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GROUPS WITH FINITELY MANY NORMALIZERS OF INFINITE INDEX

(Dedicated to Hermann Heineken on his 70's birthday)

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Abstract

A famous theorem of B. H. Neumann states that a group G is central-by-finite if and only if all normalizers of subgroups of G have finite index, and Y. D. Polovickii proved that this is also equivalent to the property that the group G has only finitely many normalizers of subgroups. Here the structure of groups in which all but finitely many normalizers of (infinite) subgroups have finite index is investigated, and the above results are extended to this more general situation.

1. Introduction

A subgroup X of a group G is called *almost normal* if it has finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ has finite index in G. In a famous paper of 1955, Neumann [8] proved that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to abelian subgroups (see [2]). This result suggests that the

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behaviour of normalizers of subgroups with a given relevant property has a strong influence on the structure of the group. In fact, more recently Polovickii [9] has shown that if an *FC*-group G has finitely many normalizers of infinite abelian subgroups, then the factor group $G/Z(G)$ is finite (recall that G is an *FC-group* if it has finite conjugacy classes of elements). Since it is very easy to prove that any group with finitely many normalizers of cyclic subgroups has the property *FC*, it follows from Polovickii's theorem that a group has finitely many normalizers of abelian subgroups if and only if it is central-by-finite (see also [12]). Groups with finitely many normalizers of non-abelian subgroups have recently been described in [1].

The aim of this paper is to study groups in which only finitely many normalizers of (infinite) subgroups have infinite index. In particular, our first result generalizes both Neumann's and Polovickii's theorems.

Theorem A. *Let G be a group in which all but finitely many normalizers of abelian subgroups have finite index. Then the factor group $G/Z(G)$ is finite.*

Since every infinite subgroup of the locally dihedral 2-group has finite index, the above theorem cannot be improved imposing the condition just to infinite subgroups. However, the only obstructions in this direction are Cernikov groups, at least in the case of locally finite groups; note also that our next result does not hold for arbitrary periodic groups, as all Tarski groups (i.e., infinite simple groups whose proper non-trivial subgroups have prime order) obviously satisfy the required condition.

Theorem B. *Let G be a locally finite group in which all but finitely many normalizers of infinite subgroups have finite index. Then either G is a Cernikov group or $G/Z(G)$ is finite.*

Of course, the class of groups described by Theorem B contains all locally finite groups whose infinite subgroups are almost normal, and locally (soluble-by-finite) groups with this latter property have been completely described by Kurdachenko et al. [5]. Actually, it follows easily from Theorem B that if a locally finite group G has finitely many normalizers of infinite non-(almost normal) subgroups, then all infinite subgroups of G are almost normal.

We finally consider the case of non-periodic groups. A statement similar to Theorem B cannot be proved in this situation, as the consideration of the infinite dihedral group shows. On the other hand, we shall prove that the derived subgroup of any non-periodic group with all but finitely many normalizers of infinite subgroups having finite index must be polycyclic-by-finite; this property will be used to prove our last main result, concerning groups with large periodic subgroups.

Theorem C. *Let G be a non-periodic group in which all but finitely many normalizers of infinite subgroups have finite index. If G contains an infinite periodic subgroup, then the factor group $G/Z(G)$ is finite.*

Most of our notation is standard and can for instance be found in [10].

2. Proofs

The following result will be essential for our purposes; it was proved by Neumann [7] in the more general situation of groups covered by cosets of subgroups.

Lemma 1. *Let the group $G = X_1 \cup \cdots \cup X_t$ be the union of finitely many subgroups $X_1, \ldots, X_t$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_t$ has finite index in G.*

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates. Thus a group G is an FC-group if and only if it coincides with its FC-centre.

Lemma 2. *Let G be an FC-group in which all but finitely many normalizers of infinite abelian subgroups have finite index. Then the factor group $G/Z(G)$ is finite.*

Proof. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite index of infinite abelian subgroups of G. By Lemma 1 the set

$$N_G(X_1) \cup \cdots \cup N_G(X_k)$$

is properly contained in G. Let x be an element of

$$G \setminus (N_G(X_1) \cup \cdots \cup N_G(X_k)),$$

and consider any infinite abelian subgroup A of the centralizer $C_G(x)$; as x normalizes A, the normalizer $N_G(A)$ must have finite index in G. Thus all abelian subgroups of $C_G(x)$ are almost normal, and so $C_G(x)$ is central-by-finite (see [2]). On the other hand, the index $|G : C_G(x)|$ is finite, so that G is an abelian-by-finite FC-group and hence $G/Z(G)$ is finite.

