Far East J. Appl. Math. **26**(1) (2007), 111-119

REGULAR FACTORS IN 2-CONNECTED $K_{1,n}$-FREE GRAPHS

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Abstract

Let $n\,(\geq 3)$ and r be integers with $r \geq n-1$. We show that if G is a 2-connected $K_{1,n}$-free graph with $r|V(G)|$ even, and the minimum degree of G is at least

$$\max\left\{\frac{(nr-(n-1))^2}{4(n-1)r} + (n-1) - \frac{(n-1)((n-4)r+(n-1))}{2r} + \frac{(n-1)^3}{4r},\right.$$
$$\left.\frac{(nr-(n-1))^2}{4(n-1)r} + (n-1)\right\},$$

then G has an r-factor.

1. Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. Let G be a graph. Then we let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For disjoint subsets X and Y of $V(G)$, we let $E(X, Y)$ denote the set of edges of G joining X and Y. A vertex x is often identified with $\{x\}$, for example, when

2000 Mathematics Subject Classification: 05C70.

Keywords and phrases: regular factor, $K_{1,n}$-free graph, minimum degree.

Received October 15, 2006

$x \notin B$, we write $E(x, B)$ for $E(\{x\}, B)$. For $x \in V(G)$, we let $\deg_G(x)$ denote the degree of x in G, and we let $N(x) = N_G(x)$ denote the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For a subset X of $V(G)$, we let $N(X)$ denote the union of $N(x)$ as x ranges over X. A spanning subgraph F of a graph G with $\deg_F(v) = r$ for all $v \in V(G)$ is called an *r-factor*. A graph G is said to be $K_{1,n}$*-free* if it contains no $K_{1,n}$ as an induced subgraph.

Ota and Tokuda proved the following theorem:

Theorem A [2]**.** *Let $n (\geq 3)$ and r be positive integers. If r is odd, then we assume that $r \geq n-1$. Let G be a connected $K_{1,n}$-free graph with $r|V(G)|$ even, and suppose that the minimum degree of G is at least*

$$\left(n + \frac{n-1}{r}\right)\left\lceil \frac{n}{2(n-1)} r \right\rceil - \frac{n-1}{r}\left(\left\lceil \frac{n}{2(n-1)} r \right\rceil\right)^2 + n - 3.$$

Then G has an r-factor.

If we let $r = 2$ in Theorem A, then we obtain the following theorem:

Theorem B. *Let $n (\geq 3)$ be an integer. Let G be a connected $K_{1,n}$-free graph, and suppose that the minimum degree of G is at least $2n-2$. Then G has a 2-factor.*

The lower bounds on the minimum degree in Theorems A and B are sharp. On the other hand, as for Theorem B, Aldred et al. showed that if we confine ourselves to 2-connected graphs, then we can improve the bound on the minimum degree as follows:

Theorem C [1]**.** *Let $n (\geq 3)$ be an integer. Let G be a 2-connected $K_{1,n}$-free graph, and suppose that the minimum degree of G is at least n. Then G has a 2-factor.*

In this paper, we show that for 2-connected graphs, we can improve the bound on the minimum degree in Theorem A as follows:

Theorem. *Let $n (\geq 3)$ and r be integers with $r \geq n-1$. Let G be a*

2-connected $K_{1,n}$-free graph with $r|V(G)|$ even, and suppose that the minimum degree of G is at least

$$\delta := \max\left\{\frac{(nr-(n-1))^2}{4(n-1)r} + (n-1) - \frac{(n-1)((n-4)r+(n-1))}{2r} + \frac{(n-1)^3}{4r}, \frac{(nr-(n-1))^2}{4(n-1)r} + (n-1)\right\}.$$

Then G has an r-factor.

Let n, r, δ be as in Theorem, and assume that $n \geq 5$. Then $\delta = (nr-(n-1))^2/4(n-1)r + (n-1)$. The bound δ is sharp in the following sense. Assume that r is a multiple of $n-1$, and n and $r/(n-1)$ are odd. Set $d = (nr-(n-1))/2(n-1)$. Then d is an integer, and $\delta = (n-1)d^2/r + (n-1)$. Set $\delta' = \lceil \delta \rceil - 1$. Let p be a positive integer. We define a graph G of order $2p((\delta'-d)+(n-1)(d+1))$ as follows. Let L_i $(1 \leq i \leq 2p)$ be $2p$ disjoint copies of the complete graph of order $\delta'-d$. Let $M_{i,j}$ $(1 \leq i \leq 2p, 1 \leq j \leq n-1)$ be $2p(n-1)$ disjoint copies of the complete graph of order $d+1$ which are disjoint from $\bigcup_{1\leq i\leq 2p} L_i$. Write $V(L_i) = \{v_{i,1}, \ldots, v_{i,\delta'-d}\}$. Now define G by

