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GENERALIZED MONOTONE ITERATIVE METHOD AND FIRST ORDER INITIAL VALUE PROBLEMS

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Abstract

A unified approach to monotone iterative technique, concerning the existence of coupled maximal and minimal solutions of the dynamical systems, is extended to the case which involves a nonlinear term admitting a composition of a non-monotone part and two monotone parts.

1. Introduction

The investigation of the constructive procedure for obtaining the solutions of the nonlinear problems is undergoing rapid development, as the method of lower and upper solutions coupled with the monotone iterative technique is applied in a wide variety of fields.

An excellent and comprehensive introduction [2], presented some results of the monotone method, in which the nonlinear term is

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nonincreasing or nondecreasing, or could be made nondecreasing by adding a linear function. Recently, the monotone method was effectively generalized and extended for the nonlinear term involving the difference of two monotone functions and we refer the reader to [1, 3, 4]. This motivates us to attempt a generalized monotone method under two types of coupled upper and lower solutions, with less restrictive assumptions on the nonlinear term. That is, by not demanding the nonlinear term to be the difference of two monotone parts, we should consider the following initial value problem:

$$u' = F(t, u) = f(t, u) - g(t, u) + h(t, u), \quad u(0) = u_0, \quad t \in J = [0, T], \tag{1.1}$$

where $T > 0$, $F \in C[J \times R, R]$, $f(t, u)$ and $g(t, u)$ are nondecreasing in u, and $h(t, u)$ is non-monotone in u.

Consequently, the paper is organized as follows. In Section 2, the possibility of upper and lower solutions for the initial value problem (1.1), which are going to play a fundamental role in the subsequent parts, should be introduced. In Section 3, by employing the diverse iterative schemes, we consider coupled upper and lower solutions of Type (Ia) and (Ib), and discuss two sequences converging to coupled minimal and maximal solutions, respectively.

2. Preliminaries

Consider the initial value problem

$$u' = f(t, u) - g(t, u) + h(t, u), \quad u(0) = u_0, \quad t \in J \equiv [0, T], \tag{2.1}$$

where $f, g, h \in C[J \times R, R]$, f and g are nondecreasing in u, uniformly in t.

Definition 2.1. With respect to problem (2.1), the *functions* $\alpha, \beta \in C^1[J, R]$ are

(i) natural lower and upper solutions if

$$\alpha' \leq f(t, \alpha) - g(t, \alpha) + h(t, \alpha), \;\; \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \beta) - g(t, \beta) + h(t, \beta), \;\; \beta(0) \geq u_0,$$

(ii) coupled lower and upper solutions of Type (Ia) if

$$\alpha' \leq f(t, \alpha) - g(t, \beta) + h(t, \alpha), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \beta) - g(t, \alpha) + h(t, \beta), \ \ \beta(0) \geq u_0,$$

(iii) coupled lower and upper solutions of Type (Ib) if

$$\alpha' \leq f(t, \alpha) - g(t, \beta) + h(t, \beta), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \beta) - g(t, \alpha) + h(t, \alpha), \ \ \beta(0) \geq u_0,$$

(iv) coupled lower and upper solutions of Type (IIa) if

$$\alpha' \leq f(t, \beta) - g(t, \alpha) + h(t, \beta), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \alpha) - g(t, \beta) + h(t, \alpha), \ \ \beta(0) \geq u_0,$$

(v) coupled lower and upper solutions of Type (IIb) if

$$\alpha' \leq f(t, \beta) - g(t, \alpha) + h(t, \alpha), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \alpha) - g(t, \beta) + h(t, \beta), \ \ \beta(0) \geq u_0,$$

(vi) coupled lower and upper solutions of Type (IIIa) if

$$\alpha' \leq f(t, \alpha) - g(t, \alpha) + h(t, \beta), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \beta) - g(t, \beta) + h(t, \alpha), \ \ \beta(0) \geq u_0,$$

(vii) coupled lower and upper solutions of Type (IIIb) if

$$\alpha' \leq f(t, \beta) - g(t, \beta) + h(t, \alpha), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \alpha) - g(t, \alpha) + h(t, \beta), \ \ \beta(0) \geq u_0,$$

(viii) coupled lower and upper solutions of Type (IIIc) if

$$\alpha' \leq f(t, \beta) - g(t, \beta) + h(t, \beta), \ \ \alpha(0) \leq u_0,$$

$$\beta' \geq f(t, \alpha) - g(t, \alpha) + h(t, \alpha), \ \ \beta(0) \geq u_0,$$

respectively.

