Far East J. Math. Sci. (FJMS) **25**(1) (2007), 59-72
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ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A NEW MULTIPLIER INTEGRAL OPERATOR

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(Received October 17, 2006)

Abstract

In this paper, we introduce a class of analytic functions $M_\lambda^n[A, B]$ defined by a new integral operator $I_\lambda^n f$, where $I_\lambda^0 f(z) = f(z)$, $I_\lambda^1 f(z) = \dfrac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} f(t)\, dt = I_\lambda f(z)$, $\lambda > 0$, $I_\lambda^2 f(z) = I_\lambda(I_\lambda^1 f(z))$, ..., $I_\lambda^n f(z) = I_\lambda(I_\lambda^{n-1} f(z))$, $n \in \mathbb{N}$. Using differential subordinations, certain results concerning inclusion relation, integral operator defined on this class and other results are given.

1. Introduction

Let $\mathcal{H}(U)$ be the set of all analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U). \tag{1.1}$$

2000 Mathematics Subject Classification: 30C45.

Keywords and phrases: univalent functions, integral operator, differential subordination.

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The Hadamard product or Convolution of two power series $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined [2] as the power series $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.

An analytic function $f(z)$ on U is said to be *subordinate* to an analytic function $g(z)$ on U (written $f(z) \prec g(z)$) if $g(z)$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$, but if $g(z)$ is not univalent we say that $f(z)$ is *subordinate* to $g(z)$ [4], if $f(z) = g(\phi(z))$, $z \in U$, for some analytic function $\phi(z)$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, $z \in U$. For an analytic function $f(z)$ given by (1.1), Sălăgean [6] defined the integral operator $I^n f$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = \int_0^z \frac{f(t)}{t}\, dt = If(z)$$

$$\vdots$$

$$I^n f(z) = I(I^{n-1} f(z)), \ z \in U.$$

Thus

$$I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k.$$

Further, the integral operator I^n can be written

$$I^n f(z) = (\underbrace{h * h * \cdots * h}_{n\, times} * f)(z),$$

where $h(z) = -\log(1 - z)$.

For an analytic function $f(z)$ given by (1.1), Al-Oboudi differential operator D_λ^n, $n \in \mathbb{N}_0$, $\lambda \geq 0$ is defined [1] by

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$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$$

$$\vdots$$

$$D_\lambda^n f(z) = D_\lambda (D_\lambda^{n-1} f(z)), \quad n \in \mathbb{N}.$$

Thus

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k, \quad n \in \mathbb{N}_0.$$

Now we shall define an integral operator, which generalize Sălăgean integral operator.

2. Definitions

Definition 2.1. Let $f(z) \in \mathcal{A}$. We define the integral operator $I_\lambda^n f$, $n \in \mathbb{N}_0$, $\lambda > 0$, by

$$I_\lambda^0 f(z) = f(z),$$

$$I_\lambda^1 f(z) = \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} f(t) dt = I_\lambda f(z)$$

$$I_\lambda^2 f(z) = \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^1 f(t) dt$$

$$\vdots$$

$$I_\lambda^n f(z) = \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^{n-1} f(t) dt, \quad n \in \mathbb{N}. \tag{2.1}$$

Remark 2.1. If $f(z) \in \mathcal{A}$, then $I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{(1 + \lambda(k-1))^n} z^k$.

Remark 2.2. If $\lambda = 1$, then we get the Sălăgean integral operator.

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Remark 2.3. If $f(z) \in \mathcal{A}$, then $I_\lambda^n(D_\lambda^n f(z)) = D_\lambda^n(I_\lambda^n f(z)) = f(z)$.

Remark 2.4. If $f(z) \in \mathcal{A}$, then $I_\lambda^n(I_\lambda^m f(z)) = I_\lambda^{n+m} f(z)$, $n, m \in \mathbb{N}_0$.

