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GREEN FUNCTION FOR BOUNDARY VALUE PROBLEM OF 2*M*-TH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH FREE BOUNDARY CONDITION

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Abstract

Green function for a $2M$-order linear ordinary differential equation with free boundary condition is found by means of the so-called symmetric orthogonalization method. Its properties are also studied in detail.

1. Introduction

Let $f(x)$ be a given function satisfying the following solvability condition:

$$\int_{-1}^{1} f(x)\varphi_i(x)dx = 0 \quad (0 \le i \le M-1). \tag{1.1}$$

We consider the following boundary value problem:

$$\begin{cases} (-1)^M u^{(2M)} = f(x) \ \ (-1 < x < 1), \\ u^{(i)}(\pm 1) = 0 \ \ (M \le i \le 2M-1), \\ \int_{-1}^{1} u(x)\varphi_i(x)dx = 0 \ \ (0 \le i \le M-1), \end{cases} \tag{1.2}$$

where $\varphi_i(x)$ are normalized Legendre polynomials defined as follows:

$$\varphi_i(x) = \sqrt{i+\frac{1}{2}}P_i(x) = \sqrt{i+\frac{1}{2}}\,\frac{(-1)^i}{2^i\, i!}\left(\frac{d}{dx}\right)^i(1-x^2)^i. \tag{1.3}$$

The above set of functions $\{\varphi_i(x)\}$ $(i = 0, 1, \ldots, M-1)$ are eigenfunctions corresponding to the eigenvalue $\lambda = 0$ of the following eigenvalue problems:

$$\begin{cases} (-1)^M u^{(2M)} = \lambda u \ \ (-1 < x < 1), \\ u^{(i)}(\pm 1) = 0 \ \ (M \le i \le 2M-1). \end{cases} \tag{1.4}$$

The solution to (1.2) is given as follows:

$$u(x) = \int_{-1}^{1} G(x,\ y) f(y)dy,$$

where $G(x, y)$ is a Green function constructed in the following procedures. We start with the following proto Green function:

$$G_0(x, y) = \frac{(-1)^M}{2} K_0(|x - y|),$$

where $K_0(x)$ is a monomial defined by

$$K_0(x) = \frac{x^{2M-1}}{(2M-1)!}.$$

We also introduce its successive derivatives $K_j(x)$ defined by

$$K_j(x) = \left(\frac{d}{dx}\right)^j K_0(x) = \frac{x^{2M-1-j}}{(2M-1-j)!} \quad (1 \le j \le 2M-1).$$

The proto Green function $G_0(x, y)$ can also be viewed as a fundamental solution of the differential operator $(-1)^M (d/dx)^{2M}$. We can construct $G(x, y)$ by symmetric orthogonalization method [3] as

$$G(x, y) = G_0(x, y) - \sum_{i=0}^{M-1} \varphi_i(x) \int_{-1}^{1} \varphi_i(\xi) G_0(\xi, y) d\xi$$

$$- \sum_{i=0}^{M-1} \int_{-1}^{1} \varphi_i(\eta) G_0(x, \eta) d\eta \varphi_i(y)$$

$$+ \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \varphi_i(x) \int_{-1}^{1} \int_{-1}^{1} \varphi_i(\xi) G_0(\xi, \eta) \varphi_j(\eta) d\xi d\eta \varphi_j(y).$$

The main purpose of this paper is to find an explicit formula of $G(x, y)$ and investigate its properties as a reproducing kernel.

