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STRONG CONVERGENCE OF MODIFIED MANN ITERATIONS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY SMOOTH BANACH SPACES

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Abstract

In this paper, we use the modified Mann iteration method introduced by Kim and Xu [7] to show the strong convergence for asymptotically nonexpansive mappings in uniformly smooth Banach spaces. The results presented in this paper affirmatively answered the question proposed by Kim and Xu [8].

1. Introduction

Let K be a nonempty closed convex subset of real normed linear space E. A mapping $T: K \to K$ is said to be *nonexpansive* if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in K$. A mapping $T: K \to K$ is called *asymptotically nonexpansive* if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \| \tag{1.1}$$

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for all $x, y \in K$ and each $n \geq 1$. A mapping $T : K \to K$ is said to be *uniformly asymptotically regular* if $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$ for all $x \in K$. In this paper, we use $F(T)$ to denote the fixed point set of T, i.e., $F(T) = \{x \in K : Tx = x\}$.

The interest and importance of construction of fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas, such as image recovery and signal processing (see, e.g., [1, 2, 12, 13]).

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see, e.g., [3, 5, 6, 15]), using the famous Mann iteration process or the Ishikawa iteration process.

Mann iteration procedure is a sequence $\{x_n\}$ which is generated in the following recursive way

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0 \tag{1.2}$$

where the initial value $x_0 \in K$ is chosen arbitrarily. However, the Mann iterations for nonexpansive mapping have only weak convergence even in Hilbert space.

Some attempts to modified the Mann iteration (1.1) so that strong convergence is guaranteed have recently been made, such as Nakajo and Takahashi [10] and Kim and Xu [7]. In [7], Kim and Xu introduced a modified Mann iteration sequence $\{x_n\}$ in the following way: For arbitrary $x_0 \in K$

$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$

$$x_{n+1} = \beta_n u + (1 - \beta_n) y_n, \quad n \geq 0 \tag{1.3}$$

where $u \in K$ is an arbitrary (but fixed) element in K, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. In uniformly smooth Banach space, they obtained the following strong convergence theorem for nonexpansive mappings.

Theorem 1 [7]. *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in* (0, 1), *the following conditions are satisfied*:

(1) $\alpha_n \to 0, \beta_n \to 0$;

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n = \infty$;

(3) $\sum_{n=0}^{\infty} | \alpha_n - \alpha_{n-1} | < \infty, \sum_{n=0}^{\infty} | \beta_n - \beta_{n-1} | < \infty$

the sequence $\{x_n\}$ is defined by (1.3). *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Being an important generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972, who proved that if K is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on K, then T has a fixed point.

In 2006, Kim and Xu [8] proposed whether the modified Mann iterations (1.3) can be adapted to asymptotically nonexpansive mappings similar to nonexpansive mappings in [8]. In this paper, we construct the following iteration method to approximate a fixed point of asymptotically nonexpansive mapping T: Fixed a $u \in K$, for arbitrary $x_1 \in K$

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ x_{n+1} &= \beta_n u + (1 - \beta_n) y_n, \quad n \geq 1 \end{aligned} \tag{1.4}$$

and obtain the strong convergence of $\{x_n\}$ under some appropriate assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, which affirmatively answered the question proposed by Kim and Xu [8]. Meanwhile, we prove that the fixed point set $F(T)$ of T is sunny nonexpansive retract of K if K is a nonempty bounded closed convex subset of a uniformly smooth Banach space E.

2. Preliminaries

For the sake of convenience, we restate the following concepts and lemmas.

A closed convex subset K of E is said to be *retract* if there exists continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : E \to E$ is said to be a *retraction* if $P^2 = P$. A mapping P from E onto K is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t > 0$. A subset K of E is said to be *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E on K.

Note. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P. In addition, if E is a Hilbert space, then the metric projection P_K is a sunny nonexpansive retraction from E to any closed convex subset K of E.