Our attempt is to create iterative techniques converging maximal and minimal solutions for the initial value problem (2.1). We shall take a unified approach, considering all eight possible types of coupled lower and upper solutions, α and β described in the above definition. In fact, corresponding to each type of lower and upper solutions, after the line of brief analysis [3], we could immediately conclude that only four of these possible sequences are meaningful. The four sequences considered are

$$\begin{cases} \alpha'_{n+1} = f(t, \alpha_n) - g(t, \alpha_n) + h(t, \alpha_n), \ \alpha_{n+1}(0) = u_0, \\ \beta'_{n+1} = f(t, \beta_n) - g(t, \alpha_n) + h(t, \beta_n), \ \beta_{n+1}(0) = u_0, \end{cases} \tag{2.2}$$

$$\begin{cases} \alpha'_{n+1} = f(t, \alpha_n) - g(t, \alpha_n) + h(t, \beta_n), \ \alpha_{n+1}(0) = u_0, \\ \beta'_{n+1} = f(t, \beta_n) - g(t, \alpha_n) + h(t, \alpha_n), \ \beta_{n+1}(0) = u_0, \end{cases} \tag{2.3}$$

$$\begin{cases} \alpha'_{n+1} = f(t, \beta_n) - g(t, \alpha_n) + h(t, \beta_n), \ \alpha_{n+1}(0) = u_0, \\ \beta'_{n+1} = f(t, \alpha_n) - g(t, \alpha_n) + h(t, \alpha_n), \ \beta_{n+1}(0) = u_0, \end{cases} \tag{2.4}$$

$$\begin{cases} \alpha'_{n+1} = f(t, \beta_n) - g(t, \alpha_n) + h(t, \alpha_n), \ \alpha_{n+1}(0) = u_0, \\ \beta'_{n+1} = f(t, \alpha_n) - g(t, \beta_n) + h(t, \beta_n), \ \beta_{n+1}(0) = u_0. \end{cases} \tag{2.5}$$

Theorem 2.1. *For initial value problem* (2.1), *assume that*

(A1) $\alpha_0, \beta_0 \in C^1[J, R]$ *are coupled lower and upper solutions of Type* (Ia) *such that* $\alpha_0 \le \beta_0$, $t \in J$;

(A2) $h(t, u_1) - h(t, u_2) \ge M(u_1 - u_2)$, *for* $\alpha_0 \le u_2 \le u_1 \le \beta_0$, $M \ge 0$.

Then there exist monotone sequences $\alpha_n(t)$ *and* $\beta_n(t)$ *on* J *such that* $\alpha_n(t) \to \alpha(t)$ *and* $\beta_n(t) \to \beta(t)$ *uniformly and monotonically and* (α, β) *are coupled minimal and maximal solutions to problem* (2.1), *respectively. That is*, (α, β) *satisfy*

$$\begin{cases} \alpha' = f(t, \alpha) - g(t, \beta) + h(t, \alpha), \ \alpha(0) = u_0, \\ \beta' = f(t, \beta) - g(t, \alpha) + h(t, \beta), \ \beta(0) = u_0, \end{cases} \tag{2.6}$$

where the iterative scheme is defined by (2.2).

Proof. Since f, g and h are continuous functions on $J \times R$, the solutions to (2.2) exist and are unique on J for all $n \ge 1$.

We first claim that $\alpha_1(t) \geq \alpha_0(t)$ on J.

