}_{i\in\Lambda}$ *be an open cover of a scheme X, and D be a Cartier divisor on X such that* $|D| \subset \bigcup_{i\in\Lambda'} U_i$ *and* $|D| \cap U_i = \varnothing$ *for all* $i \notin \Lambda'$, *where* $\Lambda' \subset \Lambda$. *Write local equations for D as* u_i *on* U_i *for* $i \in \Lambda'$ *and* 1 *on* U_i *for* $i \notin \Lambda'$. *Let* F_1 *and* F_2 *be vector bundles of rank* f_1 *and* f_2, *respectively, on X. Suppose* F_1 *splits on* $U = \bigcup_{i\in\Lambda'} U_i$ *and write* $F_1|_U = L_1 \oplus \cdots \oplus L_{f_1}$. *Let* $\phi : F_1 \to F_2$ *be a morphism of rank r on* $X - |D|$ *dropping rank by* $k \geq 1$ *on* $|D|$. *Fix (locally) free bases for* F_1 *and* F_2, *respecting the splitting* $F_1|_U = L_1 \oplus \cdots \oplus L_{f_1}$, *so* ϕ *has a matrix representation* $[\phi]$. *Consider* $[\phi]$ *decomposed into submatrices whose dimensions are indicated below by subscripts*

$$[\phi] = \begin{bmatrix} A_{r, f_1 - r} & B_{r,r} \\ C_{f_2 - r, f_1 - r} & D_{f_2 - r, r} \end{bmatrix}.$$

Assume for all $i \in \Lambda'$, *the submatrices* $B_{r,r}(U_i)$ *and* $D_{f_2-r,r}(U_i)$ *have the following form (where the dependence of the entries* $h_{s,t}$ *on i has been suppressed):*

$$B_{r,r}(U_i) = \begin{bmatrix} h_{1,1} & \cdots & h_{1,r-k} & u_i^{M_1} h_{1,r-k+1} & \cdots & u_i^{M_k} h_{1,r} \\ & & & \ddots & & \\ h_{r,1} & \cdots & h_{r,r-k} & u_i^{M_1} h_{r,r-k+1} & \cdots & u_i^{M_k} h_{r,r} \end{bmatrix},$$

and

$$D_{f_2-r,r}(U_i) = \begin{bmatrix} h_{r+1,1} & \cdots & h_{r+1,r-k} & u_i^{M_1} h_{r+1,r-k+1} & \cdots & u_i^{M_k} h_{r+1,r} \\ & & & \ddots & & \\ h_{f_2,1} & \cdots & h_{f_2,r-k} & u_i^{M_1} h_{f_2,r-k+1} & \cdots & u_i^{M_k} h_{f_2,r} \end{bmatrix}.$$

Let G be the partial tensor of F_1 *and* $L = \mathcal{O}_X(D)$ *by the sequence* $M_1, \ldots, M_k$ *such that*

$$G(U) = L_1 \oplus \cdots \oplus L_{f_1-k} \oplus (L_{f_1-k+1} \otimes L^{M_1}) \oplus \cdots \oplus (L_{f_1} \otimes L^{M_k}).$$

Then there is a morphism $\psi : G \to F_2$ *such that*

(1) $\mathrm{im}(\phi) \subset \mathrm{im}(\psi)$ *and*

(2) $\mathrm{im}(\phi_x) = \mathrm{im}(\psi_x)$ *for all* $x \notin |D|$.

Proof. Notice columns $f_1 - k + 1, \ldots, f_1$ of $[\phi]$ vanish on $|D|$ with vanishing orders $M_1, \ldots, M_k$. Define

$$[\psi_i] = \begin{bmatrix} A(U_i) & B_i' \\ C(U_i) & D_i' \end{bmatrix},$$

where for all $i \in \Lambda$

$$[B_i'] = \begin{bmatrix} h_{1,1} & \cdots & h_{1,r} \\ & \ddots & \\ h_{r,1} & \cdots & h_{r,r} \end{bmatrix}$$

and

$$[D_i'] = \begin{bmatrix} h_{r+1,1} & \cdots & h_{r+1,r} \\ & \ddots & \\ h_{f_2,1} & \cdots & h_{f_2,r} \end{bmatrix}.$$

For $i \in \Lambda$, write $[\Theta_i]$ for the following matrix:

$$[\Theta_i] = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & u_i^{M_1} & & \\ & & & & \ddots & \\ 0 & & & & & u_i^{M_k} \end{bmatrix}.$$

