Far East J. Math. Sci. (FJMS) **25**(1) (2007), 145-158
This paper is available online at http://www.pphmj.com

A NOTE ON BINARY RELATIONS BETWEEN NON-EMPTY FINITE SETS

WI-GON PARK

(Received November 9, 2006)

Abstract

We introduce some notations for finite relations. Using these, we define an "odd composition" and "even composition" which refine the ordinary composition of finite relations. We then present some consequences including a ring with binary operations of the symmetric difference and odd composition.

1. Preliminaries and Conventions

In this note, we use the typical ideas and notation of set theory described in the literature [2, 4, 5].

A binary relation is a set of ordered pairs. A function f is the binary relation f such that if $(x, y) \in f$ and $(x, z) \in f$, this implies $y = z$. If f is a function, the unique y such that $(x, y) \in f$ is the value of f at x. The Cartesian product of two sets X and Y is the set of all ordered pairs such that $X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$. If $X = \varnothing$ or $Y = \varnothing$, then $X \times Y$ is the empty set $\varnothing$.

We adopt the following conventions. A binary relation is called simply a *relation*. Let R be a relation (including functions). Then the notation xRy is equivalent to $(x, y) \in R$. If f is a function, then the notation $(x)f = y$ is used for the value y of f at x. Let R and S be relations. Then

2000 Mathematics Subject Classification: 03E20.

Keywords and phrases: non-empty finite set, composition relation, symmetric difference.

the composition (or *ordinary* composition) of R and S, denoted by $R \circ S$, is the relation such that (x, y) belongs to $R \circ S$ if xRz, zSy, and $z \in (\mathrm{ran}(R) \cap \mathrm{dom}(S))$. Let f and g be functions such that $\mathrm{ran}(f) \subseteq \mathrm{dom}(g)$. Then the composition (or *ordinary* composition) of f and g, denoted by $f \circ g$, is the function with $\mathrm{dom}(f \circ g) = \mathrm{dom}(f)$ such that $(x)(f \circ g) = ((x)f)g$ for every $x \in \mathrm{dom}(f)$.

Let A and B be sets. Then we adopt the following conventions. A relation R from A to B is a subset of $A \times B$. The empty relation $\varnothing$ is denoted by the symbol O. A function f from A to B is the relation f from A to B such that $\mathrm{dom}(f) = A$ and if $(x, y) \in f$ and $(x, z) \in f$, this implies $y = z$. A relation on A is a relation from A to A. A function on A is a function from A to A.

Let A and B be sets. Then the symmetric difference of A and B, denoted by $A + B$, is the set such that $A + B = (A - B) \cup (B - A)$. If R is a relation from A to B, then the inverse of R, denoted by R^{-1}, is the relation from B to A such that $R^{-1} = \{(y, x) | (x, y) \in R\}$. Let $R, S \subseteq A \times B$. Then $(R^{-1})^{-1} = R$; $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$; $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$; $(R - S)^{-1} = R^{-1} - S^{-1}$; and $(R + S)^{-1} = R^{-1} + S^{-1}$.

To reduce the parentheses in expressions with a sequence of symbols, we adopt the usual conventions. In this case the symbols "$\in$, $\notin$, $=$, $\neq$, $\subseteq$" are dominant. However, the two symbols "$\rightarrow$, $\leftrightarrow$" are more dominant symbols.

2. Introducing Some Notation for Finite Relations

We introduce the following notation for finite relations.

Definition 2.1. Let A and B be non-empty finite sets, with $R \subseteq A \times B$, $a \in A$ and $b \in B$.

• The symbol aR is the set such that "$y \in aR$ if $(a, y) \in R$ for some $y \in B$" or "$aR = \varnothing$ if $(a, y) \notin R$ for every $y \in B$".

• The symbol Rb is the set such that "$x \in Rb$ if $(x, b) \in R$ for some $x \in A$" or "$Rb = \varnothing$ if $(x, b) \notin R$ for every $x \in A$".

