Far East J. Appl. Math. **26**(2) (2007), 203-211

POTENTIAL THEORY METHOD AND VOLTERRA-FREDHOLM INTEGRAL EQUATION

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Abstract

In this work, the potential theory method is used to obtain the eigenvalue and eigenfunction of Volterra-Fredholm integral equation of the first kind. The Volterra integral term is measured with respect to time, while Fredholm term is measured with respect to position. The kernel of Fredholm integral term is considered in the logarithmic function form.

1. Introduction

The mechanics mixed problems of continuous media have been studied by many authors (see [5, 6, 10]). Prostenko and Prostenko [11] used the potential theory method for solving the problem about the contact of a thin plate in the form of an infinite strip lying on an elastic frictionless half-space in a three dimensional formulation. In [4], Abdou and Hassan obtained the spectral relationships for the Fredholm integral equation of the first kind with logarithmic kernel. Also in [1] the eigenvalue and the eigenfunction are obtained for the Fredholm integral equation of the first kind with Carleman kernel. The importance of Carleman kernel came from the work of Arytiunian [7] who has shown

2000 Mathematics Subject Classification: 45D01, 45L04.

Keywords and phrases: potential theory method, Volterra-Fredholm integral equation, logarithmic function, Chebyshev polynomial.

Received February 5, 2005

that, the contact problem of approximation reduce to a Fredholm integral equation of the first kind with Carleman kernel. Abdou in [2, 3], using potential theory method, obtained the spectral relationships for the Fredholm integral equation of the first kind with generalized potential kernel and Macdonald kernel, respectively.

In this work, our aim is solving the boundary value problem of Volterra-Fredholm integral equation of the first kind in the space $L_2(\Omega) \times C[0, T], T < \infty,$ Ω is the domain of integration with respect to position. The Volterra integral term is measured with respect to time, while the Fredholm integral term is measured with respect to position. Using a numerical method the Volterra-Fredholm integral equation transformed to a linear system of Fredholm integral equation. Using potential theory method, the Fredholm integral system can be solved as a system of partial differential equation.

2. Volterra-Fredholm Integral Equation

Consider the following integral equation

$$\int_0^t \int_{-1}^1 F(t, \tau) k(x, y) \phi(y, \tau) dy d\tau = \pi[\gamma(t) - f_*(x)] = \pi f(x, t) \qquad (2.1)$$

under the condition

$$\int_{-1}^1 \phi(x, t) dx = P(t). \qquad (2.2)$$

Here, the given function $F(t, \tau),$ which represents the kernel of Volterra integral term, is positive and continuous in the class $C[0, T],$ for all values of the time $t, \tau \in [0, T], T < \infty.$ The function $k(x, y),$ which behaved badly in the domain $[-1, 1],$ is called the *kernel* of Fredholm integral term. The given function $f(x, t)$ is continuous with its partial derivatives with respect to position and time, and belongs to the space $L_2[-1, 1] \times C[0, T].$ The unknown function $\phi(x, t)$ is called the *potential function* of the integral equation, and its result will be discussed in the space $L_2[-1, 1] \times C[0, T].$

In order to guarantee the existence of unique solution of (2.1) under the condition (2.2), we assume the following:

(i) The kernel of position $k(x, y) \in C([-1, 1] \times [-1, 1])$ and satisfies the following $\left[\int_{-1}^{1}\int_{-1}^{1} |k(x, y)|^2 dydx\right]^{\frac{1}{2}} = A,\ A$ is a constant.

(ii) For all values of $t, \tau \in [0, T], T < \infty,$ the function $F(t, \tau)$ is a positive continuous and satisfies $F(t, \tau) < B,\ B$ is a constant.

(iii) The continuous function $f(x, t) \in L_2[-1, 1] \times C[0, T]$ and its norm $\| f(x, t) \| = \max_t \int_0^t (f^2(x, \tau)dx)^{1/2} d\tau.$

(iv) The unknown function $\phi(x, t)$ satisfies Hölder condition with respect to time

$$|\phi(x, t_1) - \phi(x, t_2)| \leq E(x)|t_1 - t_2|^{\upsilon} \qquad 0 < \upsilon < 1$$

and Lipschitz condition with respect to position

$$|\phi(x_1, t) - \phi(x_2, t)| \leq H(t)|x_1 - x_2|,$$

where $E(x)$ and $H(t)$ are continuous functions in x and t, respectively.

The integral equation (2.1) is investigated from the contact problem of a rigid surface (G, υ) having an elastic material, where G is the displacement magnitude, υ is Poisson's coefficient. If a stamp of length 2 units, whose surface is described by the function $f_*(x)$, is impressed into an elastic layer surface of a stamp by a variable force $P(t)$, $0 \leq t \leq T < \infty$, with eccentricity of application $e(t)$, then its rigid displacement is $\gamma(t)$. The function $F(t, \tau)$ represents the resistance force of material in the domain contact through the time $t \in [0, T]$.