Remark 2.5. If $f(z) \in \mathcal{A}$ and $g(z) = z\,{}_2F_1\left(1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1; z\right)$, then

$$I_\lambda^n f(z) = (\underbrace{g * g * \cdots * g}_{n\,times} * f)(z), \tag{2.2}$$

where the function ${}_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1) \cdot b(b+1)}{2! \cdot c(c+1)} z^2 + \cdots$, for any real or complex numbers a, b and c, $(c \neq 0, -1, -2, \ldots)$ is called the *hypergeometric series* which represents an analytic function in U. If $\lambda = 1$, then $z\,{}_2F_1\left(1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1; z\right) = -Log(1 - z)$.

Remark 2.6. If $f(z) \in \mathcal{A}$ and the integral operator $I_\lambda^n f(z)$ is given by (2.1), then

$$I_\lambda^n f(z) = (1 - \lambda) I_\lambda^{n+1} f(z) + \lambda z (I_\lambda^{n+1} f(z))'. \tag{2.3}$$

Using the operator I_n^λ, we now introduce a subclass of $\mathcal{A}$ as follows:

Definition 2.2. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{M}_\lambda^n[A, B]$ $(-1 \leq B < A \leq 1)$ if and only if

$$\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.4}$$

for $z \in U$, where the symbol '$\prec$' stand for subordination. Let

$$\mathcal{M}_\lambda^n[1 - 2\alpha, -1] = \mathcal{M}_\lambda^n(\alpha),$$

where $\mathcal{M}_\lambda^n(\alpha)$ denotes the class of functions $f(z) \in \mathcal{A}$, which satisfies the condition

$$\operatorname{Re}\left(\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)}\right) > \alpha, \quad (0 \leq \alpha < 1, \lambda > 0, z \in U).$$

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3. Preliminary Lemmas

Lemma 3.1 [3]. *Let* $\beta, \gamma \in \mathbb{C}$, *let* $h(z) \in \mathcal{H}(U)$ *be convex univalent in* U, *with* $h(0) = 1$ *and* $\operatorname{Re}(\beta h(z) + \gamma) > 0$, $z \in U$ *and let* $p(z) \in \mathcal{H}(U)$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. *Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{3.1}$$

If the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1$$

has a univalent solution $q(z)$, *then*

$$p(z) \prec q(z) \prec h(z),$$

and $q(z)$ *is the best dominant (i.e.,* $p(z) \prec q(z)$ *for all* $p(z)$ *satisfying* (3.1) *and if* $p(z) \prec s(z)$, *then* $q(z) \prec s(z)$).

Lemma 3.2 [5]. *If* $-1 \leq B < A \leq 1$, $B > 0$ *and the complex number* γ *satisfy that* $\operatorname{Re}(\gamma) \geq -\beta \dfrac{1-A}{1-B}$, *then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz}$$

has a univalent solution in U *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1 + Bz)^{\beta\left(\frac{A-B}{B}\right)}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}(1 + Bt)^{\beta\left(\frac{A-B}{B}\right)} dt} - \dfrac{\gamma}{\beta}, & B \neq 0 \\[2ex] \dfrac{z^{\beta+\gamma} e^{\beta A z}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1} e^{\beta A t} dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \tag{3.2}$$

If $p(z)$ *is analytic in* U *and satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in U)$$

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then

$$p(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}$$

and $q(z)$ *is the best dominant.*

Lemma 3.3 [8]**.** *Let* μ *be a positive measure on the unit interval* $[0, 1]$*. Let* $g(t, z)$ *be an analytic function in U, for each* $t \in [0, 1]$*, and integrable in* t *for each* $z \in U$ *and for almost all* $t \in [0, 1]$*, and suppose that* $\operatorname{Re} g(t, z) > 0$ *on* U*,* $g(t, -r)$ *is real and* $\operatorname{Re} \frac{1}{g(t, z)} \geq \frac{1}{g(t, -r)}$*, for* $|z| \leq r$ *and* $t \in [0, 1]$*. If* $g(z) = \int_0^1 g(t, z) d\mu(t)$*, then* $\operatorname{Re} \frac{1}{g(z)} \geq \frac{1}{g(-r)}$*,* $|z| \leq r$*.*

