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EXISTENCE OF SOLUTIONS OF A DIFFERENTIAL INCLUSION WITH NONLINEAR BOUNDARY CONDITIONS BY THE LOWER AND UPPER SOLUTIONS METHOD

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Abstract

We prove by the lower and upper solutions method, the existence of solutions of the differential inclusion boundary value problem

$$\begin{cases} u''(t) \in [\varphi(t, x), \psi(t, x)] & \forall\, t \in I, \\ g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)), \end{cases}$$

where φ and ψ are two maps such that φ is upper semicontinuous and ψ is lower semicontinuous, g_1 and g_2 are continuous maps which verify some monotony conditions.

1. Problem Statement

Given a set A, the set of all subsets of A is denoted by 2^A. $\varnothing$ designates the empty set and for a given set Ω, $C^k(\Omega)$ is the set of k times continuously differentiable functions $f : \Omega \to \mathbb{R}$.

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Let $a < b$ be two real numbers,

$$I = [a, b] = \{t \in \mathbb{R} : a \leq t \leq b\},$$

$$\mathring{I} =]a, b[= \{t \in \mathbb{R} : a < t < b\},$$

and $\varphi, \psi : I \times \mathbb{R} \to \mathbb{R}$ be two maps such that φ is upper semicontinuous (u.s.c.) and ψ is lower semicontinuous (l.s.c.) satisfying

$$\varphi(t, x) \leq \psi(t, x) \quad \forall (t, x) \in I \times \mathbb{R}.$$

Consider the set value map $F : I \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \varnothing$ defined by

$$F(t, x) = [\varphi(t, x), \psi(t, x)] = \{v \in \mathbb{R} : \varphi(t, x) \leq v \leq \psi(t, x)\}.$$

Our objective is to extend the domain of application of the lower and upper solutions method to the class of the differential inclusion boundary value problems of the type

$$\begin{cases} u''(t) \in F(t, u(t)) \quad \forall t \in I, \\ g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)), \end{cases} \tag{1}$$

where $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous maps that verify:

(H1) : $\forall x \in \mathbb{R}$, the function $y \to g_1(x, y)$ is decreasing;

(H2) : $\forall x \in \mathbb{R}$, the function $y \to g_2(x, y)$ is increasing.

The lower and upper solutions method was initiated by Dragoni [8] in 1931 for a Dirichlet problem. Since then, a large number of contributions have enriched the theory, notably its extension to the Nagumo's conditions [12] in 1937 and to the Carathéodory's conditions [9] in 1938. More recent results have been found by Mawhin and Schmitt [11], Adjé [1, 2, 3], Coster and Habets [5, 6] and Frigon [10].

Our contribution in this paper consists in defining notions of lower and upper solutions for the differential inclusion boundary value problem (1) and to establish a result concerned with the existence of solutions.

2. Lower and Upper Solutions

Here we give a definition of the notion of lower and upper solutions of the problem (1) that permits to establish that the existence of a lower solution α and a upper solution β such that $\alpha(\beta t) \leq \alpha(t)$ $\forall t \in I$, guarantees the existence of a solution $u \in C^2(I)$ of the problem (1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ $\forall t \in I$.

Definition 1. (1) A function $\alpha \in C^2(\mathring{I}) \cap C^1(I)$ is a *lower solution* of (1) if

(i) $\forall t \in \mathring{I}$, $\alpha''(t) \geq \psi(t, \alpha(t))$;

(ii) $g_1(\alpha(a), \alpha'(a)) \leq 0$, $g_2(\alpha(b), \alpha'(b)) \leq 0$.

(2) A function $\beta \in C^2(\mathring{I}) \cap C^1(I)$ is a *upper solution* of (1) if

(i) $\forall t \in \mathring{I}$, $\beta''(t) \leq \varphi(t, \beta(t))$;

(ii) $g_1(\beta(a), \beta'(a)) \geq 0$, $g_2(\beta(b), \beta'(b)) \geq 0$.

Theorem 1. *Assume that there exist a lower solution* α *and a upper solution* β *of the problem* (1) *such that* $\alpha(t) \leq \beta(t)$ $\forall t \in I$.

Then the problem (1) *has at least one solution* $u \in C^2(I)$ *satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \forall t \in I.$$

3. Proof of Theorem 1

The proof will be done in four steps. First, we show that the set value map F admits a continuous selection f, then we justify that the solution of the selection problem is the solution of the original problem. To solve the selection problem, we introduce a modified problem whose solutions are set between α and β and therefore are solutions of the selection problem. Finally we show that the modified problem has a solution.

Step 1. Existence of a continuous selection

We are going to prove that F admits at least one continuous selection. We know that $\forall (t, x) \in I \times \mathbb{R}$, $F(t, x) = [\varphi(t, x), \psi(t, x)]$ is convex and close. We need to show again that F is l.s.c. in accordance with the selection theorem of Michael. Let U be an open subset of $\mathbb{R}$, $(t_n, x_n)_{n \in \mathbb{N}^*}$ a sequence of elements from $I \times \mathbb{R}$ converging towards $(t_o, x_o) \in I \times \mathbb{R}$ and such that $F(t_o, x_o) \cap U \neq \emptyset$. Then

$$\overline{\lim_{n \to \infty}} \varphi(t_n, x_n) \leq \varphi(t_o, x_o) \leq \psi(t_o, x_o) \leq \underline{\lim_{n \to \infty}} \psi(t_n, x_n),$$

we deduce that there exists $N_o \in \mathbb{N}$ such that for $n \geq N_o$,

$$F(t_n, x_n) \cap U \neq \emptyset.$$

So $F^{-1}(U)$ is an open subset of $I \times \mathbb{R}$. Thus F is l.s.c. by the selection theorem of Michael (see [7, p. 303]), there exists a continuous function $f : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\forall (t, x) \in I \times \mathbb{R}, \quad f(t, x) \in F(t, x) = [\varphi(t, x), \psi(t, x)].$$