We can now prove our first main result.

Proof of Theorem A. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite index of abelian subgroups of G, and let F be the FC-centre of G. Clearly,

$$G = F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)$$

and so it follows from Lemma 1 that $G = F$ is an FC-group. Thus $G/Z(G)$ is finite by Lemma 2.

Lemma 3. *Let G be a group and let x be an element of G such that every infinite subgroup of G normalized by x is almost normal. Then either x belongs to the FC-centre of G or its centralizer $C_G(x)$ satisfies the minimal condition on abelian subgroups.*

Proof. Assume that $C_G(x)$ contains an abelian subgroup A which does not satisfy the minimal condition. If x has infinite order, by hypothesis we have that $\langle x \rangle$ is almost normal in G and so x lies in the FC-centre F of G. Suppose that x has finite order. If A contains an element a of infinite order, then $\langle x, a \rangle$ is almost normal in G and $\langle x \rangle$ is characteristic in $\langle x, a \rangle$, so that $\langle x \rangle$ is almost normal in G and $x \in F$. Thus we may also suppose that A is periodic, so that its socle is infinite and there exist infinite subgroups A_1, A_2 of A such that

$$A_1 \cap A_2 = \langle A_1, A_2 \rangle \cap \langle x \rangle = \{1\}.$$

Then $x = \langle x, A_1 \rangle \cap \langle x, A_2 \rangle$ is almost normal in G and so x belongs to F also in this case.

Our next lemma shows that every locally finite group with finitely many normalizers of infinite non-(almost normal) subgroups is a finite extension of a locally soluble group.

Lemma 4. *Let G be a locally finite group in which all but finitely many normalizers of infinite subgroups have finite index. Then G contains a locally soluble subgroup of finite index.*

Proof. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite index of abelian subgroups of G. By Lemma 2 it can be assumed that the FC-centre F of G is properly contained in G, so that also

$$F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)$$

is a proper subset. If x is an element of

$$G \setminus (F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)),$$

it follows from Lemma 3 that the centralizer $C_G(x)$ is a Cernikov group. Write

$$\langle x \rangle = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle,$$

where each $\langle x_i \rangle$ is a non-trivial primary component of $\langle x \rangle$. Assume first that $C_G(x_i)$ is finite for some i; in this case, G is (locally soluble)-by-finite by a result of Hartley [4]. Thus we may suppose that all subgroups $C_G(x_1), \ldots, C_G(x_t)$ are infinite. As $x \in C_G(x_i) \leq N_G(C_G(x_i))$, it follows that $N_G(C_G(x_i))$ has finite index in G for all i, so that also

$$K = \bigcap_{i=1}^{t} N_G(C_G(x_i))$$

is a subgroup of finite index. Moreover, $N_G(C_G(x_i))/C_G(x_i)$ has finite many normalizers of infinite index and hence it is central-by-finite by Theorem A. Therefore $K/C_G(x)$ is abelian-by-finite and so G is soluble-by-finite. The lemma is proved.

Lemma 5. *Let G be a group, and let A be an infinite abelian normal subgroup of G with prime exponent p. If all subgroups of A are almost normal in G, then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A such that*

$$\langle a_n \mid n \in \mathbb{N} \rangle^G = \operatorname*{Dr}_{n \in \mathbb{N}} \langle a_n \rangle^G.$$

Proof. Let a_1 be any non-trivial element of A, and suppose by induction that elements $a_1, \ldots, a_n$ of A have been chosen in such a way that

$$\langle a_1, \ldots, a_n \rangle^G = \langle a_1 \rangle^G \times \cdots \times \langle a_n \rangle^G.$$

Clearly, $\langle a_1, \ldots, a_n \rangle^G$ is finite, and hence A contains an infinite subgroup B such that $A = \langle a_1, \ldots, a_n \rangle^G \times B.$ Since B is almost normal in G, the core B_G of B has finite index in A and so is infinite. If $a_{n+1} \in B_G \setminus \{1\},$ then we have

$$\langle a_1, \ldots, a_n, a_{n+1} \rangle^G = \langle a_1 \rangle^G \times \cdots \times \langle a_n \rangle^G \times \langle a_{n+1} \rangle^G$$

and the lemma is proved.