Then $[g_{i,j}^{F_2}] \cdot [\psi_j] \cdot [\Theta_j] = [g_{i,j}^{F_2}] \cdot [\phi(U_j)] = [\phi(U_i)] \cdot [g_{i,j}^{F_1}] = [\psi_i] \cdot [\Theta_i] \cdot [g_{i,j}^{F_1}]$. In other words, $[g_{i,j}^{F_2}] \cdot [\psi_j] = [\psi_i] \cdot [\Theta_i] \cdot [g_{i,j}^{F_1}] \cdot [\Theta_j]^{-1} = [\psi_i] \cdot [g_{i,j}^{G}]$. Thus there is a vector bundle morphism $\psi : G \to F_2$ extending all the ψ_i. Since the u_i are units on $X - |D|$, condition (2) is satisfied. To check condition (1), suppose $y = (y_1, \ldots, y_{f_2}) \in (\mathrm{im}(\phi))(U_i)$ for some $i \in \Lambda'$. That is, there is some $(x_1, \ldots, x_{f_1})$ such that $\phi(U_i)(x_1, \ldots, x_{f_1}) = (y_1, \ldots, y_{f_2})$. Then by comparing the matrices of $\phi(U_i)$ and $\psi(U_i)$, one observes

$$\psi(U_i)(x_1, \ldots, x_{f_1 - k}, u_i^{M_1} x_{f_1 - k + 1}, \ldots, u_i^{M_k} x_{f_1}) = (y_1, \ldots, y_{f_2}).$$

Lemma 5. *Let X be a nonsingular, quasi-projective scheme of dimension $n \geq 2$ over a field K. Suppose E is a coherent sheaf which is nice over a finite set of distinct closed points $\{x_1, \ldots, x_k\} \subset X$. Suppose E has locally free resolution*

$$0 \to F_2 \overset{\phi}{\to} F_1 \to E \to 0,$$

where F_1 and F_2 have rank f_1 and f_2, respectively. Let $p : \widetilde{X} \to X$ be the blow up of X along the first nonzero Fitting ideal of E. Let $e_i = p^{-1}(x_i)$ and suppose $[\phi]$ has column vanishing sequence $M_1^i, \ldots, M_{f_1}^i$ on e_i for $i = 1, \ldots, k$. Let G be the partial tensor of $p^* F_2$ and $L_i = \mathcal{O}_{\widetilde{X}}(e_i)$ by the sequence $M_1^i, \ldots, M_{f_1}^i$ for $i = 1, \ldots, k$. Then Lemma 4 gives a sheaf morphism $\psi : G \to p^* F_1$. Moreover ψ is injective and

$$\mathrm{coker}(\psi) \cong (p^* E)^{**}.$$

Proof. Fix an index i in $1, \ldots, k$, and write $x = x_i$, $e = e_i$, $L = L_i$, and $M_1, \ldots, M_{f_1} = M_1^i, \ldots, M_{f_1}^i$. Since E is not locally free at x, $[\phi]$ drops rank at this point, say by $r \geq 1$. Let U be an affine open subset of X containing x such that $F_1(U) \cong \mathcal{O}_X(U)^{f_1}$, $F_2(U) \cong \mathcal{O}_X(U)^{f_2}$, and $U \cap \{x_1, \ldots, x_k\} = \{x\}$. Then the f_2 row by f_1 column matrix $[\phi(U)]$ has elements of $\mathcal{O}_X(U)$ as entries. Evaluated at the point x, they produce a rank $f_1 - r$ matrix of elements of the field K. Thus there is a finite sequence of K-linear elementary row operations giving the matrix $[\phi(U)]$ the following form (where I_s and $0_{s,t}$ denote the s by s identity and s by

t zero matrices, respectively):

$$\begin{bmatrix} I_{f_1-r} & 0_{f_1-r,r} \\ 0_{f_2-(f_1-r),f_1-r} & 0_{f_2-(f_1-r),r} \end{bmatrix}.$$