Lemma 2.1 [16]**.** *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following*

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \quad n \geq 0$$

where $\{\beta_n\} \subset (0, 1)$ *and* $\{\delta_n\}$ *are such that* (1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$; (2) $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \leq 0$. *Then* $\{a_n\}$ *converges to zero.*

Lemma 2.2 [11]**.** *Let E be a real Banach space. Then the following inequality holds*: *for all* $x, y \in E$, *we have*

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.3 [14]**.** *Let E be a uniformly smooth Banach space and K be a convex subset of E. Let C be a nonempty subset of K and P be a retraction from K onto C. Then P is sunny and nonexpansive if and only if for each* $u \in K$ *and* $q \in C$, $\langle Pu - u, J(Pu - q)\rangle \leq 0$.

Theorem LX [9]**.** *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E and* $T : K \to K$ *be an* *asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1, \infty)$. *Fix a* $u \in K$ *and* $\{Z_n\}$ *defined by*

$$Z_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n} T^n Z_n,$$

where $\{t_n\} \subset (0, 1)$ *is a sequence such that* $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. *If* $\lim_{n\to\infty} \| Z_n - TZ_n \| = 0$, *then* $\{Z_n\}$ *converges strongly to a fixed point of* T.

3. Main Results

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a Banach space* E *and* $T : K \to K$ *be an asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1, \infty)$. *For a fixed* $u \in K$ *and arbitrary* $x_1 \in K$, *suppose that* $\{x_n\}$ *is generated by* (1.4), *where* $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ *and* $\{k_n\}$ *satisfy the following conditions*:

(1) $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(2) $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=2}^{\infty} | \alpha_n - \alpha_{n-1} | < \infty$, $\sum_{n=2}^{\infty} | \beta_n - \beta_{n-1} | < \infty$;

(3) $\lim_{n\to\infty} \frac{\| T^n x_{n-1} - T^{n-1} x_{n-1} \|}{\beta_n} = 0$.

If $F(T)$ *is nonempty, then* $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.

Proof. Firstly, we show that $\{x_n\}$ is bounded. Taking $p \in F(T)$, then

$$\begin{aligned} \| y_n - p \| &\le \alpha_n \| x_n - p \| + (1 - \alpha_n) k_n \| x_n - p \| \\ &\le k_n \| x_n - p \| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \| x_{n+1} - p \| &\le \beta_n \| u - p \| + (1 - \beta_n) k_n \| x_n - p \| \\ &\le k_n \max\{\| u - p \|, \| x_n - p \|\}. \end{aligned} \tag{3.2}$$

By induction, we have

$$\| x_{n+1} - p \| \leq k_1 k_2 \cdots k_n \max\{\| u - p \|, \| x_1 - p \|\}.$$

Setting $k_n - 1 = u_n$. Thus

$$\| x_{n+1} - p \| \leq (1 + u_1)(1 + u_2) \cdots (1 + u_n) \max\{\| u - p \|, \| x_1 - p \|\}$$

$$< e^{\sum_{n=1}^{\infty} u_n} \max\{\| u - p \|, \| x_1 - p \|\}.$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} u_n < \infty$. Therefore $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{T^n x_n\}$.

It follows from (1.4) that

$$\| y_n - T^n x_n \| = \alpha_n \| x_n - T^n x_n \| \to 0 \quad \text{as } n \to \infty, \tag{3.3}$$

$$\| x_{n+1} - y_n \| = \beta_n \| u - y_n \| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Further, we have

$$\| x_{n+1} - T^n x_n \| \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Secondly, we prove that $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$.

$$\| x_{n+1} - x_n \| = \| \beta_n u + (1 - \beta_n) y_n - \beta_{n-1} u + (1 - \beta_{n-1}) y_{n-1} \|$$

$$\leq | \beta_n - \beta_{n-1} | \| u \| + (1 - \beta_n) \| y_n - y_{n-1} \| + | \beta_n - \beta_{n-1} | \| y_{n-1} \|, \tag{3.6}$$

where

$$\| y_n - y_{n-1} \| = \| \alpha_n x_n + (1 - \alpha_n) T^n x_n - \alpha_{n-1} x_{n-1} - (1 - \alpha_{n-1}) T^{n-1} x_{n-1} \|$$