$$V(G) = \left(\bigcup_{1\leq i\leq 2p} V(L_i)\right) \cup \left(\bigcup_{1\leq i\leq 2p, 1\leq j\leq n-1} V(M_{i,j})\right)$$

$$E(G) = \left(\bigcup_{1\leq i\leq 2p} E(L_i)\right) \cup \left(\bigcup_{1\leq i\leq 2p, 1\leq j\leq n-1} E(M_{i,j})\right)$$

$$\cup \left(\bigcup_{1\leq i\leq 2p} \left\{v_{i,k}x \,\middle|\, 1 \leq k \leq \delta'-d-1,\ x \in \bigcup_{1\leq j\leq n-1} V(M_{i,j})\right\}\right)$$

$$\cup \left(\bigcup_{1\leq i\leq 2p} \left\{v_{i,\delta'-d}x \,\middle|\, x \in \bigcup_{1\leq j\leq n-2} V(M_{i,j})\right\}\right)$$

$$\cup \left(\bigcup_{1 \le i \le 2p-1} \{v_{i,\delta'-d}x \mid x \in V(M_{i+1,n-1})\} \right)$$

$$\cup \{v_{2p,\delta'-d}x \mid x \in V(M_{1,n-1})\}.$$

Then G is 2-connected and $K_{1,n}$-free, and has minimum degree δ', but we easily see that G does not have an r-factor (for example, if we apply Theorem D, which we state in Section 2, with $S = \bigcup_{1 \le i \le 2p} V(L_i)$ and $T = \bigcup_{1 \le i \le 2p, 1 \le j \le n-1} V(M_{i,j})$, then we obtain $\theta(S, T) = -2pr(\delta - \delta') < 0$).

2. Proof of Theorem

The following criterion for the existence of an r-factor is essential for our proof:

Theorem D [3]. *Let r be a positive integer, and let G be a graph. Then G has an r-factor if and only if*

$$\theta(S, T) := r|S| + \sum_{x \in T} (\deg_{G-S}(x) - r) - h(S, T) \ge 0$$

for all disjoint subsets S and T of $V(G)$, where $h(S, T)$ denotes the number of components C of $G - S - T$ such that $r|V(C)| + |E(T, V(C))|$ is odd (such components are referred to as odd components).

Let n, r, δ, G be as in Theorem. Suppose that G does not have an r-factor. Then by Theorem D, there exist disjoint subsets S and T of $V(G)$ such that $\theta(S, T) < 0$. We choose S and T so that $|S \cup T|$ is as large as possible. Let $\mathcal{H}(S, T) = \{C \mid C$ is an odd component of $G - S - T\}$. Note that $h(S, T) = |\mathcal{H}(S, T)|$.

Claim 1. $|V(C)| \ge 2$ for each $C \in \mathcal{H}(S, T)$.

Proof. Suppose that there exists $C \in \mathcal{H}(S, T)$ such that $|V(C)| = 1$, and write $V(C) = \{w\}$. If $|E(T, w)| = r$, then $r|V(C)| + |E(T, V(C))| = 2r$, which means that C is not an odd component, a contradiction. Thus

$|E(T, w)| \neq r$. If $|E(T, w)| \leq r-1$, then since $\mathcal{H}(S, T \cup \{w\}) = \mathcal{H}(S, T) - \{C\}$, we get

$$\begin{aligned}\theta(S, T \cup \{w\}) &= \theta(S, T) + (\deg_{G-S}(w) - r) \\ &\quad - (h(S, T \cup \{w\}) - h(S, T)) \\ &= \theta(S, T) + (|E(T, w)| - r) + 1 \\ &\leq \theta(S, T) < 0,\end{aligned}$$

which contradicts the maximality of $|S \cup T|$. Thus $|E(T, w)| \geq r+1$. This implies

$$\begin{aligned}\sum_{x \in T} \deg_{G-(S \cup \{w\})}(x) &= \sum_{x \in T} \deg_{G-S}(x) - |E(T, w)| \\ &\leq \sum_{x \in T} \deg_{G-S}(x) - (r+1).\end{aligned}$$

