Far East J. Math. Sci. (FJMS) **25**(1) (2007), 73-81
This paper is available online at http://www.pphmj.com

GENERALIZED-CONVERGENCE OF FILTERS ON GENERALIZED TOPOLOGICAL SPACES

WON KEUN MIN

(Received August 31, 2006)

Abstract

We introduce the concepts of generalized-convergence of filters, g^*-sets and g^*-continuity. And we investigate characterizations of g^*-sets and g^*-continuity by using the concept of generalized-convergence of filters.

1. Introduction

Let X be a nonempty set. A subclass $\tau \subset P(X)$ is called a *generalized topology* on X [1] if $\emptyset \in \tau$ and τ is closed under arbitrary union. (X, τ) is called a *generalized topological space*. The members of τ are called *generalized open sets*. The complement of generalized open sets are called *generalized closed sets*. Let (X, τ) be a generalized topological space and $S \subset X$. The *generalized-closure* of S, denoted by $c_g(S)$, is the intersection of generalized closed sets including S. And the *generalized-interior* of S, denoted by $i_g(S)$, the union of generalized open sets included in S. Let (X, τ) and (Y, μ) be generalized topological spaces. A function $f : X \to Y$ is called (τ, μ)-*continuous* if the inverse image of each generalized open set of Y is a generalized open set in X. Let X be a topological. Let S be a

2000 Mathematics Subject Classification: 54C08, 54A10, 54A20.

Keywords and phrases: g^*-sets, g^*-continuous, generalized-neighborhood filters, generalized-convergence of filters.

subset of X. Then the closure (resp. interior) of S will be denoted by $cl(S)$ (resp. $\text{int}(S)$). A subset S of X is said to be *preopen* [2] if $S \subset \text{int}(cl(S))$. The complement of a preopen set is said to be *preclosed*.

A subset $P(x)$ of X is called a *pre-neighborhood* of a point $x \in X$ [4] if there exists a preopen set S such that $x \in S \subset P(x)$. In [3], the author introduced the concept of pre-convergence of filters. Now we recall the concept of pre-convergence of filters. Let $PO(x) = \{A \in PO(X) : x \in A\}$ and let $P_x = \{A \subset X :$ there exists $\mu \subset PO(x)$ such that μ is finite and $\bigcap \mu \subset A\}$. Then P_x is called the *pre-neighborhood filter* at x. For any filter $\mathcal{F}$ on X, we say that $\mathcal{F}$ *pre-converges* to x if $\mathcal{F}$ is finer than the pre-neighborhood filter P_x at x. Here $\dot{x}$ is a filter generated by the singleton set $\{x\}$.

In this paper, we introduce the concept of generalized open neighborhood filter and generalized-convergence of filters on a generalized topological space. Also we introduce g^*-sets which are defined by the generalized open neighborhood filter. By using the concept of generalized-convergence of filters, we investigate some properties of g^*-sets and g^*-continuity.

2. Generalized-convergence of Filters

Definition 2.1. Let (X, τ) be a generalized topological space and let $G(x) = \{A \in \tau : x \in A\}$ for each $x \in X$. Then we call $\mathcal{G}_x = \{A \subset X :$ there exists $\mu \subset G(x)$ such that μ is finite and $\bigcap \mu \subset A\}$ the *generalized-neighborhood filter* at x.

Definition 2.2. Let (X, τ) be a generalized topological space and let $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ *generalized-converges* to x if $\mathcal{F}$ is finer than the generalized-neighborhood filter $\mathcal{G}_x$.

Definition 2.3. Let (X, τ) be a generalized topological space and let $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ has a *generalized-cluster point* x if every $F \in \mathcal{F}$ meets each $U \in G(x)$.

Remark. It is evident from the definition that a filter $\mathcal{F}$ has a generalized-cluster point x if and only if $x \in \bigcap c_g(F)$ for every $F \in \mathcal{F}$.

Definition 2.4. Let (X, τ) be a generalized topological space and let $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ has a *strong generalized-cluster point* x if every $F \in \mathcal{F}$ meets each $U \in \mathcal{G}_x$.

Every strong generalized-cluster point of a generalized topological space is a generalized-cluster point. But the converse is not always true.