$$\leq \alpha_n \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n-1} | \| x_{n-1} \| + (1 - \alpha_n) \| T^n x_n - T^n x_{n-1} \|$$

$$+ | \alpha_n - \alpha_{n-1} | \| T^{n-1} x_{n-1} \| + (1 - \alpha_{n-1}) \| T^n x_{n-1} - T^{n-1} x_{n-1} \|$$

$$\leq k_n \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n-1} | (\| x_{n-1} \| + \| T^{n-1} x_{n-1} \|)$$

$$+ \| T^n x_{n-1} - T^{n-1} x_{n-1} \|. \tag{3.7}$$

Since $\{x_n\}$, $\{y_n\}$ and $\{T^n x_n\}$ are bounded, there exists $M > 0$ such that $\max\{\| u \|, \| y_n \|, \| x_n \|, \|T^n x_n\|\} \leq M$ for all positive integers n. Thus, from (3.6) and (3.7), we have

$$\| x_{n+1} - x_n \| \leq (1 - \beta_n)k_n \| x_n - x_{n-1} \| + 2M| \beta_n - \beta_{n-1} | + 2M| \alpha_n - \alpha_{n-1} | + \| T^n x_{n-1} - T^{n-1} x_{n-1} \|. \quad (3.8)$$

Since $\{\beta_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\lim_{n\to\infty} \frac{k_n - 1}{\beta_n} = 0$, i.e., $k_n - 1 = o(\beta_n)$, thus there exists positive real sequence $\{d_n\}$ such that $k_n - 1 = \beta_n d_n$, where $\lim_{n\to\infty} d_n = 0$. It follows from (3.8) that

$$\| x_{n+1} - x_n \| \leq [1 - \beta_n(1 - d_n + \beta_n d_n)] \| x_n - x_{n-1} \| + 2M(| \beta_n - \beta_{n-1} | + | \alpha_n - \alpha_{n-1} |) + \| T^n x_{n-1} - T^{n-1} x_{n-1} \|.$$

Since $\lim_{n\to\infty} d_n = 0$, there exist $\delta \in (0, 1)$ and positive integer N such that $1 - d_n + \beta_n d_n > \delta$ as $n \geq N$. Therefore, as $n \geq N$

$$\| x_{n+1} - x_n \| \leq (1 - \delta\beta_n) \| x_n - x_{n-1} \| + 2M(| \beta_n - \beta_{n-1} | + | \alpha_n - \alpha_{n-1} |) + \| T^n x_{n-1} - T^{n-1} x_{n-1} \|.$$

It follows from Lemma 2.1 that $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$. In addition,

$$\| x_n - T^n x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T^n x_n \|.$$

Further, it follows from (3.5) that $\lim_{n\to\infty} \| x_n - T^n x_n \| = 0$.

Finally, we show that $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. Since

$$\begin{aligned} \| x_n - Tx_n \| &\leq \| Tx_n - T^{n+1} x_n \| + \| T^{n+1} x_n - T^{n+1} x_{n+1} \| \\ &\quad + \| T^{n+1} x_{n+1} - x_{n+1} \| + \| x_{n+1} - x_n \| \\ &\leq k_1 \| x_n - T^n x_n \| + k_{n+1} \| x_{n+1} - x_n \| \\ &\quad + \| T^{n+1} x_{n+1} - x_{n+1} \| + \| x_{n+1} - x_n \|. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. The proof is completed.