Let A and B be non-empty finite sets and let f be a function from A to B. Then by the properties of a function and Definition 2.1, af is the singleton $\{(a)f\}$ for every $a \in A$. If $b \in B$ and $b \notin \operatorname{ran}(f)$, then fb is an empty set $\varnothing$.

We next present some properties of finite relations relating to Definition 2.1.

Proposition 2.2. *Let A and B be non-empty finite sets, with $R \subseteq A \times B$, $a \in A$ and $b \in B$. Then*:

1. *$y \in aR$ if and only if aRy, and $x \in Rb$ if and only if xRb.*

2. *$aR = R^{-1}a$ and $Rb = bR^{-1}$.*

Proof. (1) Let $y \in aR$. From Definition 2.1, it follows that $y \in B$ and $(a, y) \in R$. Then $y \in aR \to aRy$. Conversely, let aRy. This implies that $(a, y) \in R$ and $y \in B$. By Definition 2.1, $y \in aR$. Then $aRy \to y \in aR$. Thus, we have $y \in aR \leftrightarrow aRy$. In a similar fashion, we can also prove $x \in Rb \leftrightarrow xRb$.

(2) Let $y \in aR$. From (1), it follows that $y \in aR \leftrightarrow aRy \leftrightarrow yR^{-1}a \leftrightarrow y \in R^{-1}a$. This implies that $aR = R^{-1}a$. Similarly, let $x \in Rb$. From (1), it follows that $x \in Rb \leftrightarrow xRb \leftrightarrow bR^{-1}x \leftrightarrow x \in bR^{-1}$. This implies that $Rb = bR^{-1}$. □

Let p and q be two statements. Then, by "*exclusive* p or q", we mean that only one of the two statements is true, but not both.

Proposition 2.3. *Let A and B be non-empty finite sets. Let $R, S \subseteq A \times B$, with $a \in A$ and $b \in B$. Then $R = S$ if and only if "$aR = aS$ for every $a \in A$" or "$Rb = Sb$ for every $b \in B$".*

Proof. First we show that $R = S$ if and only if $aR = aS$ for every $a \in A$. Let $R = S$. Suppose that $a'R \neq a'S$ for some $a' \in A$. Then there is some element $b \in B$ such that exclusive $b \in a'R$ or $b \in a'S$. Then, by Proposition 2.2(1), exclusive $(a', b) \in R$ or $(a', b) \in S$. This is a contradiction because of the hypothesis $R = S$. Hence, we can say that if $R = S$, then $aR = aS$ for every $a \in A$.

To prove the converse, let $aR = aS$ for every $a \in A$. Suppose that $R \neq S$. Then there are some elements $x \in A$ and $y \in B$ such that exclusive $(x, y) \in R$ or $(x, y) \in S$. Then, by Proposition 2.2(1), exclusive $y \in xR$ or $y \in xS$. This is a contradiction because, by the hypothesis, $xR = xS$. Hence, we can say that if $aR = aS$ for every $a \in A$, then $R = S$.

Thus, $R = S$ if and only if $aR = aS$ for every $a \in A$. In a similar fashion, we can also prove $R = S$ if and only if $Rb = Sb$ for every $b \in B$. □

Proposition 2.4. *Let A and B be non-empty finite sets. Let $R, S \subseteq A \times B$ with $a \in A$ and $b \in B$. Then*:

1. $a(R \cup S) = aR \cup aS$ *and* $(R \cup S)b = Rb \cup Sb$.

2. $a(R \cap S) = aR \cap aS$ *and* $(R \cap S)b = Rb \cap Sb$.

3. $a(R - S) = aR - aS$ *and* $(R - S)b = Rb - Sb$.

4. $a(R + S) = aR + aS$ *and* $(R + S)b = Rb + Sb$.

Proof. (1) First we show that $a(R \cup S) = aR \cup aS$.

To prove $a(R \cup S) \subseteq aR \cup aS$, let $y \in a(R \cup S)$. By Proposition 2.2(1), $(a, y) \in R \cup S$ and thus $(a, y) \in R$ or $(a, y) \in S$. By Proposition 2.2(1), $y \in aR$ or $y \in aS$ and thus $y \in aR \cup aS$. Hence, $a(R \cup S) \subseteq aR \cup aS$.