Lemma 3.4 [7]**.** *For real or complex numbers a, b and c* $(c \neq 0, -1, -2, \ldots)$*,* $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$*, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z), \tag{3.3}$$

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z), \tag{3.4}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{3.5}$$

$$(b+1) {}_2F_1(1, b; b+1; z) = (b+1) + bz \, {}_2F_1(1, b+1; b+2; z). \tag{3.6}$$

Lemma 3.5. *Let* $h(z)$ *be convex univalent in U, with* $h(0) = 1$ *and* $\operatorname{Re}(h(z)) > 0$*, and let* $f(z) \in \mathcal{A}$*. Then*

$$\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)} \prec h(z) \Rightarrow \frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)} \prec h(z), \quad z \in U. \tag{3.7}$$

Proof. Let

$$p(z) = \frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)}. \tag{3.8}$$

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Then $p(z)$ is analytic in U and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Logarithmic differentiation of both sides of (3.8) gives

$$\frac{\lambda z p'(z)}{p(z)} = \frac{\lambda z (I_\lambda^{n+1} f(z))'}{I_\lambda^{n+1} f(z)} - \frac{\lambda z (I_\lambda^{n+2} f(z))'}{I_\lambda^{n+2} f(z)}.$$

From (2.3) we obtain

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}\, p(z)} = \frac{I_\lambda^{n} f(z)}{I_\lambda^{n+1} f(z)}.$$

Using Lemma 3.1, for $\beta = \frac{1}{\lambda}$, $\gamma = 0$ we obtain

$$\frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)} \prec h(z).$$

4. Main Results

Theorem 4.1.

$$\mathcal{M}_\lambda^n[A, B] \subseteq \mathcal{M}_\lambda^{n+1}[A, B].$$

Proof. Let $f(z) \in \mathcal{M}_\lambda^n[A, B]$. Then from (2.4) and Lemma 3.5, by choosing $h(z) = \frac{1 + Az}{1 + Bz}$, we have

$$\frac{I_\lambda^{n} f(z)}{I_\lambda^{n+1} f(z)} \prec \frac{1 + Az}{1 + Bz} \Rightarrow \frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)} \prec \frac{1 + Az}{1 + Bz},$$

hence $f(z) \in \mathcal{M}_\lambda^{n+1}[A, B]$, which is the required result.

We also have a better result than Theorem 4.1.

Theorem 4.2. *Let* $(-1 \le B < A \le 1)$, $\lambda > 0$ *and* $n \in \mathbb{N}_0$. *If* $f(z) \in \mathcal{M}_\lambda^n[A, B]$, *then*

$$\frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)} \prec \frac{1}{Q(z)} = \widetilde{q}(z), \tag{4.1}$$

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where

$$Q(z) = \begin{cases} \dfrac{1}{\lambda}\displaystyle\int_0^1 t^{\frac{1}{\lambda}-1}\left(\dfrac{1+Bzt}{1+Bz}\right)^{\frac{1}{\lambda}\left(\frac{A-B}{B}\right)} dt, & B \neq 0 \\ \dfrac{1}{\lambda}\displaystyle\int_0^1 t^{\frac{1}{\lambda}-1} e^{\frac{1}{\lambda}A(t-1)z} dt, & B = 0 \end{cases} \tag{4.2}$$

and $\widetilde{q}(z)$ *is the best dominant of* (4.1). *Furthermore, if* $1 + \dfrac{A}{\lambda B} > 0$ *with* $B < 0$, *then*

$$\mathcal{M}_\lambda^n[A, B] \subset \mathcal{M}_\lambda^{n+1}(\rho_1(A, B, \lambda)), \tag{4.3}$$

where

$$\rho_1(A, B, \lambda) = \left({}_2F_1\left(1, \frac{1}{\lambda}\left(\frac{B-A}{B}\right); \frac{1}{\lambda}+1; \frac{B}{B-1}\right)\right)^{-1}.$$

The result is best possible.