This sequence of row and operations corresponds to changing the local trivializations of F_1 and F_2. Since X is a nonsingular, quasi-projective scheme of dimension n, one may shrink U to a smaller affine open set as necessary so there are functions $u_1, \ldots, u_n \in \mathcal{O}_X(U)$ giving local coordinates near x. Choose them so the first nonzero Fitting ideal I of E is degree $d > 0$ homogeneous in $u_1, \ldots, u_n$. Write $U = \mathrm{Spec}(R)$ so $I(U)$ is generated by a finite number of degree d homogeneous polynomials in $u_1, \ldots, u_n$ over R, say by the polynomials $p_1, \ldots, p_t$. Since I is supported at x, one has $u_1^d, \ldots, u_n^d \in I(U)$. Let $\{U_i\}_{i\in\Lambda} \cup U$ be an open cover of X such that $x \notin U_i$ for all $i \in \Lambda$. Write $W = p^{-1}(U)$ and $W_i = p^{-1}(U_i)$ for $i \in \Lambda$. On W the blow up $\widetilde{X}$ has equations $\{u_i t_j = u_j t_i \mid i = 1, \ldots, n$ and $j = 1, \ldots, n\} \subset X \times \mathbb{P}^{n-1}$ ([3] 1.4). Let V_i be the open set $\{t_i \neq 0\} \cap W$ for $i = 1, \ldots, n$. Then the divisor e has equations u_i on V_i. Since $I(U)$ is generated by degree d polynomials $(p^{-1}I \cdot \mathcal{O}_{\widetilde{X}})(W \cap V_i) \subset (u_i^d)$. Since $u_i^d \in I(U)$, one has $(p^{-1}I \cdot \mathcal{O}_{\widetilde{X}})(W \cap V_i) = (u_i^d)$. Thus $(p^{-1}I \cdot \mathcal{O}_{\widetilde{X}})(W)$ $= (e^d)$. The hypotheses of Lemma 4 hold so there is a morphism $\psi' : G'$ $\to p^*F_1$, where G' is the partial tensor of p^*F_2 with L by the sequence $M_1, \ldots, M_{f_1}$. To see ψ' is injective, notice ϕ has rank f_1 on $X - \{x\}$ so $p^*\phi$ has rank f_1 on $\widetilde{X} - |e|$. ψ' has the same rank as $p^*\phi$ on $\widetilde{X} - |e|$. Choose a point $y \in |e|$ and suppose $y \in V_i$ for some fixed index i. The first nonzero Fitting ideal of E is generated by the $f_1 \times f_1$ minor determinants of $[\phi]$. Since the Fitting ideal is degree d homogeneous with respect to $u_1, \ldots, u_n$, $[\phi]$ has at least one $f_1 \times f_1$ minor determinant, call it $\prod$, of degree exactly d in $u_1, \ldots, u_n$. Compared to $[\phi]$, columns $j = f_1 - r$ $+ 1, \ldots, f_1$ of $[\psi']$ lack a factor of $u_i^{M_j}$. Hence the corresponding minor determinants of $[\psi']$ lack a factor of u_i^M, where $M = M_1 + \cdots + M_{f_1}$. Therefore the $f_1 \times f_1$ minor determinant of $[\psi']$ corresponding to $\prod$ has degree $d - M = 0$ in u_i so it is nonzero at y. Thus ψ' has rank f_1 at y. Since y was chosen arbitrarily in $|e|$, ψ' has rank f_1 everywhere. Repeat this argument for each $x_i \in \{x_1, ..., x_k\}$ to obtain an injective morphism $\psi : G \to p^*F_1$, where G is the partial tensor of p^*F_2 and L_i by the sequence $M_1^i, ..., M_{f_1}^i$ for $i = 1, ..., k$.

Let $f : p^*F_2 \to G$ be the morphism given by Lemma 3. ψ forms a commutative square with $1_{p^*F_2}$

$$\begin{array}{ccccccccc} & & p^*F_2 & \xrightarrow{p^*\phi} & p^*F_1 & \longrightarrow & p^*E & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow 1_{p^*F_1} & & \downarrow h & & \\ 0 & \longrightarrow & G & \xrightarrow{\psi} & p^*F_1 & \longrightarrow & \operatorname{coker}(\psi) & \longrightarrow & 0. \end{array}$$

There is a unique induced morphism $h : p^*E \to \operatorname{coker}(\psi)$ which is surjective by Lemma 5. To see $\ker(h)$ is torsion, let W_0 be a nonempty affine open subset of $\widetilde{X}$ such that $W_0 \cap |e_i| = \varnothing$ for $i = 1, ..., k$. Take sections over W_0 to obtain a diagram of modules with exact rows, where $f(W_0)$ is a module isomorphism

$$\begin{array}{ccccccccc} & & p^*F_2(W_0) & \longrightarrow & p^*F_1(W_0) & \longrightarrow & p^*E(W_0) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow 1_{p^*F_1} & & \downarrow h(W_0) & & \\ 0 & \longrightarrow & G(W_0) & \longrightarrow & p^*F_1(W_0) & \longrightarrow & (\operatorname{coker}(\psi))(W_0) & \longrightarrow & 0. \end{array}$$

By Lemma 5, $h(W_0)$ is an isomorphism. $\ker(h)$ is torsion since it is supported on a proper closed subset of $\widetilde{X}$. Thus there is an exact sequence of sheaves

$$0 \to \ker(h) \to p^*E \xrightarrow{h} \operatorname{coker}(\psi) \to 0.$$

Since $\ker(h)$ is torsion, applying $\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(\cdot, \mathcal{O}_{\widetilde{X}})$ gives

$$0 \to (\mathrm{coker}(\psi))^* \to (p^*E)^* \to (\ker(h))^* = 0.$$

$\mathrm{coker}(\psi) \cong (\mathrm{coker}(\psi))^{**}$ since $\mathrm{coker}(\psi)$ is a vector bundle so applying $\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(\cdot, \mathcal{O}_{\widetilde{X}})$ again gives the desired result.

Proof of Theorem 1. The theorem follows from Lemma 3 and the Whitney sum formula applied to the following short exact sequence from Lemma 5:

$$0 \to G \overset{\psi}{\to} p^*F_1 \to \mathrm{coker}(\psi) \to 0.$$

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