For the purpose, let $p(t) = \alpha_1(t) - \alpha_0(t)$, $t \in J$. Note that $p(0) = \alpha_1(0) - \alpha_0(0) \geq 0$. Then

$$p'(t) = \alpha_1'(t) - \alpha_0'(t)$$

$$\geq f(t, \alpha_0) - g(t, \beta_0) + h(t, \alpha_0) - f(t, \alpha_0) + g(t, \beta_0) - h(t, \alpha_0) = 0,$$

which implies $p(t) \geq p(0) \geq 0$ and hence $\alpha_1(t) \geq \alpha_0(t)$, $t \in J$. Similarly, we can show $\beta_1(t) \leq \beta_0(t)$ on J.

Assume that, for some n, $\alpha_n(t) \leq \alpha_{n+1}(t)$ and $\beta_n(t) \geq \beta_{n+1}(t)$ on J.

Next, consider $p(t) = \alpha_{n+2}(t) - \alpha_{n+1}(t)$. Again, $p(0) = \alpha_{n+2}(0) - \alpha_{n+1}(0) = 0$. And in view of (A2),

$$p'(t) = \alpha_{n+2}'(t) - \alpha_{n+1}'(t)$$

$$= f(t, \alpha_{n+1}) - g(t, \beta_{n+1}) + h(t, \alpha_{n+1}) - f(t, \alpha_n) + g(t, \beta_n) - h(t, \alpha_n)$$

$$\geq M(\alpha_{n+1} - \alpha_n) \geq 0.$$

It then follows that $\alpha_{n+1}(t) \leq \alpha_{n+2}(t)$ on J. Thus, by induction, we get $\alpha_n(t) \leq \alpha_{n+1}(t)$ for all $n \geq 1$, $t \in J$. Similarly, we can show that $\beta_n(t) \leq \beta_{n+1}(t)$ for all $n \geq 1$, $t \in J$.

Now, we shall show that $\alpha_1(t) \leq \beta_1(t)$ on J. Setting $p(t) = \beta_1(t) - \alpha_1(t)$. Again, $p(0) \geq 0$. Then

$$p'(t) = \beta_1'(t) - \alpha_1'(t)$$

$$= f(t, \beta_0) - g(t, \alpha_0) + h(t, \beta_0) - f(t, \alpha_0) + g(t, \beta_0) - h(t, \alpha_0)$$

$$\geq M(\beta_0 - \alpha_0) \geq 0$$

on J, for all $n \geq 1$.

Thus we have $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0$ on J. Using standard argument, we can show that the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$

converge uniformly and monotonically on J and we denote $\alpha = \lim_{n\to\infty} \alpha_n$ and $\beta = \lim_{n\to\infty} \beta_n$. Therefore, $\alpha(t)$ and $\beta(t)$ satisfy the initial value problem (2.6).

The final step is to show that $\alpha(t)$ and $\beta(t)$ are coupled minimal and maximal solutions of (2.1). That is, if $u(t)$ is any solution of (2.1) such that $\alpha_0(t) \le u(t) \le \beta_0(t)$ on J, then we get

$$\alpha_0(t) \le \alpha(t) \le u(t) \le \beta(t) \le \beta_0(t), \quad t \in J.$$

Suppose that for some n, $\alpha_n(t) \le u(t) \le \beta_n(t)$ on J. Let $p(t) = u(t) - \alpha_{n+1}(t)$. Note that $p(0) = 0$, we obtain $p(t) \ge 0$ implying $u(t) \ge \alpha_{n+1}(t)$, $t \in J$. Similarly, $u(t) \le \beta_{n+1}(t)$ on J. Clearly, $\alpha_{n+1}(t) \le u(t) \le \beta_{n+1}(t)$, $t \in J$. Again, by induction $\alpha_n(t) \le u(t) \le \beta_n(t)$, for all $n \ge 1$, $t \in J$. Now, taking the limit as $n \to \infty$, we get $\alpha(t) \le u(t) \le \beta(t)$, $t \in J$. This completes the proof.

Theorem 2.2. *For initial value problem* (2.1), *assume that conditions of Theorem* 2.1 *hold. Then for any solution* $u(t)$ *of* (2.1) *with* $\alpha_0(t) \le u(t) \le \beta_0(t)$, $t \in J$, *there exist the alternating sequences* $\{\alpha_{2n}, \beta_{2n+1}\}$ *and* $\{\beta_{2n}, \alpha_{2n+1}\}$ *satisfying*

$$\begin{aligned} \alpha_0 &\le \beta_1 \le \cdots \le \alpha_{2n} \le \beta_{2n+1} \le u \le \alpha_{2n+1} \\ &\le \beta_{2n} \le \cdots \le \alpha_1 \le \beta_0, \quad t \in J, \end{aligned} \tag{2.7}$$

for all $n \ge 1$, *where the iterative is given by* (2.4).