Proof. Let $f \in \mathcal{M}_\lambda^n[A, B]$, and let

$$p(z) = \frac{I_\lambda^{n+1} f(z)}{I_\lambda^{n+2} f(z)}. \tag{4.4}$$

Then p is analytic in U and $p(0) = 1$. Making use of the logarithmic differentiation on both sides of (4.4) and simplifying the resulting equation, we deduce that

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}p(z)} = \frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)} \prec \frac{1+Az}{1+Bz}. \tag{4.5}$$

By Lemma 3.2, we obtain

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is the best dominant of (4.5) and is given by (3.2) for $\beta = \dfrac{1}{\lambda}$ and $\gamma = 0$. Now, rewriting $q(z)$ by changing the variables, we get

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$$p(z) \prec Q^{-1}(z) = \tilde{q}(z).$$

Next we show that

$$\inf_{|z|<1} \operatorname{Re}\{\tilde{q}(z)\} = \tilde{q}(-1). \tag{4.6}$$

If we set $a = \frac{1}{\lambda}\left(\frac{B-A}{B}\right)$, $b = \frac{1}{\lambda}$ and $c = b + 1$, then $c > b > 0$. From (4.2) by using (3.3) and (3.4) we see that, for $B \neq 0$

$$Q(z) = {}_2F_1\left(1, a; 1 + \frac{1}{\lambda}; \frac{Bz}{1+Bz}\right). \tag{4.7}$$

To prove (4.6), we show that $\operatorname{Re}\left(\frac{1}{Q(z)}\right) \geq \frac{1}{Q(-1)}$, $z \in U$. For $1 + \frac{A}{\lambda B} > 0$ and $B < 0$ (so that $c > a > 0$) we can rewrite (4.7) as

$$Q(z) = \int_0^1 g(t, z) d\mu(t),$$

where μ is a positive measure on $[0, 1]$, $g(t, z)$ is an analytic function in U for $t \in [0, 1]$ and integrable in t for each $z \in U$ and for almost all $t \in [0, 1]$.

For $B < 0$, we have $\operatorname{Re}\{g(t, z)\} > 0$, $g(t, -r)$ is real for $0 \leq r < 1$, $t \in [0, 1]$ and

$$\operatorname{Re}\frac{1}{g(t, z)} = \operatorname{Re}\frac{1 + (1-t)Bz}{1 + Bz} \geq \frac{1 - (1-t)Br}{1 - Br} = \frac{1}{g(t, -r)},$$

for $|z| \leq r < 1$ and $t \in [0, 1]$.

Therefore by using Lemma 3.3 and by letting $r \to 1^-$, we obtain

$$\operatorname{Re}\frac{1}{Q(z)} \geq \frac{1}{Q(-1)}, \quad |z| < 1.$$

This by (4.1) leads to (4.3)

If we put $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Theorem 4.2, then we obtain

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Corollary 4.1. *Let* $0 \leq \alpha < 1, \lambda > 0, 2\alpha > 1 - \lambda$ *and* $n \in \mathbb{N}_0$. *Then*

$$\mathcal{M}_\lambda^n(\alpha) \subset \mathcal{M}_\lambda^{n+1}(\beta(\alpha, \lambda)),$$

where $\beta(\alpha, \lambda)$ *is given by*

$$\beta(\alpha, \lambda) = \left\{ {}_2F_1\left(1, \frac{2}{\lambda}(1-\alpha); \frac{1}{\lambda}+1; \frac{1}{2}\right)\right\}.$$

The result is the best possible.