Example 2.5. Consider a set $S = \{(a, b] : a, b \in R\} \cup \{[c, d) : c, d \in R\}$ on the real number set R; then $\tau(S) = \{\bigcup S^* : S^*$ is any subclass of $S\}$ is a generalized topology. Let $x \in X$ be an arbitrary point. Then the filter $\mathcal{F} = \{R - \{x\}, R\}$ generalized-clusters at x. But x is not a strong generalized-cluster point of $\mathcal{F}$, since $\{x\} \in \mathcal{G}_x$.

Theorem 2.6. *Let (X, τ) be a generalized topological space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ has x as a strong generalized-cluster point if and only if there is a finer filter $\mathcal{H}$ than $\mathcal{F}$ such that $\mathcal{H}$ generalized-converges to x.*

Proof. If $\mathcal{F}$ has a point x as a strong generalized-cluster point, then the collection $\mathcal{H} = \{G \cap F : G \in \mathcal{G}_x, F \in \mathcal{F}\}$ is a filter base for a filter which is finer than both $\mathcal{F}$ and $\mathcal{G}_x$. Thus $\mathcal{H}$ generalized-converges to x.

Conversely, if $\mathcal{F} \subset \mathcal{H}$ and $\mathcal{H}$ generalized-converges to x, then $\mathcal{F}$ strongly generalized-clusters at x.

We recall that if $\mathcal{F}$ is a filter on X and $f : X \to Y$ is a function, then $f(\mathcal{F})$ is the filter on Y having for a filter base the collection of $f(F)$ for every $F \in \mathcal{F}$.

Theorem 2.7. *Let (X, τ) and (Y, μ) be generalized topological spaces. If a function $f : X \to Y$ is (τ, μ)-continuous at $x \in X$, then whenever a filter $\mathcal{F}$ on X generalized-converges to x in X, $f(\mathcal{F})$ generalized-converges to $f(x)$ in Y.*

Proof. Suppose f is (τ, μ)-continuous at x and a filter $\mathcal{F}$ generalized-converges to x. If V is an element of the generalized-neighborhood filter $\mathcal{G}_{f(x)}$, there are generalized open sets $V_1, \ldots, V_n$ such that $f(x) \in V_1 \cap \cdots \cap V_n \subset V$. And since f is (τ, μ)-continuous, there are generalized open sets $U_1, \ldots, U_n$ in X such that $x \in U_1 \cap \cdots \cap U_n$ and $f(U_1) \subset V_1, \ldots, f(U_n) \subset V_n$. Thus we get $f(U_1 \cap \cdots \cap U_n) \subset V$, and since $U_1 \cap \cdots \cap U_n$ is an element of the generalized-neighborhood filter $\mathcal{G}_x$ and $\mathcal{F}$ is a finer filter than $\mathcal{G}_x$, we can say the set V contained in $\mathcal{G}_{f(x)}$ is also an element of $f(\mathcal{F})$.

3. g^*-sets

Definition 3.1. Let (X, τ) be a generalized topological space. A subset U of X is called a g^*-*set* in X if either U is empty or $U \in \mathcal{G}_x$ for all $x \in U$.

The class of all g^*-sets induced by the generalized topology τ will be denoted by $g^*_\tau(X)$ (simply $g^*(X)$).

Remark. From the concept of generalized-neighborhood filters and g^*-sets, we can easily show every generalized open set is a g^*-set, but the converse is not always true.

Example 3.2. From Example 2.5, we can say any subset $\{x\}$ in X is a g^*-*set* but not a generalized open set.

Definition 3.3. Let (X, τ) be a generalized topological space. The g^*-interior of a set A in X, denoted by $g^*I(A)$, is the union of all g^*-sets contained in A.

Theorem 3.4. *Let* (X, τ) *be a generalized topological space and* $A \subset X$. *Then*

(a) $g^*I(A) = \{x \in A : A \in \mathcal{G}_x\}$.

(b) *A is g^*-set if and only if* $A = g^*I(A)$.

Proof. (a) For each $x \in g^*I(A)$, there exists a g^*-set U such that $x \in U$ and $U \subset A$. From the concept of g^*-sets, the subset U is in the generalized-neighborhood filter $\mathcal{G}_x$. And since $\mathcal{G}_x$ is a filter, $A \in \mathcal{G}_x$.

Conversely, let $A \in \mathcal{G}_x$, then there exist $U_1, \ldots, U_n \in G(x)$ such that $U = U_1 \cap \cdots \cap U_n \subset A$. Thus U is a g^*-set containing x. Therefore we get $x \in g^*I(A)$.