3. System of Fredholm Integral Equation

To obtain the solution of (2.1) under the condition (2.2), we divide the interval $[0, T], 0 \leq t \leq T < \infty$, as $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, where

$t = t_l,\ l = 0, 1, 2, \ldots, N,$ and get

$$\int_0^{t_l}\int_{-1}^{1} F(t_l, \tau)k(x, y)\phi(y, \tau)dyd\tau = f(x, t_l). \tag{3.1}$$

Hence, we have

$$\sum_{j=0}^{l} u_j F(t_l, t_j)\int_{-1}^{1} k(x, y)\phi(y, t_j)dy + O(\hbar_l^{p+1}) = f(x, t_l),$$

$$(\hbar_k^{p+1} \to 0,\ p > 0), \tag{3.2}$$

where $\hbar_l = \max_j h_j,\ h_j = t_{j+1} - t_j$.

The values of u_j and p are depending on the number of derivatives of $F(t, \tau)$ with respect to time. For example if $F(t, \tau) \in C^4[0, T]$, then we have $p = 4,\ l \approx 4,$ also $u_0 = \frac{1}{2}h_0,\ u_4 = \frac{1}{2}h_4,\ u_i = h_i,\ i = 1, 2, 3.$ More information for characteristic points and the quadrature coefficients are found in [8, 9].

Using the following notations

$$\phi_j(x) = \phi(x, t_j),\ \ f_l(x) = f(x, t_l),\ \ F_{l,j} = F(t_l, t_j),\ \ l = 0, 1, \ldots, N, \tag{3.3}$$

the formula (3.2) can be adapted in the form

$$\sum_{j=0}^{l} u_j F_{l,j}\int_{-1}^{1} k(x, y)\phi_j(y)dy = f_l(x). \tag{3.4}$$

Also, the boundary condition (2.2) becomes

$$\int_{-1}^{1} \phi_l(x)dx = P_l. \tag{3.5}$$

4. Contact Problem with Logarithmic Kernel

Consider the system of integral equations

$$\sum_{j=0}^{l} u_j F_{l,j}\int_{-1}^{1} k\left(\frac{x-y}{\lambda}\right)\phi_j(y)dy = \pi f_l(x), \quad \lambda \in (0, \infty), \tag{4.1}$$

$$k(z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuz} du, \quad i = \sqrt{-1} \tag{4.2}$$

under the static condition

$$\int_{-1}^{1} \phi_j(x) dx = P_j < \infty. \tag{4.3}$$

The boundary value problem (4.1)-(4.3) represents the contact problem of a strip occupying the region $0 \le y \le h$, made of material satisfies Hook's law (see [6]). The strip, in the absence of mass force, lies without fraction on a rigid support, a system of rectangular stamps is impressed into the boundary of a strip $y = h$. Assume the frictional forces in the contact area between the stamps and the strip are small, so it can be neglected. Also, assume the resistance force of the material is a function of time, and the width of the area of contact is independent of the magnitude of the force applied.

As in [1, p. 32], we can write the kernel in the form

$$k(z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuz} du = -\ln\left|\tanh \frac{\pi z}{4}\right|, \quad \left(z = \frac{x - y}{\lambda},\ \lambda \in (0, \infty)\right). \tag{4.4}$$

If $\lambda \to \infty$ and $(x - y)$ is very small, so that it satisfies the condition $\tanh \upsilon = \upsilon$, then we have

$$\ln\left|\tanh \frac{\pi z}{4}\right| = \ln|z| - d, \quad \left(d = \ln \frac{4\lambda}{\pi}\right). \tag{4.5}$$

Here, the kernel (4.2) takes the form

$$k(z) = [-\ln|x - y| + d].$$

Hence, equation (4.1) becomes

$$\sum_{j=0}^{l} u_j F_{l,j} \int_{-1}^{1} \left[\ln \frac{1}{|x - y|} + d\right] \phi_j(y) dy = \pi f_l(x) \tag{4.6}$$

under the condition (4.2).

For solving equation (4.6), using potential theory method, we introduce the logarithmic potential function

$$U_l(x,\ \upsilon) = \sum_{j=0}^{l} u_j F_{l,\,j} \int_{-1}^{1} \left[\ln \frac{1}{\sqrt{(x-y)^2 + \upsilon^2}} + d \right] \phi_j(y) dy. \tag{4.7}$$

Equations (4.7) and (4.3) reduce to the Dirichlet boundary value problem

$$\begin{aligned}
&\Delta U_l(x,\ \nu) = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \nu^2} \quad ((x,\ \nu) \notin (-1,\ 1)),\\
&U_l(x,\ \nu)|_{\nu=0} = \pi f_l(x), \quad (x \in (-1,\ 1)),\\
&U_l(x,\ y) \approx u_j F_{j,\,l} P_l\left(\ln \frac{1}{r} + d \right), \quad r = \sqrt{x^2 + \nu^2},\\
&P_l\left(\ln \frac{1}{r} + d \right) \to \text{ finite term} \quad (\text{as } r \to \infty).
\end{aligned} \tag{4.8}$$

