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ON THE FIRST DERIVED LIMITS

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Abstract

In this note, another characteristic feature of the first derived limits is discussed.

1. Introduction and Result

Let $\{G_n\} = (G_n, g_n^{n+1}, \mathbb{N})$ be an inverse tower of (possibly non-abelian) groups G_n and homomorphisms $g_n^{n+1} : G_{n+1} \to G_n$ indexed by the set of all nonnegative integers $\mathbb{N}$. We consider a left action of $\prod G_n$ on $\prod G_n$ by the formula

$$(\ldots, s_n, s_{n+1}, \ldots) \circ (\ldots, t_n, t_{n+1}, \ldots) = (\ldots, s_n t_n g_n^{n+1}(s_{n+1}^{-1}), \ldots).$$

We define the first derived limit, $\lim^1\{G_n\}$, of an inverse tower as the set of orbits of $\prod G_n$ under this action in the sense of Bousfield-Kan [1, p. 251]. We can also define the inverse limit, $\lim\{G_n\}$, of the inverse tower $\{G_n\}$ by using this action:

$$\lim\{G_n\} = \left\{ g \in \prod G_n \mid g \circ * = * \right\}.$$

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Moreover, the set $\lim^1\{G_n\} = \prod G_n/\sim$ can be viewed as the quotient set of the direct product $\prod G_n$ by an equivalence relation $\sim$ defined as follows: For $x = (\dots, x_n, \dots)$, $y = (\dots, y_n, \dots) \in \prod G_n$, one has $x \sim y$ if and only if there exists an element $s = (\dots, s_n, \dots) \in \prod G_n$ such that $y = s \circ x$.

Let Γ^n, $n \geq 0$ be the set of all increasing sequences $\bar{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_n)$, $\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n$, $\gamma_i \in \Gamma$ and let $\bar{\gamma}_j \in \Gamma^{n-1}$, $0 \leq j \leq n$ be obtained from $\bar{\gamma} \in \Gamma^n$ by deleting the jth factor γ_j, i.e., $\bar{\gamma}_j = (\gamma_0, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n)$. And for each $\bar{\gamma} \in \Gamma^n$, we associate an abelian group $A_{\bar{\gamma}}$ by the abelian group A_{γ_0} of the first index γ_0 in the category of abelian groups, i.e., $A_{\bar{\gamma}} = A_{\gamma_0}$.

Let $\{A_\gamma\} = (A_\gamma, a_{\gamma\gamma'}, \Gamma)$ be an inverse system of abelian groups A_γ and group homomorphisms $a_{\gamma\gamma'} : A_{\gamma'} \to A_\gamma$, $\gamma \leq \gamma'$ over the directed set Γ. We define an n-cochain group $C^n(\{A_\gamma\})$, $n \geq 0$ of $\mathfrak{A}$ by

$$C^n(\{A_\gamma\}) = \prod_{\bar{\gamma} \in \Gamma^n} A_{\bar{\gamma}},\ n \geq 0,$$

where $A_{\bar{\gamma}} = A_{\gamma_0}$ as just mentioned above.

Let $pr_{\bar{\gamma}} : C^n(\{A_\gamma\}) \to A_{\bar{\gamma}}$ be a projection. If y is an element of $C^n(\{A_\gamma\})$, then we denote the element $y_{\bar{\gamma}}$ of $A_{\bar{\gamma}}$ by $y_{\bar{\gamma}} = pr_{\bar{\gamma}}(y)$. The coboundary operator $\delta^n : C^{n-1}(\{A_\gamma\}) \to C^n(\{A_\gamma\})$, $n \geq 1$ is defined by

$$(\delta^n y)_{\bar{\gamma}} = a_{\gamma_0\gamma_1}(y_{\bar{\gamma}_0}) + \sum_{j=1}^{n} (-1)^j y_{\bar{\gamma}_j},$$

where $y \in C^{n-1}(\{A_\gamma\})$. For $n = 0$, if we put $\delta^0 = 0 : 0 \to C^0(\{A_\gamma\})$, then we have a cochain complex

$$(C^*(\{A_\gamma\}),\,\delta) : 0 \to C^0(\{A_\gamma\}) \overset{\delta^1}{\to} C^1(\{A_\gamma\}) \to \cdots$$

$$\to C^{n-1}(\{A_\gamma\}) \overset{\delta^n}{\to} C^n(\{A_\gamma\}) \to \cdots.$$

We now define the nth *derived limit* (see [4]) denoted by $H^n(\{A_\gamma\})$ of the inverse system $\{A_\gamma\} = (A_\gamma, a_{\gamma\gamma'}, \Gamma)$ of abelian groups by the cohomology group of the above cochain complex $(C^*(\{A_\gamma\}),\,\delta)$. That is to say

$$H^n(\{A_\gamma\}) = \ker(\delta^{n+1})/\mathrm{im}(\delta^n).$$

Let $\{D_\lambda\} = (D_\lambda,\, d_{\lambda\lambda'},\, \Lambda)$ and $\{F_\gamma\} = (E_\gamma,\, e_{\gamma\gamma'},\, \Gamma)$ be inverse systems in any category $\mathfrak{C}$. We say that $s = \{\varphi,\, s_\gamma : \gamma \in \Gamma\} : \{D_\lambda\} \to \{F_\gamma\}$ is a *rigid system map* from $\{D_\lambda\}$ to $\{F_\gamma\}$ if $\varphi : \Gamma \to \Lambda$ is an increasing function, $s_\gamma : D_{\varphi(\gamma)} \to E_\gamma$, $\gamma \in \Gamma$ is a morphism in the category $\mathfrak{C}$, and for any $\gamma \le \gamma'$ in Γ the following diagram

$$\begin{array}{ccc} D_{\varphi(\gamma)} & \xleftarrow{d_{\varphi(\gamma)\varphi(\gamma')}} & D_{\varphi(\gamma')} \\ s_\gamma \downarrow & & s_{\gamma'} \downarrow \\ E_\gamma & \xleftarrow{e_{\gamma\gamma'}} & E_{\gamma'} \end{array}$$

is commutative. Moreover, we can make a category inv-$\mathfrak{C}$ of inverse systems in $\mathfrak{C}$ and rigid system maps. The rigid system map is called a *level system map* provided $\Gamma = \Lambda$ and φ is an identity map id_Λ on Λ. In this note, we are interested in the case of level system maps indexed by the set of all nonnegative integers $\mathbb{N}$.