Theorem 3.2. *Let K be a nonempty bounded closed convex subset of a uniformly smooth Banach space E and $T : K \to K$ be a uniformly asymptotically regular and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. For a fixed $u \in K$ and arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is generated by* (1.4). *If $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{k_n\}$ satisfy the conditions as in Lemma* 3.1, *then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Under the conditions of our Theorem 3.2, it follows from Corollary 1 in [9] that the fixed point set $F(T)$ of T is nonempty. To show our result, let sequence $\{Z_n\}$ be defined as in Theorem LX and we may choose that $\{t_n\} \subset (0, 1)$ satisfies the conditions as in Theorem LX, i.e., $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Since K is bounded and $Z_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n} T^n Z_n$, we have

$$\| Z_n - T^n Z_n \| = \left(1 - \frac{t_n}{k_n}\right)\| u - T^n Z_n \| \to 0 \quad \text{as} \quad n \to \infty \tag{3.9}$$

and

$$\begin{aligned} \| Z_n - TZ_n \| &\leq \| Z_n - T^n Z_n \| + \| T^n Z_n - T^{n+1} Z_n \| + \| T^{n+1} Z_n - TZ_n \| \\ &\leq (1 + k_1) \| Z_n - T^n Z_n \| + \| T^n Z_n - T^{n+1} Z_n \|. \end{aligned}$$

Thus, it follows from (3.9) and the uniformly asymptotically regularity of T that $\lim_{n\to\infty} \| Z_n - TZ_n \| = 0$. Therefore $\{Z_n\}$ converges strongly to a fixed point of T by Theorem LX. We may set $q = Pu = s - \lim_{n\to\infty} Z_n$.

We claim that $\limsup_{n\to\infty} \langle Pu - u, J(Pu - x_n)\rangle \leq 0$

$$\begin{aligned} \| Z_m - x_n \|^2 &= \left\| \left(1 - \frac{t_m}{k_m}\right)(u - x_n) + \frac{t_m}{k_m}(T^m Z_m - x_n) \right\|^2 \\ &\leq \left(\frac{t_m}{k_m}\right)^2 \| T^m Z_m - x_n \|^2 + 2\left(1 - \frac{t_m}{k_m}\right)\langle u - x_n, J(z_m - x_n)\rangle \end{aligned}$$

$$\leq \left(\frac{t_m}{k_m}\right)^2 [k_m^2 \| Z_m - x_n \|^2$$

$$+ (2k_m \| Z_m - x_n \| + \| T^m x_n - x_n \|) \| T^m x_n - x_n \|]$$

$$+ 2\left(1 - \frac{t_m}{k_m}\right) \langle u - Z_m, J(Z_m - x_n) \rangle + 2\left(1 - \frac{t_m}{k_m}\right) \| Z_m - x_n \|^2.$$

Setting $L_n = \frac{t_m^2}{k_m^2}(2k_m \| Z_m - x_n \| + \| T^m x_n - x_n \|)$. Since K is bounded and $\lim_{n\to\infty} k_n = 1$, there exists $M > 0$ such that $\max\{L_n, k_m^2 \| Z_m - x_n \|^2\} \leq M$ for all positive integers m and n. Thus we have

$$2\left(1 - \frac{t_m}{k_m}\right) \langle Z_m - u, J(Z_m - x_n) \rangle$$

$$\leq \left(1 - \frac{t_m}{k_m}\right)^2 M + M \| T^m x_n - x_n \|$$

$$\leq \left(1 - \frac{t_m}{k_m}\right)^2 M + M(\| x_n - Tx_n \| + \| Tx_n - T^2 x_n \| + \cdots$$

$$+ \| T^{m-1} x_n - T^m x_n \|)$$

$$\leq \left(1 - \frac{t_m}{k_m}\right)^2 M + M(1 + k_1 + k_2 + \cdots + k_{m-1}) \| x_n - Tx_n \|.$$

Hence

$$\langle Z_m - u, J(Z_m - x_n) \rangle$$

$$\leq \frac{1 - \frac{t_m}{k_m}}{2} M + \frac{M(1 + k_1 + k_2 + \cdots + k_{m-1}) \| x_n - Tx_n \|}{1 - \frac{t_m}{k_m}}. \tag{3.10}$$

Letting $n \to \infty$ in (3.10), it follows from Lemma 3.1 that

$$\limsup_{n\to\infty} \langle Z_m - u, J(Z_m - x_n) \rangle \leq \frac{1 - \frac{t_m}{k_m}}{2} M. \tag{3.11}$$

Since K is bounded and E is uniformly smooth Banach space, the duality mapping J is norm-to-norm uniformly continuous on bounded subset of E. Letting $m \to \infty$ in (3.11), it is easily found that the two limits can be interchanged. Therefore, $\limsup_{n\to\infty}\langle Pu - u, J(Pu - x_n)\rangle \le 0$.