(b) Obvious.

Theorem 3.5. *Let (X, τ) be a generalized topological space. Then the class $g^*(X)$ of all g^*-sets in X is a topology on X.*

Proof. Since $\varnothing$ is a generalized open set, it is also a g^*-set in X. X is a g^*-set because for every x in X, $\mathcal{G}_x$ contains X.

Let A and B be non-disjoint g^*-sets. For $x \in A \cap B$, since $\mathcal{G}_x$ is a filter, the intersection of A and B is also an element of $\mathcal{G}_x$. Thus $A \cap B$ is a g^*-set.

For each $\alpha \in I$, let $A_\alpha \in g^*(X)$ and $U = \bigcup A_\alpha$. For each $x \in U$, there exists a subset A_α of U such that $x \in A_\alpha$ and $A_\alpha \in \mathcal{G}_x$, and since $\mathcal{G}_x$ is a filter, U is an element of the filter $\mathcal{G}_x$. Thus $U = \bigcup A_\alpha$ is a g^*-set.

Let (X, τ) be a generalized topological space. For a subset B of X, we call B a g^*-*closed set* if the complement of B is a g^*-set. From Theorem 3.5, the intersection of any family of g^*-closed sets is a g^*-closed set and the union of finitely many g^*-closed sets is a g^*-closed set.

Definition 3.6. Let (X, τ) be a generalized topological space and $A \subset X$. Then $g^*cl(A) = \{x \in X : A \cap U \neq \varnothing \text{ for all } U \in \mathcal{G}_x\}$. We call $g^*cl(A)$ the g^*-*closure* of A.

Now we can get the following.

Remark. It is evident from Definition 2.4 that a filter $\mathcal{F}$ has a strong generalized-cluster point x if and only if $x \in \bigcap g^* cl(F)$ for every $F \in \mathcal{F}$.

Theorem 3.7. *Let (X, τ) be a generalized topological space. For $A \subset X$,*

(1) $A \subset g^* cl(A)$.

(2) *A is g^*-closed if and only if $A = g^* clA$.*

(3) $g^* I(A) = X - g^* cl(X - A)$.

(4) $g^* cl(A) = X - g^* I(X - A)$.

Theorem 3.8. *Let (X, τ) be a generalized topological space. Then $x \in g^* cl(A)$ if and only if there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F}$ generalized-converges to x.*

Proof. Let $x \in g^* cl(A)$. Then the collection $\mathcal{B} = \{U \cap A : U \in \mathcal{G}_x\}$ is a filter base. The filter $\mathcal{F}$ generated by filter base $\mathcal{B}$ generalized-converges to x and $A \in \mathcal{F}$. Suppose that there is a filter $\mathcal{F}$ generalized-converging to x such that $A \in \mathcal{F}$. Since $\mathcal{F}$ contains $\mathcal{G}_x$ and $\mathcal{F}$ is a filter, for all $U \in \mathcal{G}_x$, $U \cap A \neq \emptyset$. Thus $x \in g^* cl(A)$.

Theorem 3.9. *Let (X, τ) be a generalized topological space. A set G is g^*-closed if and only if whenever $\mathcal{F}$ generalized-converges to x and $G \in \mathcal{F}$, $x \in G$.*

4. g-continuity and g^*-continuity

Definition 4.1. Let (X, τ) and (Y, μ) be generalized topological spaces. A function $f : X \to Y$ is called *g-continuous* if the inverse image of each generalized open set of Y is a g^*-set in X.

Theorem 4.2. *Let (X, τ) and (Y, μ) be generalized topological spaces and let $f : X \to Y$ be a function. Then the following statements are* *equivalent*:

(1) *f is g-continuous*.

(2) *The inverse image of each generalized closed set in Y is* g^**-closed*.

(3) $g^*cl(f^{-1}(B)) \subset f^{-1}(c_g(B))$ *for every* $B \subset Y$.

(4) $f(g^*cl(A)) \subset c_g(f(A))$ *for every* $A \subset X$.

(5) $f^{-1}(i_g(B)) \subset g^*I(f^{-1}(B))$ *for every* $B \subset Y$.

Definition 4.3. Let (X, τ) and (Y, μ) be generalized topological spaces. A function $f : X \to Y$ is called g^**-continuous* if the inverse image of each g^*-set of Y is an g^*-set in X.