Proof of Theorem B. Consider the normalizers $N_G(X_1), \ldots, N_G(X_k)$ of infinite subgroups of G which are not almost normal, and let F be the FC-centre of G. Assume that the statement is false and choose the counterexample G in such a way that k is smallest possible. By a result of Šunkov [11], G contains an abelian subgroup which does not satisfy the minimal condition and hence $k > 0$ (see [3, Lemma 2.1]). Since by Lemma 2 the result is true for FC-groups, it follows from Lemma 1 that $F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)$ is a proper subset of G. Consider an element x of

$$G \setminus (F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)).$$

As G is (locally soluble)-by-finite by Lemma 4, it follows from a result of Zaicev [13] that G contains an abelian subgroup A which is not a Cernikov group such that $A^x = A.$ Without loss of generality, A can be replaced by its socle, so that it can be assumed that A is a direct product of infinitely many subgroups of prime order. If a is any element of $A \setminus (N_G(X_1) \cup \cdots \cup N_G(X_k)),$ the centralizer $C_G(a)$ does not satisfy the minimal condition on abelian subgroups and hence a belongs to F by Lemma 3. Therefore A is contained in the set $F \cup N_G(X_1) \cup \cdots \cup N_G(X_k).$ Assume that $\langle x \rangle^A$ is finite, so that A contains infinite subgroups A_1, A_2

such that

$$A_1 \cap A_2 = \langle A_1, A_2 \rangle \cap \langle x \rangle^A = \{1\};$$

then both subgroups $\langle x \rangle^A A_1$ and $\langle x \rangle^A A_2$ are almost normal in G and hence also

$$\langle x \rangle^A = \langle x \rangle^A A_1 \cap \langle x \rangle^A A_2$$

has finitely many conjugates. It follows that $\langle x \rangle^G = (\langle x \rangle^A)^G$ is finite by Dietzmann's lemma, and this contradiction shows that $\langle x \rangle^A$ is infinite. In particular, $\langle x, A \rangle$ is not an FC-group, and the minimal choice of k yields that $\langle x, A \rangle \cap N_G(X_i)$ has infinite index in $\langle x, A \rangle$ for each $i = 1, \ldots, k$. Thus the subgroups $A \cap N_G(X_1), \ldots, A \cap N_G(X_k)$ have infinite index in A, and so A is contained in F by Lemma 1. On the other hand, $F/Z(F)$ is finite by Lemma 2 and hence replacing A by the socle of $Z(F)$ it can be assumed that A is normal in G. Moreover, as A is not contained in $N_G(X_1) \cup \cdots \cup N_G(X_k)$, every (infinite) subgroup of A is almost normal in G, so that it follows from Lemma 5 that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A such that

$$\langle a_n \mid n \in \mathbb{N} \rangle^G = \underset{n \in \mathbb{N}}{\mathrm{Dr}} \langle a_n \rangle^G.$$

Thus A contains infinite G-invariant subgroups H and K such that

$$H \cap K = \langle H, K \rangle \cap \langle x \rangle = \{1\}.$$

Clearly, the subgroups $\langle x, H \rangle$ and $\langle x, K \rangle$ are almost normal in G, so that also $\langle x \rangle = \langle x, H \rangle \cap \langle x, K \rangle$ is almost normal and x belongs to F. This last contradiction completes the proof of the theorem.

Corollary 6. *Let G be a locally finite group in which all but finitely many normalizers of infinite subgroups have finite index. Then every infinite subgroup of G is almost normal.*

Proof. By Theorem B it can be assumed that G is a Cernikov group. Let X be any infinite subgroup of G, and let J be the finite residual of X.

Then J is infinite and it follows from Theorem A that the factor group $N_G(J)/J$ is central-by-finite, so that in particular X is almost normal in $N_G(J)$. Clearly, $N_G(J)$ has finite index in G and hence X is almost normal in G.

Lemma 7. *Let G be a locally nilpotent group with finitely many normalizers of infinite non-(almost normal) subgroups. Then either G is a Cernikov group or $G/Z(G)$ is finite.*

Proof. Suppose that G is not a Cernikov group, so that $C_G(x)$ is not a Cernikov group for any element x of G (see [6]). It follows from Lemma 3 that

$$G = F \cup N_G(X_1) \cup \cdots \cup N_G(X_k),$$

where F is the FC-centre of G and $N_G(X_1), \ldots, N_G(X_k)$ are the normalizers of all infinite subgroups of G which are not almost normal. Thus $G = F$ is an FC-group by Lemma 1, and hence $G/Z(G)$ is finite by Theorem A.