We also have $\mathcal{H}(S \cup \{w\}, T) = \mathcal{H}(S, T) - \{C\}$. Consequently,

$$\begin{aligned}&\theta(S \cup \{w\}, T) \\ &= \theta(S, T) + r + \left(\sum_{x \in T} \deg_{G-(S \cup \{w\})}(x) - \sum_{x \in T} \deg_{G-S}(x)\right) \\ &\quad - (h(S \cup \{w\}, T) - h(S, T)) \\ &\leq \theta(S, T) + r - (r+1) + 1 = \theta(S, T) < 0,\end{aligned}$$

which again contradicts the maximality of $|S \cup T|$. □

Set

$$\mathcal{H}_1 = \{C \in \mathcal{H}(S, T) \,|\, |N(V(C)) \cap T| \geq 2\},$$

$$\mathcal{H}_2 = \{C \in \mathcal{H}(S, T) \,|\, |N(V(C)) \cap T| = 1, |E(T, V(C))| \geq 2\},$$

$$\mathcal{H}_3 = \{C \in \mathcal{H}(S, T) \,|\, |E(T, V(C))| \leq 1\},$$

$$a = |\mathcal{H}_1|, \quad b = |\mathcal{H}_2|, \text{ and } c = |\mathcal{H}_3|.$$

Thus

$$a + b + c = h(S, T). \tag{2.1}$$

For each $x \in T$, let

$$\alpha(x) = |\{C \in \mathcal{H}_1 \mid E(x, V(C)) \neq \emptyset\}|,$$

$$\beta(x) = |\{C \in \mathcal{H}_2 \mid E(x, V(C)) \neq \emptyset\}|.$$

Then

$$a \leq \sum_{x \in T} \alpha(x)/2 \quad \text{and} \quad b = \sum_{x \in T} \beta(x). \tag{2.2}$$

We define $x_1, x_2, \ldots, x_m$ and $N_1, N_2, \ldots, N_m$ inductively as follows: Assume for the moment that $T \neq \emptyset$. We let $x_1 \in T$ be the vertex such that $\deg_{G-S}(x_1) - \alpha(x_1)/2 - \beta(x_1)$ is minimum, and set $N_1 = (N(x_1) \cup \{x_1\}) \cap T$. Let now $i \geq 2$, and assume that $x_1, \ldots, x_{i-1}$ and $N_1, \ldots, N_{i-1}$ have been defined. If $T - \bigcup_{j<i} N_j \neq \emptyset$, then we let $x_i \in T - \bigcup_{j<i} N_j$ be the vertex such that $\deg_{G-S}(x_i) - \alpha(x_i)/2 - \beta(x_i)$ is minimum, and set $N_i = (N(x_i) \cup \{x_i\}) \cap \left(T - \bigcup_{j<i} N_j\right)$; if $T - \bigcup_{j<i} N_j = \emptyset$, then we let $m = i - 1$ and terminate this procedure. When $T = \emptyset$, we simply define $m = 0$. By the definition of x_i and N_i $(1 \leq i \leq m)$, $\{x_1, x_2, \ldots, x_m\}$ is an independent set of vertices in G, and T is the disjoint union of $N_1, N_2, \ldots, N_m$. We here prove the following claim:

Claim 2. $(n-1)|S| \geq \sum_{i=1}^{m} |E(x_i, S)| + c.$

Proof. In the case where $S = \emptyset$, since G is 2-connected, we have $c = 0$, which implies the desired inequality. Thus we may assume that $S \neq \emptyset$. Write $\mathcal{H}_3 = \{C_1, C_2, \ldots, C_c\}$. Since G is 2-connected, it follows from Claim 1 that for each $i\,(1 \leq i \leq c)$, there exists $v_i w_i \in E(G)$ such that $v_i \in S$, $w_i \in V(C_i)$ and $E(w_i, T) = \emptyset$. Let $A = \{x_1, x_2, \ldots, x_m, w_1, w_2, \ldots, w_c\}$. Then A is independent in G. Since G is $K_{1,n}$-free, this implies that every vertex $v \in S$ is adjacent to at most $n-1$ vertices in A. Therefore

$$(n-1)|S| \geq |E(A, S)| = \sum_{i=1}^{m} |E(x_i, S)| + \sum_{i=1}^{c} |E(w_i, S)|$$

$$\geq \sum_{i=1}^{m} |E(x_i, S)| + c,$$

as desired. □

By the definition of x_i and N_i $(1 \leq i \leq m)$, $\deg_{G-S}(x) - \alpha(x)/2 - \beta(x) \geq \deg_{G-S}(x_i) - \alpha(x_i)/2 - \beta(x_i)$ for every vertex $x \in N_i$, and hence

$$\sum_{x \in N_i} \left(\deg_{G-S}(x) - \frac{\alpha(x)}{2} - \beta(x) \right) \geq |N_i| \left(\deg_{G-S}(x_i) - \frac{\alpha(x_i)}{2} - \beta(x_i) \right). \quad (2.3)$$