For a function $f \in \mathcal{A}$, the function $\mathcal{F}_\mu(z)$ is defined by

$$\mathcal{F}_\mu(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt, \tag{4.8}$$

where $\mu + 1 > 0, z \in U$ and

$$\begin{aligned}\mathcal{F}_\mu(z) &= I^1_{\frac{1}{\mu+1}} f(z) \\ &= \left(z + \sum_{k=2}^\infty \frac{\mu+1}{\mu+k} z^k\right) * f(z) \\ &= z + \sum_{k=2}^\infty \frac{\mu+1}{\mu+k} a_k z^k.\end{aligned}$$

It is easy to see that

$$z(I_\lambda^n \mathcal{F}_\mu(z))' = (\mu+1)I_\lambda^n f(z) - \mu I_\lambda^n \mathcal{F}_\mu(z). \tag{4.9}$$

Theorem 4.3. *Let* $-1 \leq B < A \leq 1$ *and*

$$\lambda(\mu+1) - 1 \geq -\frac{1-A}{1-B}. \tag{4.10}$$

(i) *If* $f(z) \in \mathcal{M}_\lambda^n[A, B]$, *then the function* $\mathcal{F}_\mu(z)$ *defined by* (4.8) *satisfies*

$$\frac{I_\lambda^n \mathcal{F}_\mu(z)}{I_\lambda^{n+1}\mathcal{F}_\mu(z)} \prec \frac{1}{Q_1(z)} - (\lambda(\mu+1)-1) = \tilde{q}_1(z), \tag{4.11}$$

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where

$$Q_1(z) = \begin{cases} \int_0^1 \frac{t^{\mu}}{\lambda}\left(\frac{1+Bzt}{1+Bz}\right)^{\frac{1}{\lambda}\left(\frac{A-B}{B}\right)} dt, & B \neq 0 \\ \int_0^1 \frac{t^{\mu}}{\lambda} e^{A(t-1)z} dt, & B = 0 \end{cases} \tag{4.12}$$

and $q_1(z)$ *is the best dominant of* (4.11).

(ii) *If in addition to* (4.10), $\frac{A}{B} > 1 - \lambda(\mu+2)$ *with* $B < 0$, *then for* $f(z) \in \mathcal{M}_{\lambda}^{n}[A, B]$ *we have* $\mathcal{F}_{\mu}(z) \in \mathcal{M}_{\lambda}^{n}(\rho_2(A, B, \lambda, \mu))$, *where*

$$\rho_2(A, B, \lambda, \mu) = \frac{\lambda(\mu+1)}{{}_2F_1\left(1, \frac{(B-A)/B}{\lambda}; \mu+2; \frac{B}{B-1}\right)} - (\lambda(\mu+1)-1).$$

The result is best possible.

Proof. Let $f(z) \in \mathcal{M}_{\lambda}^{n}[A, B]$ and

$$p(z) = \frac{I_{\lambda}^{n}\mathcal{F}_{\mu}(z)}{I_{\lambda}^{n+1}\mathcal{F}_{\mu}(z)}. \tag{4.13}$$

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Then $p(z)$ is analytic in U and $p(0) = 1$. Using (2.3) and (4.9) in (4.13), we get

$$\frac{\lambda z(I_{\lambda}^{n+2}\mathcal{F}_{\mu}(z))' + (1-\lambda)(I_{\lambda}^{n+2}\mathcal{F}_{\mu}(z))}{\lambda z(I_{\lambda}^{n+1}f(z))' + (1-\lambda)(I_{\lambda}^{n+2}f(z))} = \frac{\lambda(\mu+1)}{p(z) + (\lambda(\mu+1)-1)}. \tag{4.14}$$

Since $f(z) \in \mathcal{M}_{\lambda}^{n}[A, B]$, we note that $I_{\lambda}^{n+1}f(z) \neq 0$ in U. Logarithmic differentiation of both sides of (4.14) with using (2.3), we have

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}p(z) + \frac{1}{\lambda}(\lambda(\mu+1)-1)} = \frac{I_{\lambda}^{n}f(z)}{I_{\lambda}^{n+1}f(z)} \prec \frac{1+Az}{1+Bz}. \tag{4.15}$$

Using Lemma 3.2, we obtain

$$p(z) \prec \widetilde{q}_1(z) = \frac{1}{Q_1(z)} - (\lambda(\mu+1)-1) \prec \frac{1+Az}{1+Bz},$$

where $Q_1(z)$ is given by (4.12) and $\widetilde{q}_1(z)$ is the best dominant of (4.15).