To prove $aR \cup aS \subseteq a(R \cup S)$, let $y \in aR \cup aS$. Then $y \in aR$ or $y \in aS$. By Proposition 2.2(1), $(a, y) \in R$ or $(a, y) \in S$ and thus $(a, y) \in R \cup S$. Then, by Proposition 2.2(1), $y \in a(R \cup S)$ and hence $aR \cup aS \subseteq a(R \cup S)$.

Thus, we have $a(R \cup S) = aR \cup aS$.

On the other hand, from Proposition 2.2(2) and the result above, it follows that $(R \cup S)b = b(R \cup S)^{-1} = b(R^{-1} \cup S^{-1}) = bR^{-1} \cup bS^{-1} = Rb \cup Sb$. Thus, we have $(R \cup S)b = Rb \cup Sb$.

In a similar fashion, we can also prove (2) and (3).

(4) By definition of the symmetric difference, $R + S = (R - S) \cup (S - R)$. Then, from (1), (3), and the definition of +, it follows that

$$\begin{aligned} a(R + S) &= a((R - S) \cup (S - R)) = a(R - S) \cup a(S - R) \\ &= (aR - aS) \cup (aS - aR) = aR + aS. \end{aligned}$$

Thus, we have $a(R + S) = aR + aS$.

On the other hand, from Proposition 2.2(2) and the result above, it follows that

$$\begin{aligned} (R + S)b &= b(R + S)^{-1} = b(R^{-1} + S^{-1}) \\ &= bR^{-1} + bS^{-1} = Rb + Sb. \end{aligned}$$

Thus, we have $(R + S)b = Rb + Sb$. □

3. Odd and Even Composition of Finite Relations

Let Q be a finite set. Then $|Q|$ is the number of elements of Q. We define the following notation: $|Q| = odd$ means that $|Q|$ is an odd number. $|Q| = even$ means that $Q \neq \varnothing$ and $|Q|$ is an even number.

We define the odd composition and even composition of finite relations. We also redefine the ordinary composition of finite relations as a normal composition within the context of this paper.

Definition 3.1. Let A, B, and C be non-empty finite sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$, with $a \in A$ and $c \in C$. Let $Q = aR \cap Sc$.

• The normal composition of R and S, denoted by $R \circ S$, is defined as follows. For each $a \in A$ and $c \in C$, the ordered pair (a, c) belongs to $R \circ S$ if $Q \neq \varnothing$.

• The odd composition of R and S, denoted by $R \cdot S$, is defined as follows. For each $a \in A$ and $c \in C$, the ordered pair (a, c) belongs to $R \cdot S$ if $|Q| = odd$.

• The even composition of R and S, denoted by $R \odot S$, is defined as follows. For each $a \in A$ and $c \in C$, the ordered pair (a, c) belongs to $R \odot S$ if $|Q| = even$.

It should be noted that the normal composition is equivalent to the ordinary composition of finite relations. For this reason, the symbol "$\circ$" is also used for the normal composition.

From a graphical point of view, each composition of finite relations means the following. If $a(R \circ S)c$, then there is at least one path between the two points $a \in A$ and $c \in C$ through B; if $a(R \cdot S)c$, then there is an odd number of paths between the two points $a \in A$ and $c \in C$ through B; and if $a(R \odot S)c$, then there is an even number of paths between the two points $a \in A$ and $c \in C$ through B.

We next present some properties of finite relations relating to Definition 3.1.

Proposition 3.2. *Let A, B, and C be non-empty finite sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$, with $a \in A$ and $c \in C$. Then $(R \cdot S) \bigcup (R \odot S) = R \circ S$ and $(R \cdot S) \bigcap (R \odot S) = O$.*

Proof. By Definition 3.1, clearly, $R \cdot S \subseteq R \circ S$ and $R \odot S \subseteq R \circ S$. Then $(R \cdot S) \bigcup (R \odot S) \subseteq R \circ S$.