Moreover, the monotone sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ *converge to* $s(t)$ *and* $\{\beta_{2n}, \alpha_{2n+1}\}$ *converge to* $r(t)$ *on* J, *where* (s, r) *are coupled minimal and maximal solutions of* (2.1), *respectively, satisfying*

$$\begin{cases} s' = f(t, r) - g(t, s) + h(t, r), & s(0) = u_0, \\ r' = f(t, s) - g(t, r) + h(t, s), & r(0) = u_0, \end{cases} \tag{2.8}$$

with $s \le u \le r$ *on* J.

Proof. We first claim that $\alpha_1(t) \geq \alpha_0(t)$ on J.

For the purpose, let $p(t) = \alpha_1(t) - \alpha_0(t)$, $t \in J$. Note that $p(0) = \alpha_1(0) - \alpha_0(0) \geq 0$.

In view of (A2), we have

$$\begin{aligned} p'(t) &= \alpha_1'(t) - \alpha_0'(t) \\ &\geq f(t, \beta_0) - g(t, \alpha_0) + h(t, \beta_0) - f(t, \alpha_0) + g(t, \beta_0) - h(t, \alpha_0) \\ &\geq M(\beta_0 - \alpha_0) \geq 0, \end{aligned}$$

which implies $p(t) \geq p(0) \geq 0$ and hence $\alpha_1(t) \geq \alpha_0(t)$, $t \in J$. Similarly, we can show $\beta_1(t) \leq \beta_0(t)$ on J.

Next, our aim is to show that (2.7) is true. Let $u(t)$ be any solution of (2.1) such that $\alpha_0(t) \leq u(t) \leq \beta_0(t)$ on J. We begin by verifying inequalities (2.7) for $n = 1$. That is,

$$\alpha_0 \leq \beta_1 \leq \alpha_2 \leq \beta_3 \leq u \leq \alpha_3 \leq \beta_2 \leq \alpha_1 \leq \beta_0, \quad t \in J, \tag{2.9}$$

holds. Setting $p(t) = u(t) - \alpha_1(t)$. Again, $P(0) = 0$. Using (A2) and the fact that $\alpha_0(t) \leq u(t) \leq \beta_0(t)$, $t \in J$, we obtain

$$\begin{aligned} p'(t) &= u'(t) - \alpha_1'(t) \\ &= f(t, u) - g(t, u) + h(t, u) - f(t, \beta_0) + g(t, \alpha_0) - h(t, \beta_0) \\ &\leq -M(\beta_0 - u) \leq 0. \end{aligned}$$

It then follows that $u(t) \leq \alpha_1(t)$ on J. Similarly, $u(t) \geq \beta_1(t)$ on J. In order to avoid repetition, on the same pattern, we can show $u(t) \geq \alpha_2(t)$, $u(t) \leq \beta_2(t)$, $u(t) \leq \alpha_3(t)$ and $u(t) \geq \beta_3(t)$, $t \in J$.

We shall now show that $\alpha_0(t) \leq \beta_1(t) \leq \alpha_2(t) \leq \beta_3(t)$ and $\alpha_3(t) \leq \beta_2(t) \leq \alpha_1(t) \leq \beta_0(t)$, $t \in J$. Setting $p(t) = \alpha_0(t) - \beta_1(t)$. Again, $p(0) \leq 0$. Clearly, $p'(t) \leq 0$ on J. Hence, $p(t) \leq 0$ on J which yields $\alpha_0(t) \leq \beta_1(t)$ on J. Similarly, we can show that $\alpha_1(t) \leq \beta_0(t)$, $\beta_1(t) \leq \alpha_2(t)$, $\alpha_1(t) \geq \beta_2(t)$, $\alpha_2(t) \leq \beta_3(t)$, and $\alpha_3(t) \leq \beta_2(t)$ on J. Therefore, (2.9) is true.