Our next result proves that if G is any group in which all but finitely many normalizers of infinite subgroups have finite index, then G contains a subgroup M of finite index such that every infinite subgroup of M is either subnormal or almost normal in G.

Lemma 8. *Let G be a group with finitely many normalizers of infinite non-(almost normal) subgroups. Then G contains a characteristic subgroup M of finite index such that for each infinite subgroup X of M either $N_G(X)$ has finite index in G or $N_M(X)$ is normal in M.*

Proof. If X is any infinite subgroup of G which is not almost normal, the normalizer $N_G(X)$ has obviously finitely many images under automorphisms of G; in particular, the subgroup $N_G(X)$ has finitely many conjugates in G and so the index $|G : N_G(N_G(X))|$ is finite. It follows that also the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in Aut\, G} N_G(N_G(X))^{\alpha}$$

has finite index in G. Let $\mathcal{H}$ be the set of all infinite subgroups of G whose normalizer has infinite index. If X and Y are elements of $\mathcal{H}$ such that $N_G(X) = N_G(Y)$, then $M(X) = M(Y)$ and hence also

$$M = \bigcap_{X \in \mathcal{H}} M(X)$$

is a characteristic subgroup of finite index of G. Let X be any infinite subgroup of M such that the index $|G : N_G(X)|$ is infinite. Then

$$M \leq M(X) \leq N_G(N_G(X)),$$

and so $N_M(X) = N_G(X) \cap M$ is a normal subgroup of M.

Recall that the *Baer radical* of a group G is the subgroup generated by all cyclic subnormal subgroups of G. In particular, the Baer radical of any group is locally nilpotent.

Lemma 9. *Let G be a non-periodic group with finitely many normalizers of infinite non-*(*almost normal*) *subgroups. Then the commutator subgroup G' of G is polycyclic-by-finite.*

Proof. By Lemma 8, G contains a characteristic subgroup M of finite index such that every infinite subgroup of M either is subnormal or almost normal in G. Let x be an element of infinite order of M. Then x belongs either to the Baer radical B or to the FC-centre F of G. It follows that the normal closure $\langle x \rangle^G$ either is locally nilpotent or an FC-group, so that $\langle x \rangle^G / Z(\langle x \rangle^G)$ is finite by Lemmas 7 and 2, respectively, and hence $(\langle x \rangle^G)'$ is finite. Moreover, the factor group $G/\langle x \rangle^G$ has finitely many normalizers and so it is central-by-finite by Theorem A. In particular, G is soluble-by-finite. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite subgroups of G which are not almost normal. Then $N_G(X_1) \cup \cdots \cup N_G(X_k)$ is a proper subset of G by Lemma 1; if g is an element of

$$G \setminus (N_G(X_1) \cup \cdots \cup N_G(X_k)),$$

the subgroup $\langle x, g \rangle$ must be almost normal in G and hence the normal closure $\langle x, g \rangle^G$ is finitely generated. Put $N = \langle x, g \rangle^G \cap M$ and let H be the subgroup generated by all elements of infinite order of N. It follows from the first part of the proof that $H/H \cap F$ is generated by abelian normal subgroups and so it is locally nilpotent. On the other hand, as the index $|\langle x, g\rangle^G : N|$ is finite, the subgroup N is finitely generated, so that N/H is finite and H is likewise finitely generated. In particular, $H/H \cap F$ is a finitely generated nilpotent group and hence $N/H \cap F$ is polycyclic-by-finite; thus $H \cap F$ is the normal closure (in N and so also in G) of a finite subset. It follows that $H \cap F$ is a finitely generated FC-group and in particular it satisfies the maximal condition on subgroups. Therefore $\langle x, g\rangle^G$ is polycyclic-by-finite. As $G'/G' \cap \langle x\rangle^G$ is finite, it follows that also G' is polycyclic-by-finite.