To prove $R \circ S \subseteq (R \cdot S) \bigcup (R \odot S)$, let $(a, c) \in R \circ S$. Then, by the normal composition, $aR \bigcap Sc \neq \varnothing$. This implies that $|aR \bigcap Sc| = odd$ or $|aR \bigcap Sc| = even$. By the odd composition and even composition, $(a, c) \in R \cdot S$ or $(a, c) \in R \odot S$. Hence, $R \circ S \subseteq (R \cdot S) \bigcup (R \odot S)$.

Thus, we have $(R \cdot S) \bigcup (R \odot S) = R \circ S$.

To prove $(R \cdot S) \cap (R \odot S) = O$, suppose that $(a, c) \in (R \cdot S) \cap (R \odot S)$ for some $a \in A$ and $c \in C$. Then $(a, c) \in R \cdot S$ and $(a, c) \in R \odot S$. By the odd composition and even composition, $|aR \cap Sc| = odd$ and $|aR \cap Sc| = even$. This is a contradiction and hence, $(a, c) \notin (R \cdot S) \cap (R \odot S)$ for every $a \in A$ and $c \in C$. Thus, we have $(R \cdot S) \cap (R \odot S) = O$. □

Proposition 3.3. *Let A, B, and C be non-empty finite sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$, with $a \in A$ and $c \in C$. Then*:

1. *Let $R = O$ or $S = O$. Then $R \cdot S = R \odot S = O$.*

2. *Let f be a function from A to B. Then $f \cdot S = f \circ S$.*

3. *Let g be an injective function from B to C. Then $R \cdot g = R \circ g$.*

Proof. (1) It is well known that $R \circ S = O$ if $R = O$ or $S = O$. Then, by Proposition 3.2, $(R \cdot S) \cup (R \odot S) = O$. This implies that $R \cdot S = R \odot S = O$.

(2) First to show $f \odot S = O$, suppose that $(a, c) \in f \odot S$ for some $a \in A$ and $c \in C$. By the even composition, $|af \cap Sc| = even$. This is a contradiction because af is a singleton. Thus, $f \odot S = O$. Then, by Proposition 3.2, we have $f \cdot S = f \circ S$.

(3) By Proposition 3.2, clearly, $R \cdot g \subseteq R \circ g$. To prove $R \circ g \subseteq R \cdot g$, let $(a, c) \in R \circ g$. By the normal composition and injectivity of g, we have $aR \cap gc = gc$ and $|gc| = 1$. By the odd composition, this implies that $(a, c) \in R \cdot g$. Then $R \circ g \subseteq R \cdot g$. Thus, we have $R \cdot g = R \circ g$. □

Proposition 3.4. *Let A, B, and C be non-empty finite sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$, with $a \in A$ and $c \in C$. Then $(R \cdot S)^{-1} = S^{-1} \cdot R^{-1}$ and $(R \odot S)^{-1} = S^{-1} \odot R^{-1}$.*

Proof. Let $(c, a) \in (R \cdot S)^{-1}$. Then, by the odd composition and Proposition 2.2(2), we have

$$\begin{aligned}(c, a) \in (R \cdot S)^{-1} &\leftrightarrow a(R \cdot S)c \leftrightarrow |\, aR \cap Sc \,| = odd \\ &\leftrightarrow |R^{-1}a \cap cS^{-1}| = odd \\ &\leftrightarrow |cS^{-1} \cap R^{-1}a| = odd \\ &\leftrightarrow (c, a) \in S^{-1} \cdot R^{-1}.\end{aligned}$$

This implies that $(R \cdot S)^{-1} = S^{-1} \cdot R^{-1}$.

In a similar fashion, we can also prove $(R \odot S)^{-1} = S^{-1} \odot R^{-1}$. □

4. Distributivity and Associativity of the Odd Composition

The ordinary composition (normal composition) "$\circ$" is distributive over the set union "$\bigcup$". Similarly, we present the distributivity of the odd composition "$\cdot$" over the symmetric difference "+".