In order to find $G(x, y)$ explicitly, it is enough to find

$$\psi_i(x) = \int_{-1}^{1} G_0(x, \eta) \varphi_i(\eta) d\eta, \tag{1.5}$$

$$g_{ij} = \int_{-1}^{1}\int_{-1}^{1} \varphi_i(\xi) G_0(\xi, \eta) \varphi_j(\eta) d\xi d\eta$$

$$= \int_{-1}^{1} \varphi_i(\xi)\psi_j(\xi) d\xi = \int_{-1}^{1} \psi_i(\eta)\varphi_j(\eta) d\eta. \tag{1.6}$$

By using $\psi_i(x)$ and g_{ij}, Green function $G(x, y)$ is rewritten as follows:

$$G(x, y) = G_0(x, y) - \sum_{i=0}^{M-1} (\psi_i(x)\varphi_i(y) + \varphi_i(x)\psi_i(y)) + \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} g_{ij}\varphi_i(x)\varphi_j(y).$$

The main results obtained in this paper are as follows:

Theorem 1.1. *The functions* $\psi_i(x)$ $(i = 0, 1, \ldots, M-1)$ *are given as follows*:

$$\psi_i(x) = \frac{(-1)^M \sqrt{i + \frac{1}{2}}}{2(2M)!} \Bigg((-1)^i (x+1)^{2M} {}_2F_1\left(-i, i+1; 2M+1; \frac{1+x}{2}\right) + (1-x)^{2M} {}_2F_1\left(-i, i+1; 2M+1; \frac{1-x}{2}\right) \Bigg),$$

where ${}_2F_1(\alpha, \beta; \gamma; x)$ *is the Gauss hypergeometric function defined by*

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n) n!} x^n.$$

Theorem 1.2. *The coefficients* g_{ij} $(i, j = 0, 1, \ldots, M-1)$ *are given as follows*:

$$g_{ij} = \begin{cases} 0 & (i-j : odd), \\ (-1)^{M+j} 2^{2M+1} \sqrt{i+\frac{1}{2}}\sqrt{j+\frac{1}{2}} \\ \quad \sum_{k=0}^{i} \dfrac{(-1)^k (2M+k)!(i+k)!}{(2M+k-j)!(2M+k+j+1)!k!(i-k)!} & (i-j : even). \end{cases} \tag{1.7}$$

Remark 1. It is very interesting to note that the above summation in the expression of g_{ij} is rewritten in the following closed form:

$$g_{ij} = \begin{cases} 0 & (i-j : \text{odd}) \\ (-1)^{j+M}(2M)!\sqrt{i+\frac{1}{2}}\sqrt{j+\frac{1}{2}}\pi \Big/ & \\ \quad \left\{2^{2M}\Gamma\left(\frac{2M-i-j+1}{2}\right)\Gamma\left(\frac{2M+i-j+2}{2}\right)\times\right. & \\ \quad \left.\Gamma\left(\frac{2M-i+j+2}{2}\right)\Gamma\left(\frac{2M+i+j+3}{2}\right)\right\} & (i-j : \text{even}) \end{cases} \tag{1.8}$$

which is confirmed by computer software Mathematica 5.2.

This paper is organized as follows. In Section 2, we prove the above two theorems. In Section 3, we investigate some important properties of the Green function $G(x, y)$.

2. Proof of the Main Theorems

This section is devoted to the proof of the main theorems.

Proof of Theorem 1.1. Substituting (1.3) into (1.5), we have

$$\psi_i(x) = \int_{-1}^{1} G(x, \eta)\varphi_i(\eta)d\eta$$

$$= \frac{\sqrt{i+\frac{1}{2}}}{2^{i+1}i!}(-1)^{i+M}\int_{-1}^{1} K_0(|x-\eta|)D^i(1-\eta^2)^i d\eta$$

$$= \frac{\sqrt{i+\frac{1}{2}}}{2^{i+1}i!}(-1)^{i+M}$$

$$\times\left(\int_{-1}^{x} K_0(x-\eta)D^i(1-\eta^2)^i d\eta + \int_{x}^{1} K_0(\eta-x)D^i(1-\eta^2)^i d\eta\right).$$