Let X be a connected *CW*-space and let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. We denote $Y^{(n)}$ the nth Postnikov approximation of Y. By putting $G_n = [X, \Omega Y^{(n)}]$, we obtain an inverse tower $\{G_n\} = (G_n, g_n^{n+1}, \mathbb{N})$ of groups. Let us write $G_k^{(n)} = \mathrm{im}(g_k^n : G_n \to G_k)$, where $g_k^n = g_k^{k+1} \circ g_{k+1}^{k+2} \circ \cdots \circ g_{n-1}^n$. Then we have an epimorphism

$g_k^n : G_n \to G_k^{(n)}$ and a surjective level system map $\{g_k^n\} : \{G_n\} \to \{G_k^{(n)}\}$ between inverse towers of groups.

Let $\mathbb{Q}$ be the set of all rational numbers. Then we have

Theorem. *If X has a homotopy type of a suspension or if Y has a homotopy type of a loop space with* $\pi_{k+1}(Y) \otimes \mathbb{Q} = 0$, *then* $H^1(\{G_{k+1}^{(n)}\}) \cong H^1(\{G_k^{(n)}\})$.

2. Proof of Theorem

We need to find the basic roles of lim and $\lim^1$ functors from the following lemmas:

Lemma 1. *Let* $s = \{id_{\mathbb{N}}, s_n : n \in \mathbb{N}\} : \{U_n\} \to \{V_n\}$ *and* $t = \{id_{\mathbb{N}}, t_n : n \in \mathbb{N}\} : \{V_n\} \to \{W_n\}$ *be level system maps of inverse towers of groups and let the sequence*

$$0 \to U_n \xrightarrow{s_n} V_n \xrightarrow{t_n} W_n \to 0$$

be exact for each $n \in \mathbb{N}$. *Then there is a natural exact sequence of pointed sets*

$$0 \to \lim\{U_n\} \xrightarrow{s_*} \lim\{V_n\} \xrightarrow{t_*} \lim\{W_n\}$$

$$\xrightarrow{\delta} \lim{}^1\{U_n\} \xrightarrow{s_*} \lim{}^1\{V_n\} \xrightarrow{t_*} \lim{}^1\{W_n\} \to 0,$$

where δ *is a connecting function.*

Proof. See Proposition 2.3 in [1, p. 252].

Lemma 2. *Let* $\{G_n\} = (G_n, g_n^{n+1}, \mathbb{N})$ *be an inverse tower with each* G_n *finite. Then* $\lim^1\{G_n\}$ *is zero.*

Proof. See [2].

We can see the relationship (see [3]) between the derived limits as follows:

Lemma 3. *If* $\{A_n\} = (A_n, a_n^{n+1}, \mathbb{N})$ *is an inverse tower of abelian groups*, *then* $\lim^0\{A_n\} \cong H^0(\{A_n\})$ *and* $\lim^1\{A_n\} \cong H^1(\{A_n\})$.

We now see that the fibration

$$\cdots \to K(\pi_{k+1}(Y), k) \to \Omega Y^{(k+1)} \xrightarrow{p_k} \Omega Y^{(k)}$$

induces an exact sequence of groups

$$\cdots \to [X, K(\pi_{k+1}(Y), k)] \to G_{k+1} = [X, \Omega Y^{(k+1)}] \xrightarrow{p_{k*}} G_k = [X, \Omega Y^{(k)}],$$

where $K(\pi_{k+1}(Y), k)$ is the Eilenberg-MacLane space of type $(\pi_{k+1}(Y), k)$. We thus have the short exact sequence

$$0 \to [X, K(\pi_{k+1}(Y), k)]/\ker(p_{k*}) \rightarrowtail G_{k+1}^{(n)} \twoheadrightarrow G_k^{(n)} \to 0.$$

By putting $F_n = [X, K(\pi_{k+1}(Y), k)]/\ker(p_{k*})$, we obtain the short exact sequence of inverse towers

$$0 \to \{F_n\} \rightarrowtail \{G_{k+1}^{(n)}\} \twoheadrightarrow \{G_k^{(n)}\} \to 0.$$

We note that $[X, K(\pi_{k+1}(Y), k)] \cong H^k(X; \pi_{k+1}(Y))$ and this group is finite by hypothesis. We also note that $\{F_n\}$ has a trivial $\lim^1$-term because $\{F_n\}$ is an inverse tower of finite groups. By applying the six-term $\lim$-$\lim^1$ exact sequence, we can get the following exact sequence

$$0 = \lim^1\{F_n\} \to \lim^1\{G_{k+1}^{(n)}\} \xrightarrow{\cong} \lim^1\{G_k^{(n)}\} \to 0.$$

Since $G_{k+1}^{(n)}$ and $G_k^{(n)}$ are abelian by the suspension hypothesis on X or the loop space hypothesis on Y, by Lemma 3, we obtain

$$H^1(\{G_{k+1}^{(n)}\}) \cong H^1(\{G_k^{(n)}\}).$$

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