Lemma 4.1. *Let A and B be finite sets. Then* "$|\, A + B \,| = odd$" *if and only if* "*exclusive* $|\, A \,| = odd$ *or* $|\, B \,| = odd$".

Proof. Let $|\, A + B \,| = t$, $|\, A \,| = m$, $|\, B \,| = n$, and $|\, A \cap B \,| = k$, where t, m, n, and k are non-negative integers. Then $t = m + n - 2 \times k$, where +, –, and × are the usual addition, subtraction, and multiplication on numbers, respectively. By considering the equation above, the statement is proved. □

Proposition 4.2. *Let A, B, and C be non-empty finite sets. Then*:

1. *Let* $R \subseteq A \times B$ *and* $S, T \subseteq B \times C$. *Then*

$$R \cdot (S + T) = (R \cdot S) + (R \cdot T).$$

2. *Let* $R, S \subseteq A \times B$ *and* $T \subseteq B \times C$. *Then*

$$(R + S) \cdot T = (R \cdot T) + (S \cdot T).$$

Proof. (1) The following equality is well known: $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$, where X, Y, and Z are sets.

Let $(a, c) \in R \cdot (S + T)$, where $a \in A$ and $c \in C$. Then by the odd composition and Proposition 2.4(4), we have

$$(a, c) \in R \cdot (S + T) \leftrightarrow |\, aR \cap (S + T)c \,| = odd$$
$$\leftrightarrow |\, aR \cap (Sc + Tc) \,| = odd.$$

By the equality above,

$$|\, aR \cap (Sc + Tc) \,| = odd \leftrightarrow |\, (aR \cap Sc) + (aR \cap Tc) \,| = odd.$$

By Lemma 4.1,

$$|\, (aR \cap Sc) + (aR \cap Tc) \,| = odd \leftrightarrow \text{exclusive } |\, aR \cap Sc \,| = odd$$

or $|\, aR \cap Tc \,| = odd$. By the odd composition, exclusive $|\, aR \cap Sc \,| = odd$ or $|\, aR \cap Tc \,| = odd \leftrightarrow$ exclusive $(a, c) \in R \cdot S$ or $(a, c) \in R \cdot T \leftrightarrow (a, c) \in (R \cdot S) + (R \cdot T)$. This implies that

$$R \cdot (S + T) = (R \cdot S) + (R \cdot T).$$

In a similar fashion, we can also prove (2). □

The ordinary composition (normal composition) "$\circ$" is associative. Similarly, we present the associativity of the odd composition "$\cdot$".

Lemma 4.3. *Let A, B, C, and D be non-empty finite sets. Let $R \subseteq A \times B$, $M \subseteq B \times C$, and $T \subseteq C \times D$. Let M be a singleton such that $M = \{(b, c)\}$, where $b \in B$ and $c \in C$. Then*:

1. *If $a(R \cdot M) \neq \varnothing$ for some $a \in A$, then $a(R \cdot M) = \{c\}$ and $aR \cap Mc = \{b\}$.*

2. *If $(M \cdot T)d \neq \varnothing$ for some $d \in D$, then $(M \cdot T)d = \{b\}$ and $bM \cap Td = \{c\}$.*

Proof. (1) Let $a(R \cdot M) = Q$ and $Q \neq \varnothing$. Let q be an arbitrary element of Q. Then $a(R \cdot M)q$. By the odd composition, $aR \cap Mq \neq \varnothing$ and hence $Mq \neq \varnothing$. By considering the singleton M and the fact that $Mc = \{b\} \neq \varnothing$, we have $q = c$. This implies $Q = \{c\}$ because q is an arbitrary element of Q. Thus, we have $a(R \cdot M) = \{c\}$. On the other hand, $a(R \cdot M) = \{c\}$ implies $a(R \cdot M)c$. Then, from the odd composition and the fact that $Mc = \{b\}$, it follows that $aR \cap Mc = \{b\}$.