Step 2. Substitution of the initial differential inclusion by an ordinary differential equation: the selection problem

f being a continuous selection of F, every solution of the following problem

$$\begin{cases} u''(t) = f(t, u(t)) \quad \forall t \in I, \\ g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)), \end{cases} \tag{2}$$

is a solution of the problem (1). We will just to show that the problem (2) possesses at least one solution. This proof is based on the study of the modified problem

$$\begin{cases} u''(t) = f(t, \gamma(t, u(t))) + u(t) - \gamma(t, u(t)) \quad \forall t \in I, \\ u(a) = \gamma(a, u(a) + g_1(\gamma(a, u(a)), u'(a))), \\ u(b) = \gamma(b, u(b) + g_2(\gamma(b, u(b)), u'(b))), \end{cases} \tag{3}$$

where γ is the continuous function from $I \times \mathbb{R}$ into $\mathbb{R}$ defined by

$$\gamma(t, x) = \max[\alpha(t), \min(x, \beta(t))] = \begin{cases} \alpha(t) & \text{if} \quad x < \alpha(t), \\ x & \text{if} \quad \alpha(t) \leq x \leq \beta(t), \\ \beta(t) & \text{if} \quad x > \beta(t). \end{cases}$$

The continuation of the proof will be done in two steps. First we are going to show that a solution u of the problem (3) satisfies the inequality

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \forall t \in I,$$

and is therefore a solution of problem (2). Then we are going to show that (3) admits at least one solution.

Step 3. All solutions of problem (3) are wedged between α and β

Let u be a solution of (3). We will show that

$$\alpha(t) \leq u(t) \quad \forall t \in I.$$

Suppose that there exists $t_o \in I$ such that $\min_{t \in I}(u(t) - \alpha(t)) = u(t_o) - \alpha(t_o)$ < 0. Then $\gamma(t_o, u(t_o)) = \alpha(t_o)$.

- If $t_o \in \overset{\circ}{I}$, then $u'(t_o) - \alpha'(t_o) = 0$ and $u''(t_o) - \alpha''(t_o) \geq 0$.

From

$$u''(t_o) = f(t_o, \alpha(t_o)) + u(t_o) - \alpha(t_o) \leq \alpha''(t_o) + u(t_o) - \alpha(t_o),$$

we have the contradiction $0 \leq u''(t_o) - \alpha''(t_o) \leq u(t_o) - \alpha(t_o) < 0$.

- If $t_o = a$, that is to say $\min_{t \in I}(u(t) - \alpha(t)) = u(a) - \alpha(a) < 0$, then we have

$$u'(a) - \alpha'(a) \geq 0.$$

From

$$u(a) = \gamma(a, u(a) + g_1(\alpha(a), u'(a)))$$

and from (H1)

$$g_1(\alpha(a), u'(a)) \leq g_1(\alpha(a), \alpha'(a)) \leq 0,$$

we get that

$$u(a) + g_1(\alpha(a), u'(a)) \leq u(a) < \alpha(a);$$

which leads to the contradiction

$$\alpha(a) > u(a) = \gamma(a, u(a) + g_1(\alpha(a), u'(a))) = \alpha(a).$$

- If $t_o = b$, that is, to say $\min_{t \in I}(u(t) - \alpha(t)) = u(b) - \alpha(b) < 0$, then we have

$$u'(b) - \alpha'(b) \leq 0.$$

Using (H2) and the fact that $u(b) = \gamma(b, u(b) + g_2(\alpha(b), u'(b)))$, we obtain the contradiction

$$\alpha(b) > u(b) = \gamma(b, u(b) + g_2(\alpha(b), u'(b))) = \alpha(b).$$

Then $\forall t \in I$, $\alpha(t) \leq u(t)$.

In the same way, we prove that $u(t) \leq \beta(t)$ $\forall t \in I$.

Step 4. Existence of solution for the problem (3)

We are now going to show, via Schauder's fixed point theorem [7, p. 60], that (3) admits at least one solution. Let set down $X = C(I)$, $Z = C(I) \times \mathbb{R}^2$ and consider the operator $L : C^2(I) \subset X \to Z$ defined by

$$Lu = (u'' - u, u(a), u(b)).$$

L is linear and bijective and hence is a Fredholm mapping of index zero [1, p. 167]. Moreover, L^{-1} is compact. The function $N : X \to Z$ defined by

$$\begin{aligned} Nu(t) = (&f(t, \gamma(t, u(t))) - \gamma(t, u(t))), \\ &\gamma(a, u(a) + g_1(\gamma(a, u(a)), u'(a))), \\ &\gamma(b, u(b) + g_2(\gamma(b, u(b)), u'(b))), \end{aligned}$$

is continuous and bounded on $C(I)$. Indeed, the function f being continuous on the compact

$$K = I \times [\min_{t \in I} \alpha(t), \max_{t \in I} \beta(t)],$$

is bounded there. Then N is L-completely continuous mapping, so that $L^{-1}N$ is compact and by Schauder's fixed point theorem, $L^{-1}N : X \to X$ has a fixed point which is the solution of (3).

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