Remark. From Definition 4.1, we can say that every g^*-continuous function is g-continuous.

Theorem 4.4. *Let* (X, g) *and* (Y, h) *be generalized topological spaces. If* $f : X \to Y$ *is g-continuous, then it is also* g^**-continuous*.

Proof. Let U be a g^*-set in Y. For each $x \in f^{-1}(U)$, there exist $U_1, ..., U_n \in G(f(x))$ such that $f(x) \in U_1 \cap \cdots \cap U_n \subset U$. Since f is g-continuous, $f^{-1}(U_1), ..., f^{-1}(U_n)$ are g^*-sets and $x \in f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n) \subset f^{-1}(U)$. From Definition 3.3, we get $x \in g^*I(f^{-1}(U))$. Thus $f^{-1}(U)$ is a g^*-set from Theorem 3.4 (b).

Since the class of all g^*-sets in a given generalized topological space is also a topology, we get the following equivalent statements.

Theorem 4.5. *Let* (X, τ) *and* (Y, μ) *be generalized topological spaces. Then the following are equivalent*:

(1) *f is* g^**-continuous*.

(2) *The inverse image of each* g^**-closed set in Y is an* g^**-closed set*.

(3) $g^*cl(f^{-1}(V)) \subset f^{-1}(g^*cl(V))$, *for every* $V \subset Y$.

(4) $f(g^*cl(U)) \subset g^*cl(f(U))$, *for every* $U \subset X$.

(5) $f^{-1}(g^*I(B)) \subset g^*I(f^{-1}(B))$, *for every* $B \subset Y$.

Theorem 4.6. *Let* $f : (X, \tau) \to (Y, \mu)$ *be a function between generalized topological spaces. Then the following are equivalent*:

(1) *f is* g^**-continuous.*

(2) *For* $x \in X$ *and for each* $V \in \mathcal{G}_{f(x)}$, *there exists an element U in the generalized-neighborhood filter* $\mathcal{G}_x$ *such that* $f(U) \subset V$.

(3) *For each* $x \in X$, *if a filter* $\mathcal{F}$ *generalized-converges to x, then* $f(\mathcal{F})$ *generalized-converges to* $f(x)$ *in Y.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let V be an element of the generalized-neighborhood filter $\mathcal{G}_{f(x)}$ and let $\mathcal{F}$ be a filter on X generalized-converging to x. Then $f(\mathcal{G}_x) \subset f(\mathcal{F})$, and since there exists an element U in $\mathcal{G}_x$ such that $f(U) \subset V$ and $f(\mathcal{F})$ is a filter, we can say $V \in f(\mathcal{F})$. Consequently $f(\mathcal{F})$ generalized-converges to $f(x)$.

(3) $\Rightarrow$ (1) Let V be any g^*-set in Y and suppose $f^{-1}(V)$ is not empty. For each $x \in f^{-1}(V)$, since the generalized-neighborhood filter $\mathcal{G}_x$ generalized-converges to x, $f(\mathcal{G}_x)$ generalized-converges to x. Since V is a g^*-set containing $f(x)$ and $\mathcal{G}_{f(x)} \subset f(\mathcal{G}_x)$, V is an element of $f(\mathcal{G}_x)$. Thus now we can take some g^*-set U in $\mathcal{G}_x$ such that $f(U) \subset V$. Thus $U \subset f^{-1}(V)$ and $f^{-1}(V)$ is an element of $\mathcal{G}_x$. Therefore $f^{-1}(V)$ is a g^*-set in X by Theorem 3.4 (b).

Remark. Let $f : (X, \tau) \to (Y, \mu)$ be a function between generalized topological spaces. Then we can get the following diagrams:

$$(\tau, \mu)\text{-continuity} \Rightarrow \text{g-continuity} \Leftrightarrow g^*\text{-continuity.}$$

References

[1] Á. Császár, Generalized topology, generalized continuity, Acta Math. Hungar. 96(4) (2002), 351-357.

[2] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt 53 (1982), 47-53.

[3] W. K. Min, Pre-convergence of filters and $p\gamma$-continuity, Sci. Math. Jpn. 60(1) (2004), 149-156.

[4] G. B. Navalagi, Pre-neighborhoods, Math. Ed. 32(4) (1998), 201-206.

Department of Mathematics
Kangwon National University
Chuncheon, 200-701, South Korea
e-mail: wkmin@cc.kangwon.ac.kr