(2) Let $(M \cdot T)d \neq \varnothing$. Then, $(M \cdot T)d \neq \varnothing \leftrightarrow d(M \cdot T)^{-1} \neq \varnothing \leftrightarrow d(T^{-1} \cdot M^{-1}) \neq \varnothing$. Clearly, M^{-1} is the singleton $\{(c, b)\}$. Then, by equation (1), we have $d(T^{-1} \cdot M^{-1}) = \{b\}$ and $dT^{-1} \cap M^{-1}b = \{c\}$. These are equivalent to $(M \cdot T)d = \{b\}$ and $bM \cap Td = \{c\}$, respectively. □

Lemma 4.4. *Let A, B, C, and D be non-empty finite sets. Let $R \subseteq A \times B$, $M \subseteq B \times C$, and $T \subseteq C \times D$. Let M be a singleton such that $M = \{(b, c)\}$, where $b \in B$ and $c \in C$. Then $(R \cdot M) \cdot T = R \cdot (M \cdot T)$.*

Proof. We show that $(R \cdot M) \cdot T \subseteq R \cdot (M \cdot T)$ and $R \cdot (M \cdot T) \subseteq (R \cdot M) \cdot T$.

Let $(a, d) \in (R \cdot M) \cdot T$, where $a \in A$ and $d \in D$. By the odd composition, $a(R \cdot M) \cap Td \neq \varnothing$ and hence $a(R \cdot M) \neq \varnothing$. By Lemma 4.3 (1), we have $a(R \cdot M) = \{c\}$ and $aR \cap Mc = \{b\}$. This implies that $\{c\} \subseteq Td$ and $\{b\} \subseteq aR$. By the fact that $bM = \{c\}$, we have $bM \cap Td = \{c\}$. Then, by the odd composition, $(b, d) \in M \cdot T$ and hence $(M \cdot T)d \neq \varnothing$. By Lemma 4.3 (2), we have $(M \cdot T)d = \{b\}$. On the other hand, by the fact that $\{b\} \subseteq aR$, we have $aR \cap (M \cdot T)d = \{b\}$. By the odd composition, $(a, d) \in R \cdot (M \cdot T)$. Hence, $(R \cdot M) \cdot T \subseteq R \cdot (M \cdot T)$.

To prove the converse, let $(a, d) \in R \cdot (M \cdot T)$, where $a \in A$ and $d \in D$. By the odd composition, $aR \cap (M \cdot T)d \neq \varnothing$ and hence $(M \cdot T)d \neq \varnothing$. By Lemma 4.3 (2), we have $(M \cdot T)d = \{b\}$ and $bM \cap Td = \{c\}$. This implies that $\{b\} \subseteq aR$ and $\{c\} \subseteq Td$. By the fact that $Mc = \{b\}$, we have $aR \cap Mc = \{b\}$. Then, by the odd composition, $(a, c) \in R \cdot M$ and hence $a(R \cdot M) \neq \varnothing$. By Lemma 4.3 (1), we have $a(R \cdot M) = \{c\}$. On the other hand, by the fact that $\{c\} \subseteq Td$, we have $a(R \cdot M) \cap Td = \{c\}$. By the odd composition, $(a, d) \in (R \cdot M) \cdot T$. Hence $R \cdot (M \cdot T) \subseteq (R \cdot M) \cdot T$.

Thus, we have $(R \cdot M) \cdot T = R \cdot (M \cdot T)$. □

Proposition 4.5. *Let A, B, C, and D be non-empty finite sets. Let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$. Then $(R \cdot S) \cdot T = R \cdot (S \cdot T)$.*

Proof. Without loss of generality, we can write the following: $S = S_1 + S_2 + \cdots + S_n$, where S_i $(1 \leq i \leq n)$ is a singleton such that $S_i \subseteq B \times C$ and n is a positive integer.