Now, suppose that for some $n > 2$, the inequalities

$$\beta_{2n-1} \le \alpha_{2n} \le \beta_{2n+1} \le u \le \alpha_{2n+1} \le \beta_{2n} \le \alpha_{2n-1}, \quad t \in J,$$

holds. It can be shown, by employing arguments similar to the above ones, that $\beta_{2n+1} \le \alpha_{2n+2} \le \beta_{2n+3} \le u \le \alpha_{2n+3} \le \beta_{2n+2} \le \alpha_{2n+1}$ on J holds. Thus, by induction, the chain of (2.7) is valid for all $n \ge 0$.

By standard argument, we can conclude that the sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ and $\{\beta_{2n}, \alpha_{2n+1}\}$ converge to s and r, respectively, on J. Thus, the coupled system (2.8) is satisfied.

Finally, we claim that $s(t)$ and $r(t)$ are the coupled minimal and maximal solutions of (2.1), that is, if $u(t)$ is any solution of (2.1) with $\alpha_0(t) \le u(t) \le \beta_0(t)$ on J, then $\alpha_0(t) \le s(t) \le u(t) \le r(t) \le \beta_0(t)$, $t \in J$. We have already shown that $\alpha_{2n} \le \beta_{2n+1} \le u \le \alpha_{2n+1} \le \beta_{2n}$ holds on J for all $n \ge 1$. Taking the limit as $n \to \infty$, we get $s(t) \le u(t) \le r(t)$ on J. This completes the proof.

Remark 2.1. If $h(t, u) = 0$ for the initial value problem (2.1), [1] and [4] come as two special cases of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *For initial value problem* (2.1), *in addition to the hypothesis* (A1) *of Theorem* 2.1, *assume that*

$$h(t, u_1) - h(t, u_2) \le -M(u_1 - u_2), \text{ for } \alpha_0 \le u_2 \le u_1 \le \beta_0, \quad M \ge 0. \tag{2.10}$$

Further, if the iterative scheme is constructed by (2.5).

Then the conclusions of Theorem 2.2 *hold, but with* $s(t)$ *and* $r(t)$ *we have*

$$\begin{cases} s' = f(t, r) - g(t, s) + h(t, s), & s(0) = u_0, \\ r' = f(t, s) - g(t, r) + h(t, r), & r(0) = u_0. \end{cases}$$

Proof. Analogous to the proof of Theorem 2.2.

Theorem 2.4. *For initial value problem* (2.1), *assume that*

(B1) $\alpha_0, \beta_0 \in C^1[J, R]$ *are coupled lower and upper solutions of Type* (Ib) *such that* $\alpha_0 \le \beta_0$, $t \in J$;

(B2) *Condition* (2.10) *of Theorem* 2.3 *holds*.

Further, *if the iterative scheme is developed by* (2.3).

Then the conclusions of Theorem 2.1 *hold*, *but* $\alpha(t)$ *and* $\beta(t)$ *satisfy*

$$\begin{cases} \alpha' = f(t, \alpha) - g(t, \beta) + h(t, \beta), & \alpha(0) = u_0, \\ \beta' = f(t, \beta) - g(t, \alpha) + h(t, \alpha), & \beta(0) = u_0. \end{cases}$$

Proof. Analogous to the proof of Theorem 2.1.

Theorem 2.5. *For initial value problem* (2.1) *in addition to hypotheses of Theorem* 2.4, *if the iterative scheme is created by* (2.5), *then the conclusions of Theorem* 2.3 *are obtained*.

Proof. Analogous to the proof of Theorem 2.2.

Remark 2.2. If $f, g \in C[J \times R, R]$ and there exist positive constants M and N such that $f(t, u) + Mu$ and $g(t, u) + Nu$ are both monotone nondecreasing functions, for $t \in J$, then the theorems in this paper can be applied.

Remark 2.3. If $g(t, u) = 0$ and f satisfies as in [2, Theorem 1.2.1], the theorems in this paper are valid. And consider $f(t, u) = 0$, we also obtain the similar discussion.

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