Performing integration by parts i times, we have

$$\psi_i(x) = \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2^{i+1}i!}\left((-1)^i\int_{-1}^{x} K_i(x-\eta)(1-\eta^2)^i d\eta + \int_x^1 K_i(\eta-x)(1-\eta^2)^i d\eta\right)$$

$$= \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2^{i+1}(2M-1-i)!\,i!}((-1)^i I_1(x) + I_2(x)), \tag{2.1}$$

where

$$I_1(x) = \int_{-1}^{x}(x-\eta)^{2M-1-i}(1-\eta^2)^i d\eta,$$

$$I_2(x) = \int_x^1(\eta-x)^{2M-1-i}(1-\eta^2)^i d\eta.$$

We first calculate

$$I_1(x) = \int_{-1}^{x}(x-\eta)^{2M-1-i}(1+\eta)^i(1-\eta)^i d\eta.$$

Through the variable transformation $t = \frac{\eta+1}{x+1}$ and from

$$(x+1)dt = d\eta, \quad \eta = xt+t-1,$$

$$x-\eta = (x+1)(1-t), \quad 1+\eta = (x+1)t, \quad 1-\eta = 2-(x+1)t,$$

we have

$$I_1(x) = \int_0^1 (x+1)^{2M-1-i}(1-t)^{2M-1-i}(x+1)^i t^i 2^i\left(1-\frac{x+1}{2}t\right)^i (x+1)dt$$

$$= 2^i(x+1)^{2M}\int_0^1 t^i(1-t)^{2M-1-i}\left(1-\frac{x+1}{2}t\right)^i dt$$

$$= 2^i(x+1)^{2M}\,\frac{\Gamma(i+1)\Gamma(2M-i)}{\Gamma(2M+1)}\,{}_2F_1\left(-i,\, i+1;\, 2M+1;\, \frac{x+1}{2}\right),$$

where we have used the following formula:

$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; z).$$

Next we calculate

$$I_2(x) = \int_x^1 (\eta - x)^{2M-1-i}(1-\eta)^i(1+\eta)^i d\eta$$

which is rewritten through $\frac{1-\eta}{1-x} = t$ and from

$$d\eta = -(1-x)dt, \quad \eta = 1 - t + xt,$$

$$\eta - x = (1-x)(1-t), \quad 1 - \eta = (1-x)t, \quad 1 + \eta = 2 - (1-x)t$$

as follows:

$$I_2(x) = \int_0^1 (1-x)^{2M-1-i}(1-t)^{2M-1-i}(1-x)^i t^i 2^i \left(1 - \frac{1-x}{2}t\right)^i (1-x)dt$$

$$= (1-x)^{2M} 2^i \int_0^1 t^i (1-t)^{2M-1-i}\left(1 - \frac{1-x}{2}t\right)^i dt$$

$$= (1-x)^{2M} 2^i \frac{\Gamma(i+1)\Gamma(2M-i)}{\Gamma(2M+1)} {}_2F_1\left(-i, i+1; 2M+1; \frac{1-x}{2}\right).$$

From (2.1), we obtain

$$\psi_i(x) = \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2(2M)!}\Bigg((-1)^i(x+1)^{2M} {}_2F_1\left(-i, i+1; 2M+1; \frac{1+x}{2}\right)$$

$$+ (1-x)^{2M} {}_2F_1\left(-i, i+1; 2M+1; \frac{1-x}{2}\right)\Bigg)$$

which completes the proof.

Before going to the proof Theorem 1.2, we note that the function $\psi_i(x)$, although it is expressed by means of Gauss hypergeometric function, is in fact a polynomial

$$\psi_i(x) = \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2(2M)!}$$

$$\left((-1)^i(x+1)^{2M}\sum_{j=0}^{i}\frac{(-i)(-i+1)\cdots(-i+j-1)(i+1)\cdots(i+j)}{(2M+1)\cdots(2M+j)j!}\left(\frac{1+x}{2}\right)^j\right.$$

$$\left.+(1-x)^{2M}\sum_{j=0}^{i}\frac{(-i)(-i+1)\cdots(-i+j-1)(i+1)\cdots(i+j)}{(2M+1)\cdots(2M+j)j!}\left(\frac{1-x}{2}\right)^j\right)$$