By the distributivity of the odd composition,

$$R \cdot S = R \cdot (S_1 + S_2 + \cdots + S_n) = R \cdot S_1 + R \cdot S_2 + \cdots + R \cdot S_n.$$

Then, by the distributivity of the odd composition,

$$(R \cdot S) \cdot T = (R \cdot S_1) \cdot T + (R \cdot S_2) \cdot T + \cdots + (R \cdot S_n) \cdot T.$$

By Lemma 4.4,

$$(R \cdot S) \cdot T = R \cdot (S_1 \cdot T) + R \cdot (S_2 \cdot T) + \cdots + R \cdot (S_n \cdot T).$$

Then, by the distributivity of the odd composition,

$$\begin{aligned}(R \cdot S) \cdot T &= R \cdot (S_1 \cdot T + S_2 \cdot T + \cdots + S_n \cdot T) \\ &= R \cdot ((S_1 + S_2 + \cdots + S_n) \cdot T) \\ &= R \cdot (S \cdot T).\end{aligned}$$ □

In contrast to the odd composition, in general, the even composition "$\odot$" is not associative. It can be checked by the following counterexample.

Example 4.6 (Counterexample)**.** Let A, B, C, and D be non-empty finite sets. Let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$. Suppose that:

$$R = \{(a, b_1), (a, b_2)\};$$

$$S = \{(b_1, c_1), (b_1, c_2), (b_2, c_1), (b_2, c_2), (b_2, c_3)\};$$

$$T = \{(c_1, d), (c_2, d), (c_3, d)\},$$

where $a \in A$, $b_1, b_2 \in B$, $c_1, c_2, c_3 \in C$, and $d \in D$. Then $(R \odot S) \odot T = \{(a, d)\}$ and $R \odot (S \odot T) = \varnothing$.

Proof. First we compute $(R \odot S) \odot T$.

$$R \odot S : aR \cap Sc_1 = \{b_1, b_2\} \cap \{b_1, b_2\} = \{b_1, b_2\};$$

$$aR \cap Sc_2 = \{b_1, b_2\} \cap \{b_1, b_2\} = \{b_1, b_2\};$$

$$aR \cap Sc_3 = \{b_1, b_2\} \cap \{b_2\} = \{b_2\}.$$

Then, by the even composition, $R \odot S = \{(a, c_1), (a, c_2)\}$. Let $K = R \odot S$. To obtain $(R \odot S) \odot T$, we compute $K \odot T$.

$$K \odot T : aK \cap Td = \{c_1, c_2\} \cap \{c_1, c_2, c_3\} = \{c_1, c_2\}.$$

Then, by the even composition, $K \odot T = \{(a, d)\}$. Hence $(R \odot S) \odot T = \{(a, d)\}$.

Next we compute $R \odot (S \odot T)$.

$$S \odot T : b_1S \cap Td = \{c_1, c_2\} \cap \{c_1, c_2, c_3\} = \{c_1, c_2\};$$

$$b_2S \cap Td = \{c_1, c_2, c_3\} \cap \{c_1, c_2, c_3\} = \{c_1, c_2, c_3\}.$$

Then, by the even composition, $S \odot T = \{(b_1, d)\}$. Let $J = S \odot T$. To obtain $R \odot (S \odot T)$, we compute $R \odot J$.

$R \odot J : aR \cap Jd = \{b_1, b_2\} \cap \{b_1\} = \{b_1\}$. Then, by the even composition, $R \odot J = \varnothing$. Hence $R \odot (S \odot T) = \varnothing$. □

5. A Ring Related to Finite Relations

The associativity of the odd composition, together with the distributivity over the symmetric difference, induces a "ring structure" on finite relations. We present such a ring.

We first briefly review the definition and notation of a ring [1, 3]. Let A be a set. Let "+" and "·" be binary operations on A, which are called *addition* and *multiplication*, respectively. Then the ordered triplet $\langle A, +, \cdot \rangle$ is a ring if it satisfies the following conditions.

RI 1. $\langle A, + \rangle$ is an abelian group.

RI 2. Multiplication "·" is associative, and has unity.

RI 3. For each $x, y, z \in A$:

1. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

2. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.