Finally, we show that $\{x_n\}$ converges strongly to the fixed point Pu of T.

It follows from Lemma 2.2 that

$$\| x_{n+1} - Pu \|^2 = \| \beta_n(u - Pu) + (1 - \beta_n)(y_n - Pu) \|^2$$

$$\le (1 - \beta_n)^2 k_n^2 \| x_n - Pu \|^2 + 2\beta_n \langle u - Pu, J(x_{n+1} - Pu)\rangle.$$

Since $\{\beta_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty}\beta_n = \infty$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\lim_{n\to\infty}\frac{k_n - 1}{\beta_n} = 0$, i.e., $k_n - 1 = o(\beta_n)$. Hence there exists positive real sequence $\{d_n\}$ such that $k_n - 1 = \beta_n d_n$, where $\lim_{n\to\infty} d_n = 0$. Thus

$$\| x_{n+1} - Pu \|^2 \le (1 - \beta_n)(1 + 2\beta_n d_n + \beta_n^2 d_n^2)\|x_n - Pu\|^2$$

$$+ 2\beta_n \langle u - Pu, J(x_{n+1} - Pu)\rangle$$

$$= [1 - \beta_n(1 - 2d_n - \beta_n d_n^2 + 2\beta_n d_n + \beta_n^2 d_n^2)] \| x_n - Pu \|^2$$

$$+ 2\beta_n \langle u - Pu, J(x_{n+1} - Pu)\rangle.$$

Since $\lim_{n\to\infty} d_n = 0$, without loss of generality, we may assume that $1 - 2d_n - \beta_n d_n^2 + 2\beta_n d_n + \beta_n^2 d_n^2 > \frac{1}{2}$ for all positive integers n. Hence,

$$\| x_{n+1} - Pu \|^2 \le \left(1 - \frac{\beta_n}{2}\right) \| x_n - Pu \|^2 + 2\beta_n \langle u - Pu, J(x_{n+1} - Pu)\rangle.$$

It follows from Lemma 2.1 that $\lim_{n\to\infty}\| x_n - Pu \| = 0$. The proof is completed.

The fixed point set $F(T)$ of T is nonempty under the conditions in Theorem 3.2. We now prove that $F(T)$ is sunny nonexpansive retract of K.

Theorem 3.3. *Let K be a nonempty bounded closed convex subset of a uniformly smooth Banach space E and $T : K \to K$ be a uniformly asymptotically regular and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Then the fixed point set $F(T)$ of T is sunny nonexpansive retract of K.*

Proof. Since $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, we may choose a real sequence $\{t_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Let sequence $\{Z_n\}$ be defined as in Theorem LX. It is easily shown that $\lim_{n\to\infty} \| Z_n - TZ_n \| = 0$ because T is uniformly asymptotically regular. Thus, it follows from Theorem LX that $\{Z_n\}$ converges strongly to a fixed point of T. Let P be a mapping from K onto $F(T)$ and $Pu = s - \lim_{n\to\infty} Z_n$. For each $q \in F(T)$, we have (the proof of Theorem 2 of [9])

$$\langle Z_m - u, J(Z_m - q)\rangle \leq S_m d^2, \tag{3.12}$$

where d is the diameter of K and $S_m = t_m \frac{k_m - 1}{k_m - t_m} \to 0$ as $m \to \infty$. It is a fact that J is norm-to-norm uniformly continuous on bounded subset K of E since E is uniformly smooth Banach space. Therefore, letting $m \to \infty$ in (3.12), for all $q \in F(T)$ we have

$$\langle Pu - u, J(Pu - q)\rangle \leq 0.$$

It follows from Lemma 2.3 that $F(T)$ is sunny nonexpansive retract of K and P is a sunny nonexpansive retraction from K onto $F(T)$. The proof is completed.

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