$$= \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2}\left((-1)^i\sum_{j=0}^{i}\frac{(-1)^j(i+j)!}{(2M+j)!\,j!\,(i-j)!2^j}(1+x)^{2M+j}\right.$$

$$\left.+\sum_{j=0}^{i}\frac{(-1)^j(i+j)!}{(2M+j)!\,j!\,(i-j)!2^j}(1-x)^{2M+j}\right)$$

$$= \frac{(-1)^M\sqrt{i+\frac{1}{2}}}{2}((-1)^i f_i(1+x)+f_i(1-x)), \tag{2.2}$$

where $f_i(x)$ is defined as follows:

$$f_i(x) = \sum_{j=0}^{i}\frac{(-1)^j(i+j)!}{(2M+j)!\,j!\,(i-j)!2^j}x^{2M+j}.$$

Proof of Theorem 1.2. Substituting (1.3) and (2.2) into (1.6), we have

$$g_{ij} = \int_{-1}^{1}\varphi_j(x)\psi_i(x)dx$$

$$= \frac{(-1)^{j+M}\sqrt{i+\frac{1}{2}}\sqrt{j+\frac{1}{2}}}{2^{j+1}j!}$$

$$\times\left\{(-1)^i\int_{-1}^{1}(D^j(1-x^2)^j)f_i(1+x)+\int_{-1}^{1}(D^j(1-x^2)^j)f_i(1-x)dx\right\}.$$

Putting $x = -t$ and using the fact that $D^i(1 - x^2)^i$ is even or odd function according as i is even or odd, we have

$$\int_{-1}^{1} (D^j(1 - x^2)^j) f_i(1 - x) dx = (-1)^j \int_{-1}^{1} (D^j(1 - t^2)^j) f_i(1 + t) dt$$

$$= (-1)^j \int_{-1}^{1} (D^j(1 - x^2)^j) f_i(1 + x) dx.$$

Hence the following relation holds:

$$g_{ij} = \begin{cases} 0 & (i - j : \text{odd}), \\ \dfrac{(-1)^M \sqrt{i + \frac{1}{2}} \sqrt{j + \frac{1}{2}}}{2^j j!} \displaystyle\int_{-1}^{1} (D^j(1 - x^2)^j) f_i(1 + x) dx & (i - j : \text{even}). \end{cases} \tag{2.3}$$

We consider the case $i - j$ is even. Integral on the right hand side of (2.3) is calculated as follows:

$$\int_{-1}^{1} (D^j(1 - x^2)^j) \sum_{k=0}^{i} \frac{(-1)^k (i + k)!}{(2M + k)!\, k!\, (i - k)! 2^k} (1 + x)^{2M+k} dx$$

$$= \sum_{k=0}^{i} \frac{(-1)^k (i + k)!}{(2M + k)!\, k!\, (i - k)! 2^k} \int_{-1}^{1} (D^j(1 - x^2)^j)(1 + x)^{2M+k} dx$$

$$= (-1)^j \sum_{k=0}^{i} \frac{(-1)^k (i + k)!}{(2M + k)!\, k!\, (i - k)! 2^k}$$

$$(2M + k)(2M + k - 1) \cdots (2M + k - j + 1) \int_{-1}^{1} (1 - x^2)^j (1 + x)^{2M+k-j} dx$$

$$= (-1)^j \sum_{k=0}^{i} \frac{(-1)^k (i + k)!}{(2M + k - j)!\, k!\, (i - k)! 2^k} \int_{-1}^{1} (1 - x)^j (1 + x)^{2M+k} dx$$

$$= (-1)^j \sum_{k=0}^{i} \frac{(-1)^k (i + k)!}{(2M + k - j)!\, k!\, (i - k)! 2^k} 2^{2M+k+j+1} B(j + 1, 2M + k + 1)$$

$$= (-1)^j \sum_{k=0}^{i} \frac{(-1)^k (i+k)!}{(2M+k-j)!\, k!\, (i-k)!} \, 2^{2M+j+1} \, \frac{j!\, (2M+k)!}{(2M+k+j+1)!}$$

$$= (-1)^j \, 2^{2M+j+1} j! \sum_{k=0}^{i} \frac{(-1)^k (2M+k)!\, (i+k)!}{(2M+k-j)!\, (2M+k+j+1)!\, k!\, (i-k)!}.$$

Substituting the above result into (2.3), we have proved the theorem.