Proceeding as in Theorem 4.2 the second part follows.

Putting $A = 1-2\alpha$ and $B = -1$ in Theorem 4.3, we have

Corollary 4.2. *Let* $0 \le \alpha < 1$, $\lambda(\mu+1) > 2(1-\alpha)$. *If* $f(z) \in \mathcal{M}_\lambda^n(\alpha)$, *then* $\mathcal{F}_\mu(z) \in \mathcal{M}_\lambda^n(\rho_3(\alpha, \mu, \lambda))$, *where*

$$\rho_3(\alpha, \mu, \lambda) = \frac{\lambda(\mu+1)}{{}_2F_1\left(1, \frac{2}{\lambda}(1-\alpha); \mu+2; \frac{1}{2}\right)} - (\lambda(\mu+1)-1).$$

The result is best possible.

Theorem 4.4. (i) *If* $f(z) \in \mathcal{M}_\lambda^n[A, B]$, *then the function* $\mathcal{F}_\mu(z)$ *defined by* (4.8), *satisfies*

$$\frac{I_\lambda^{n+1}f(z)}{I_\lambda^{n+1}\mathcal{F}_\mu(z)} \prec \frac{1}{\lambda(\mu+1)Q_1(z)} = \widetilde{q}_2(z) \prec \frac{1+\left(\frac{A+(\lambda\mu-(1-\lambda))B}{\lambda(\mu+1)}\right)z}{1+Bz}, \quad (4.16)$$

where $Q_1(z)$ *is given by* (4.12) *and* $\widetilde{q}_2(z)$ *is the best dominant of* (4.16).

(ii) *If in addition to* (4.10), $\frac{A}{B} > 1 - \lambda(\mu+2)$, *and* $B < 0$, *then for* $f \in \mathcal{M}_\lambda^n[A, B]$, *we have*

$$\operatorname{Re}\left(\frac{I_\lambda^{n+1}f(z)}{I_\lambda^{n+1}\mathcal{F}_\mu(z)}\right) > \left[{}_2F_1\left(1, \frac{-1}{\lambda}\left(\frac{A-B}{B}\right); \mu+2; \frac{B}{B-1}\right)\right]^{-1}.$$

The result is best possible.

Proof. Let

$$p(z) = \frac{I_\lambda^{n+1}f(z)}{I_\lambda^{n+1}\mathcal{F}_\mu(z)}. \quad (4.17)$$

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Then $p(z)$ is analytic and $p(0) = 1$. Making use of the logarithmic differentiation on both sides of (4.17) and using (4.9) and (2.3), we deduce that

$$\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)} = \frac{\lambda z p'(z)}{p(z)} + ((1 - \lambda) - \lambda\mu) + \lambda(\mu + 1) p(z).$$

Let

$$P(z) = ((1 - \lambda) - \lambda\mu) + \lambda(\mu + 1) p(z).$$

Then

$$\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)} = P(z) + \frac{zP'(z)}{\frac{1}{\lambda} P(z) + \frac{1}{\lambda}(\lambda\mu - (1 - \lambda))} \prec \frac{1 + Az}{1 + Bz}.$$

By using Lemma 3.2, we deduce that

$$P(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}, \tag{4.18}$$

where $q(z)$ is the best dominant of (4.18) and is given by (3.2), for $\beta = \frac{1}{\lambda}$, and $\gamma = \frac{1}{\lambda}(\lambda\mu - (1 - \lambda))$, we have

$$((1 - \lambda) - \lambda\mu) + \lambda(\mu + 1) p(z) \prec \frac{1}{Q_1(z)} - (\lambda\mu - (1 - \lambda)) \prec \frac{1 + Az}{1 + Bz}. \tag{4.19}$$

By simplifying (4.19) we get (4.16).

To prove the second part of this theorem we proceed as in Theorem 4.2.

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