Lemma 10. *Let G be a group with finitely many normalizers of infinite non-(almost normal) subgroups. If the centre $Z(G)$ is infinite, then $G/Z(G)$ is finite.*

Proof. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite index of infinite subgroups of G, and let F be the FC-centre of G. Assume for a contradiction that $F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)$ is a proper subset of G, and consider an element x of

$$G \setminus (F \cup N_G(X_1) \cup \cdots \cup N_G(X_k)).$$

Clearly, the subgroup $\langle x\rangle$ is not almost normal in G and in particular x has finite order. If $Z(G)$ contains an element a of infinite order, the subgroup $\langle x, a\rangle$ is almost normal in G and $\langle x\rangle$ is characteristic in $\langle x, a\rangle$, a contradiction. Thus $Z(G)$ must be periodic. Suppose now that $Z(G)$ contains infinite subgroups A and B such that

$$\langle A, B\rangle \cap \langle x\rangle = A \cap B = \{1\}.$$

Clearly, the infinite subgroups $\langle x, A\rangle$ and $\langle x, B\rangle$ are almost normal in G and so $\langle x\rangle = \langle x, A\rangle \cap \langle x, B\rangle$ is likewise almost normal. This contradiction shows that $Z(G)$ satisfies the minimal condition on subgroups. Thus also the abelian subgroup $\langle x, Z(G)\rangle$ satisfies the minimal condition, and in particular it contains a finite characteristic subgroup E such that $x \in E$. On the other hand, $\langle x, Z(G) \rangle$ is almost normal in G and hence the normal closure E^G is finite, contradicting the choice of x. Therefore

$$G = F \cup N_G(X_1) \cup \cdots \cup N_G(X_k),$$

so that $G = F$ is an FC-group by Lemma 1, and hence $G/Z(G)$ is finite by Theorem A.

Our next result deals with the case of periodic-by-nilpotent groups.

Proposition 11. *Let G be a non-periodic group in which all but finitely many normalizers of infinite subgroups have finite index. If the n-th term $\gamma_n(G)$ of the lower central series of G is periodic for some positive integer n, then the factor group $G/Z(G)$ is finite.*

Proof. As G' is polycyclic-by-finite by Lemma 9, the subgroup $\gamma_n(G)$ is finite and we may consider the largest positive integer m such that $\gamma_m(G)$ is infinite. Then $\gamma_{m+1}(G)$ is finite and $\gamma_m(G)/\gamma_{m+1}(G)$ lies in the centre of $G/\gamma_{m+1}(G)$. In particular, $G/\gamma_{m+1}(G)$ has infinite centre and so it is central-by-finite by Lemma 10. It follows that G' is finite, so that G is an FC-group and hence $G/Z(G)$ is finite by Lemma 2.

Proof of Theorem C. By Lemma 8, G has a characteristic subgroup M of finite index such that every infinite subgroup of M either is subnormal or almost normal in G, and clearly M contains an infinite periodic subgroup H. In order to prove that the largest periodic normal subgroup T of G is infinite, it can obviously be assumed that H is not subnormal in G, so that there exists a normal subgroup K of G of finite index normalizing H. Then $(H \cap K)^G$ is contained in T, and hence T is infinite. Therefore all but finitely many normalizers of subgroups of G/T have finite index and hence G/T is central-by-finite by Theorem A. In particular, $G'T/T$ is finite, so that G' is periodic and it follows from Proposition 11 that $G/Z(G)$ is finite.

Corollary 12. *Let G be a group in which all but finitely many normalizers of infinite subgroups have finite index. If the hypercentre $\overline{Z}(G)$ is infinite, then either G is a Cernikov group or $G/Z(G)$ is finite.*

Proof. Assume first that G is periodic. As $\overline{Z}(G)$ is infinite, the factor group $G/\overline{Z}(G)$ is central-by-finite by Theorem A, so that G is locally finite and the statement follows from Theorem B. Suppose now that G is not periodic, and let μ be the smallest ordinal such that $Z_{\mu}(G)$ is infinite. If μ is not a limit, the subgroup $Z_{\mu-1}(G)$ is finite and $G/Z_{\mu-1}(G)$ has infinite centre; then $G/Z_{\mu-1}(G)$ is central-by-finite by Lemma 10, so that G' is periodic and $G/Z(G)$ is finite by Proposition 11. On the other hand, if μ is a limit ordinal, then the subgroup $Z_{\mu}(G)$ is periodic and hence it follows from Theorem C that G is central-by-finite.

Corollary 13. *Let G be a non-periodic group in which all but finitely many normalizers of infinite subgroups have finite index. If $\gamma_n(G)/\gamma_n(G) \cap \overline{Z}(G)$ is periodic for some positive integer n, then the factor group $G/Z(G)$ is finite.*

Proof. As $\gamma_n(G)/\gamma_n(G) \cap \overline{Z}(G)$ is periodic, either the hypercentre of G is infinite or the subgroup $\gamma_n(G)$ is periodic. Thus the statement follows from Corollary 12 and Proposition 11.

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