3. Properties of Green Functions

This section presents some important properties of the Green function.

Theorem 3.1. *The Green function* $G(x, y)$ *satisfies the following properties*:

(1) $\partial_x^{2M} G(x, y) = (-1)^{M-1} \sum_{i=0}^{M-1} \varphi_i(x)\varphi_i(y) \quad (x \neq y)$,

(2) $\partial_x^{M+i} G(x, y)|_{x=\pm 1} = 0 \quad (0 \leq i \leq M-1)$,

(3) $\partial_x^{i} G(x, y)|_{x=y-0} - \partial_x^{i} G(x, y)|_{x=y+0} = \begin{cases} 0 & (0 \leq i \leq 2M-2), \\ (-1)^{M-1} & (i = 2M-1), \end{cases}$

(4) $\int_{-1}^{1} G(x, y)\varphi_i(x)dx = 0 \quad (0 \leq i \leq M-1)$.

Proof of Theorem 3.1. We first prove (1). Taking x-derivative of $G(x, y)(x \neq y)$ $2M$ times, we have

$$\partial_x^{2M} G(x, y) = -\sum_{i=0}^{M-1} \psi_i^{(2M)}(x)\varphi_i(y).$$

Hence it is enough to find $\psi_i^{(2M)}(x)$. Differentiating (1.5) j times $(0 \leq j \leq 2M-1)$, we have

$$\psi_i^{(j)}(x) = \frac{(-1)^M}{2}\left(\frac{d}{dx}\right)^j\left(\int_{-1}^{x} K_0(x-y)\varphi_i(y)dy - \int_{x}^{1} K_0(x-y)\varphi_i(y)dy\right)$$

$$= \frac{(-1)^M}{2}\left(\int_{-1}^{x} K_j(x-y)\varphi_i(y)dy - \int_{x}^{1} K_j(x-y)\varphi_i(y)dy\right). \tag{3.1}$$

Putting $j = 2M - 1$ and differentiating once again, we obtain

$$\psi_i^{(2M)}(x) = (-1)^M \varphi_i(x)$$

which proves (1). Next we show (2) in the case $x = 1$,

$$\partial_x^{M+i} G(x, y) = \partial_x^{M+i} G_0(x, y) - \sum_{j=0}^{M-1} \psi_j^{(M+i)}(x)\varphi_j(y). \tag{3.2}$$

First term of (3.2) is calculated as follows:

$$\partial_x^{M+i} G_0(x, y) = \begin{cases} \dfrac{(-1)^i}{2} K_{M+i}(y-x) & (x < y), \\ \dfrac{(-1)^M}{2} K_{M+i}(x-y) & (y < x). \end{cases}$$

Putting $x = 1$, we have

$$\partial_x^{M+i} G_0(x, y)|_{x=1} = \frac{(-1)^M}{2} K_{M+i}(1-y).$$

Since the above function is a polynomial in y of degree $M - 1 - i$, it is expanded as follows:

$$\partial_x^{M+i} G_0(x, y)|_{x=1} = \sum_{j=0}^{M-1-i} a_j \varphi_j(y), \tag{3.3}$$

where

$$a_j = \frac{(-1)^M}{2} \int_{-1}^{1} K_{M+i}(1-\eta)\varphi_j(\eta)d\eta \quad (0 \le j \le M - 1 - i). \tag{3.4}$$

Next we consider the second term of (3.2),

$$\sum_{j=0}^{M-1} \psi_j^{(M+i)}(x)\varphi_j(y)$$

$$= \sum_{j=0}^{M-1} \varphi_j(y) \frac{(-1)^M}{2} \left(\int_{-1}^{x} K_{M+i}(x-y)\varphi_j(y)dy - \int_{x}^{1} K_{M+i}(y-x)\varphi_j(y)dy \right).$$