The solution of the integral equation (4.6) is equivalent to the solution of the Dirichlet problem (4.8). After the function $U_l(x,\ \nu)$ in (4.8) has been constructed, the density of the potential $\phi_j(x)$ will be determined from the formula

$$\phi_l(x) = -\frac{1}{\pi} \lim_{\nu \to 0} \operatorname{sgn} \nu \cdot \frac{\partial U_j(x,\ \nu)}{\partial \nu}, \quad (x \in (-1,\ 1)). \tag{4.9}$$

Assume the density source function

$$W_l(x,\ \nu) = U_l(x,\ \nu) - u_j F_{j,\,l} P_l\left(\ln \frac{1}{r} + d \right), \tag{4.10}$$

so, equation (4.8) can be written as

$$\begin{aligned}
&\Delta W_l(x,\ \nu) = 0, \quad ((x,\ \nu) \notin (-1,\ 1)),\\
&W_l(x,\ \nu)|_{\nu=0} = \pi f_l(x) - u_j F_{j,\,l} P_l(\ln|\ x\ | - d), \quad (x \in (-1,\ 1)),\\
&W_l(x,\ \nu) \to 0, \quad (\text{as } r \to \infty).
\end{aligned} \tag{4.11}$$

Consequently equation (4.9) is transferred to

$$\phi_l(x) = -\frac{1}{\pi} \lim_{\nu \to 0} \operatorname{sgn} \nu \cdot \left[\frac{\partial W_l(x,\ \nu)}{\partial \nu} - \pi u_j F_{j,\,l} P_l \delta(x) \right], \tag{4.12}$$

where $\delta(x)$ is the Dirac-delta function.

We construct the solution of the boundary value problem (4.11) by the method of conformal mapping (see [12]), that transforms a given complicated region into a simpler one.

To this end, we note that the mapping function

$$z = \frac{1}{2} w(\xi) = \frac{1}{2}(\xi + \xi^{-1}), \quad (\xi = \rho e^{i\theta},\ z = x + iy,\ c = \sqrt{-1}), \tag{4.13}$$

maps the region in (x, y) plane into the region outside the unite circle γ, such that $w'(\xi)$ does not vanish or becomes infinite outside the unite circle γ. The mapping function (4.13) maps the upper and the lower half-plane $((x, y) \in (-1, 1))$ into the lower and the upper of the semi-circle $\rho = 1$, respectively.

Moreover, the point $z = \infty$ will be mapped onto the point $\xi = 0$.

Using the parametric equation of (4.13) and under the condition (4.2), we can rewrite the density source of the logarithmic potential of equation (4.10) in the form

$$W_l(\rho, \theta) = U_{l_0}(\rho, \theta) - u_j F_{j,l} P_l(2\rho + d),$$

$$U_{l_0}(\rho, \theta) = U_l\left(\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\theta_l,\ \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\theta_l\right). \tag{4.14}$$

In view of equation (4.14) the boundary value problem of (4.11) is transformed to

$$\frac{\partial^2 W_l}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W_l}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W_l}{\partial \theta^2} = 0 \quad (\rho \leq 1,\ -\pi < \theta < \pi),$$

$$W_l(\theta, \theta_l) = 0,$$

$$W_l(1, \theta_l) = f_{l_0}(\theta_l) - u_j F_{j,l} P_l(\ln 2 + d),$$

$$f_{l_0}(\theta_l) = f_l(\cos\theta_l). \tag{4.15}$$

Consequently after using the chain rule, equation (4.12) is transformed to

$$\phi_l(\cos\theta_l) = (\pi/\sin\theta_l)^{-1}\left[u_j F_{j,l} P_l + \frac{\partial W_l}{\partial \rho}\right]_{\rho=1}. \tag{4.16}$$

To solve the Dirichlet problem of (4.15), we use the Fourier series method (see [5])

$$W_l(\rho,\ \theta_l) = \sum_{n=0}^{\infty} \alpha_{ln}\rho^n \cos\theta_l, \quad -\pi < \theta_l < \pi,$$

$$\alpha_{ln} = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{l_0}(\theta)\cos n\theta d\theta, \quad \alpha_{l0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{l_0}(\theta)d\theta. \tag{4.17}$$

Substituting (4.17) in (4.15), then using the differentiating result in (4.16) (see [5]), we obtain

$$\phi_l(\cos\theta_l) = \begin{cases} \pi\cos\theta_l(\pi\sin\theta_l)^{-1} & n = 1,\ 2,\ \ldots \\ P_l(\pi\sin\theta_l)^{-1} & n = 0 \end{cases} \tag{4.18}$$

and

$$P_l = (2\pi(\ln 2 + d))^{-1}\int_{-\pi}^{\pi} f_{l_0}(\theta)d\theta. \tag{4.19}$$

Finally, substituting (4.18) in (4.1), we have the following relationship:

$$\sum_{j=1}^{l} u_j F_{j,l}\int_{-1}^{1}\left(\ln\frac{1}{|x-y|} + d\right)\frac{T_n(\nu)}{\sqrt{1-\nu^2}}d\nu = \begin{cases} u_j F_{j,l}\pi(\ln 2 + d) & n = 0, \\ \dfrac{\pi}{n}u_j F_{j,l}T_n(x) & n \geq 1, \end{cases}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind.

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