Since $r \geq n-1$, it follows from Claim 2, (2.1), (2.2) and (2.3) that

$$\theta(S, T) \geq \frac{r}{n-1} \left(\sum_{i=1}^{m} |E(x_i, S)| + c \right)$$

$$+ \sum_{i=1}^{m} \left(\sum_{x \in N_i} (\deg_{G-S}(x) - r) \right) - (a + b + c)$$

$$\geq \frac{r}{n-1} \sum_{i=1}^{m} |E(x_i, S)| + \sum_{i=1}^{m} \left(\sum_{x \in N_i} (\deg_{G-S}(x) - r) \right) - (a + b)$$

$$\geq \frac{r}{n-1} \sum_{i=1}^{m} |E(x_i, S)| + \sum_{i=1}^{m} \left(\sum_{x \in N_i} \left(\deg_{G-S}(x) - \frac{\alpha(x)}{2} - \beta(x) - r \right) \right)$$

$$= \sum_{i=1}^{m} \left(\frac{r}{n-1} |E(x_i, S)| + \sum_{x \in N_i} \left(\deg_{G-S}(x) - \frac{\alpha(x)}{2} - \beta(x) - r \right) \right)$$

$$\geq \sum_{i=1}^{m} \left(\frac{r}{n-1} |E(x_i, S)| + |N_i| \left(\deg_{G-S}(x_i) - \frac{\alpha(x_i)}{2} - \beta(x_i) - r \right) \right).$$

Let $\theta_i = \frac{r}{n-1}|E(x_i, S)| + |N_i|\left(\deg_{G-S}(x_i) - \frac{\alpha(x_i)}{2} - \beta(x_i) - r\right)$. In order to derive a contradiction to the assumption that $\theta(S, T) < 0$, it suffices to show that $\theta_i \geq 0$ for each $i\,(1 \leq i \leq m)$. Thus we fix $i\,(1 \leq i \leq m)$, and set $d = \deg_{G-S}(x_i)$, $\alpha = \alpha(x_i)$ and $\beta = \beta(x_i)$. Then

$$|E(x_i, S)| = \deg_G(x_i) - d \geq \delta - d, \tag{2.4}$$

and

$$\begin{aligned}|N_i| &\leq |E(x_i, T - \{x_i\})| + 1\\ &\leq \deg_{G-S}(x_i) - \alpha(x_i) - 2\beta(x_i) + 1\\ &= d - \alpha - 2\beta + 1.\end{aligned} \tag{2.5}$$

If $d - \alpha/2 - \beta - r \geq 0$, then clearly $\theta_i \geq 0$. Thus we may assume that

$$d - \frac{\alpha}{2} - \beta - r < 0. \tag{2.6}$$

Set $X = \alpha/2 + \beta$. Since G is $K_{1,n}$-free, we have $\alpha + \beta \leq n - 1$, which implies

$$0 \leq X \leq n - 1. \tag{2.7}$$

It follows from (2.4), (2.5) and (2.6) that

$$\begin{aligned}\theta_i &\geq \frac{r}{n-1}(\delta - d) + (d - 2X + 1)(d - X - r)\\ &= \frac{r}{n-1}\delta + \left(d - \frac{1}{2}\left(3X + \frac{nr}{n-1} - 1\right)\right)^2\\ &\quad - \frac{1}{4}\left(3X + \frac{nr}{n-1} - 1\right)^2 + (2X - 1)(X + r)\\ &\geq \frac{r}{n-1}\delta - \frac{1}{4}\left(3X + \frac{nr}{n-1} - 1\right)^2 + (2X - 1)(X + r)\\ &= \frac{r}{n-1}\delta - \frac{X^2}{4} + \frac{(n-4)r + (n-1)}{2(n-1)}X - \frac{(nr - (n-1))^2}{4(n-1)^2} - r.\end{aligned}$$

Further by (2.7) and the definition of δ,

$$\frac{X^2}{4} - \frac{(n-4)r + (n-1)}{2(n-1)} X + \frac{(nr-(n-1))^2}{4(n-1)^2} + r$$

$$\leq \max\left\{\frac{(n-1)^2}{4} - \frac{(n-4)r + (n-1)}{2} + \frac{(nr-(n-1))^2}{4(n-1)^2} + r, \frac{(nr-(n-1))^2}{4(n-1)^2} + r\right\}$$

$$= \frac{r}{n-1}\delta.$$

Consequently, $\theta_i \geq 0$ for each $i\,(1 \leq i \leq m)$. As is mentioned earlier, this contradicts the assumption that $\theta(S, T) < 0$, and completes the proof of Theorem.

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