Putting $x = 1$, we have

$$\sum_{j=0}^{M-1} \psi_j^{(M+i)}(1)\varphi_j(y) = \sum_{j=0}^{M-1} \varphi_j(y) \frac{(-1)^M}{2} \int_{-1}^{1} K_{M+i}(1-y)\varphi_j(y)dy$$

$$= \sum_{j=0}^{M-1-i} \varphi_j(y) \frac{(-1)^M}{2} \int_{-1}^{1} K_{M+i}(1-y)\varphi_j(y)dy,$$

where we have used the fact that orthogonality relation $\int_{-1}^{1} \varphi_j(y) K_{M+i}(1-y)dy = 0$ holds for $j \geq M - i$ because $K_{M+i}(1-y)$ is a polynomial of $(M-1-i)$-th degree. Comparing the above expression with (3.3), (3.4), we can conclude that (2) holds for $x = 1$. The case $x = -1$ is proved by taking the same procedures. (3) and (4) are easy and so we omit the proof.

The following theorem, which states an important aspect of $G(x, y)$ as a reproducing kernel, is a direct consequence of Theorem 3.1.

Theorem 3.2. *$G(x, y)$ is a reproducing kernel of the following Hilbert space $(H_M, (\cdot, \cdot))$:*

$$H_M = \left\{ u^{(M)} \in L^2(-1, 1) \,\middle|\, \int_{-1}^{1} \varphi_i(x)u(x)dx = 0 \quad (0 \leq i \leq M-1) \right\},$$

$$(u, v)_M = \int_{-1}^{1} u^{(M)}(x)v^{(M)}(x)dx \quad (u, v \in H_M).$$

That is to say

(1) *$G(x, y)$, as a function in x, belongs to H_M for arbitrarily fixed $y \in [-1, 1]$.*

(2) *The following reproducing relation holds for arbitrary* $u \in H_M$,

$$u(y) = (u(\cdot), G(\cdot, y))_M = \int_{-1}^{1} u^{(M)}(x)\partial_x^M G(x, y)dx. \tag{3.5}$$

Proof of Theorem 3.2. (1) is easy to prove. We prove the reproducing relation (3.5),

$$(u(\cdot), G(\cdot, y))_M = \int_{-1}^{1} u^{(M)}(x)\partial_x^M G(x, y)dx$$

$$= \left[\sum_{i=1}^{M-1}(-1)^{i-1}u^{(M-i)}(x)\partial_x^{M-1+i}G(x, y)\right]_{x=-1}^{x=1}$$

$$+ (-1)^{M-1}\int_{-1}^{1} u'(x)\partial_x^{2M-1}G(x, y)dx.$$

The first term of the right hand side is equal to 0 owing to the property (2) of Theorem 3.1. Hence we have

$$= (-1)^{M-1}\left(\int_{-1}^{y-0} + \int_{y+0}^{1}\right)u'(x)\partial_x^{2M-1}G(x, y)dx$$

$$= (-1)^{M-1}[u(x)\partial_x^{2M-1}G(x, y)]_{x=-1}^{x=y-0} + (-1)^{M-1}[u(x)\partial_x^{2M-1}G(x, y)]_{x=y+0}^{x=1}$$

$$+ (-1)^M\int_{-1}^{1} u(x)\partial_x^{2M}G(x, y)dx$$

$$= (-1)^{M-1}u(y)\{\partial_x^{2M-1}G(x, y)|_{x=y-0} - \partial_x^{2M-1}G(x, y)|_{x=y+0}\}$$

$$+ (-1)^M\int_{-1}^{1} u(x)\partial_x^{2M}G(x, y)dx.$$

By using the properties (1) and (3) of Theorem 3.1, this is rewritten as

$$= u(y) - \sum_{i=0}^{M-1}\varphi_i(y)\int_{-1}^{1} u(x)\varphi_i(x)dx = u(y)$$

which completes the proof.

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