Let $\langle A, +, \cdot\rangle$ be a ring. The elements of A that have a multiplicative inverse are called the *units* of $\langle A, +, \cdot\rangle$. Let U be the set of the units of $\langle A, +, \cdot\rangle$. Then $\langle U, \cdot\rangle$ forms a group, which is called the *unit group* of $\langle A, +, \cdot\rangle$.

The following gives a ring with binary operations "+" and "·", where "+" is the symmetric difference and "·" is the odd composition of finite relations.

Proposition 5.1. *Let A be a non-empty finite set, and let* R_A *be the set of all relations on A. Let* S_A *be the set of all bijections on A. Let* i_A *be the identity function on A. Then*:

1. *The ordered triplet* $\langle R_A, +, \cdot\rangle$ *is a ring with multiplicative unity* i_A.

2. *Let* $\langle U_A, \cdot\rangle$ *be the unit group of the ring* $\langle R_A, +, \cdot\rangle$, *and let* $R^{\dagger}$ *be the multiplicative inverse of* $R \in U_A$. *Then* $R^{-1} \in U_A$ *and* $(R^{-1})^{\dagger} = (R^{\dagger})^{-1}$.

3. $\langle S_A, \cdot\rangle$ *is a subgroup of the unit group* $\langle U_A, \cdot\rangle$.

Proof. (1) We show that the ordered triplet $\langle R_A, +, \cdot\rangle$ satisfies all the conditions RI 1-RI 3 for a ring.

RI 1. Clearly, $\langle R_A, +\rangle$ is an abelian group with additive unity O.

RI 2. By Proposition 4.5, multiplication "·" is associative. By the properties of an odd composition, $i_A \cdot R = i_A \circ R = R$ and $R \cdot i_A = R \circ i_A = R$ for every $R \in R_A$. In other words, $i_A \cdot R = R \cdot i_A = R$ for every $R \in R_A$. This implies that i_A is the unity for the multiplication "·".

RI 3. Proof is immediate from the distributivity of the odd composition over the symmetric difference.

Thus, $\langle R_A, +, \cdot\rangle$ is a ring with multiplicative unity i_A.

(2) Let $R \in U_A$. By the definition of U_A, we have $R \cdot R^{\dagger} = R^{\dagger} \cdot R = 1_A$. Clearly, $(R \cdot R^{\dagger})^{-1} = (R^{\dagger} \cdot R)^{-1} = (1_A)^{-1}$. Then, by the properties

of an odd composition, $(R^{\dagger})^{-1} \cdot R^{-1} = R^{-1} \cdot (R^{\dagger})^{-1} = 1_A$. Then, by the definition of U_A, we have $R^{-1} \in U_A$ and $(R^{-1})^{\dagger} = (R^{\dagger})^{-1}$.

(3) By the properties of an odd composition, $f \cdot g = f \circ g$ for every $f, g \in S_A$. This implies that $\langle S_A, \cdot \rangle$ is a group that is isomorphic to the symmetric group $\langle S_A, \circ \rangle$.

On the other hand, let $f \in S_A$. Then $f^{-1} \in S_A$ because f is a bijective function on A. Then, by the properties of an odd composition, $f \cdot f^{-1} = f^{-1} \cdot f = i_A$. By the definition of U_A, we have $f \in U_A$ and hence $S_A \subseteq U_A$. This implies that $\langle S_A, \cdot \rangle$ is a subgroup of the unit group $\langle U_A, \cdot \rangle$. □

References

[1] G. Ehrlich, Fundamental Concepts of Abstract Algebra, PWS-KENT, Boston, 1991.

[2] T. Jech, Set Theory, Springer, New York, 2002.

[3] S. Lang, Algebra, 3rd ed., Addison-Wesley, New York, 1999.

[4] C. C. Pinter, Set Theory, Addison-Wesley, Reading, Mass., 1971.

[5] P. Suppes, Axiomatic Set Theory, Dover, New York, 1972,

Department of Information and Communication Engineering
Tongmyong University
Busan 608-711, South Korea
e-mail: wgpark@tu.ac.kr