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ANALYZING THE PERTURBATION OF FUZZY RELATIONAL EQUATION BASED ON max-$*$ COMPOSITION

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Abstract

With the definition the perturbation of fuzzy relational equation based on max-$*$ composition by means of the index of measurement, we discuss the condition that fuzzy relational equation and its perturbational equation possess solutions. Then we give a formula to estimate the perturbation of a fuzzy relational equation and also changes in the number of its maximal solutions and minimal solutions.

1. Introduction

In recent years, the perturbation theory of fuzzy relational equation has been applied in various fields, for example, in fuzzy control systems,

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fuzzy inference and fuzzy logic. Due to its importance, the issue of perturbable theory of fuzzy relational equation has drawn significant attention in the past few years and a lot of progress has been made [5-8, 10]. By the definition of perturbable element, Tang discussed the perturbation of fuzzy relational equation [5]. Zhang answered questions which Tang left behind [10]. However, they did not answer how the solutions of fuzzy relational equation are impacted by all perturbable elements of fuzzy matrix. In this paper, first we define the perturbation of fuzzy matrix, then discuss the perturbation of fuzzy relational equation. Finally some useful results are obtained.

Our work is organized as follows. In Section 2, we discuss the condition that perturbational fuzzy relational equation possesses solutions. In Section 3, we give a relation between the maximal solutions of fuzzy relational equations (3) and (4). Section 4 is devoted to finding the relation between the minimal solutions of (3) and (4).

Definition 1.1 [9]. Let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Then we refer mapping $A : X \to L$ as fuzzy set on X, denoted by $F(X)$. Furthermore, $R : X \times Y \to L$ is called $X \times Y$ a *fuzzy relation* on $X \times Y$, denoted by $F(X \times Y)$.

In this paper, we promise the universe of discourse is finite and $L = [0, 1]$ as well as $N = \{1, 2, \ldots, n\}$.

Definition 1.2 [4]. Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{m\times k}$, $R = (r_{ij})_{n\times k}$ be called *max-$*$ composition* of A and B, in symbols $R = A \odot B$, where $r_{il} = \bigvee_{j=1}^{m}(a_{ij} * b_{jl})(i = 1, 2, \ldots, n,\ l = 1, 2, \ldots, k)$.

There are three classes of fuzzy relational equations as follows [7]:

(1) $R = (r_{ij})_{n\times m}$ and $B = (b_1, b_2, \ldots, b_n)^T$ are given, how to find an $X = (x_1, x_2, \ldots, x_m)^T$, where $B = R \odot X$ holds.

(2) $A = (a_1, a_2, \ldots, a_m)^T$ and $B = (b_1, b_2, \ldots, b_n)^T$ are given, how to find $X = (x_{ij})_{n\times m}$, where $B = X \odot A$ holds.

(3) $A = (a_1, a_2, ..., a_n)$ and $b \in [0, 1]$ are given, how to find $X = (x_1, x_2, ..., x_m)^T$, where $b = A \odot X$ holds.

And (2) is equivalent to (3) [7]. So we only discuss the perturbation of solution (3).

Lemma 1.3 [3]. *The solution of* (3) *exists if and only if* $G(b) = \{i \in N \mid a_i \geq b\} \neq \varnothing$ *holds*.

Definition 1.4. Let $A = (a_{ij})_{m\times n}$, $A' = (a'_{ij})_{m\times n}$, if $0 \leq a'_{ij} - a_{ij} \leq \varepsilon$ holds. Then we refer A' as the *upper perturbation* of A, denoted by $A' \equiv (+\varepsilon)A$. If $0 \leq a_{ij} - a'_{ij} \leq \varepsilon$ holds, then we refer A' as the *lower perturbation* of A, denoted by $A' \equiv (-\varepsilon)A$. If $| a'_{ij} - a_{ij} | \leq \varepsilon$ holds, then we refer A' as the *total perturbation* of A, denoted by $A' \equiv (\varepsilon)A$.

Now we discuss the relation between the solutions of $\bigvee_{i=1}^{n}(a_i * x_i) = b$ and $\bigvee_{i=1}^{n}(a'_i * x_i) = b'$, denoted (3) and (4), respectively.

2. The Condition for the Existence of Solution of (4)

Lemma 2.1. *Suppose the solution of* (3) *exists*, *if* $A' = (\varepsilon)A$, $b' = (\delta)b$, *then the solution of* (4) *exists*.

Proof. By Lemma 1.2, the solution of (4) exists if and only if $G(b') = \{i \in N \mid a'_i \geq b'\} \neq \varnothing$ holds. Since $A' = (\varepsilon)A$, $b' = (\delta)b$, we have

$$(a_i - \varepsilon) - (b + \delta) \leq a'_i - b' \leq (a_i + \varepsilon) - (b - \delta). \qquad (*)$$

Because the solution of (3) exists, there at least exists an $i \in G(b)$ such that $a_i - b \geq 0$, thus $a_i - b + (\varepsilon + \delta) \geq 0$. It is easy to find that in $a'_i = a_i + \varepsilon$, $b' = b - \delta$ case, $a'_i - b' = (a_i + \varepsilon) - (b - \delta)$ holds, thus $a'_i - b' \geq 0$, furthermore $G(b') \neq \varnothing$.

Corollary 2.2. *Suppose the solution of* (3) *exists*, *then the solution of* (4) *exists*.

Lemma 2.3. *Suppose the solution of* (3) *does not exist, if* $A' = (\varepsilon)A$, $b' = (\delta)b$ *and* $\varepsilon + \delta \geq \max(b - a_i)(\forall i \in N)$, *then the solution of* (4) *exists.*

Corollary 2.4. *Suppose the solution of* (3) *does not exist, then the solution of* (4) *exists if one of the following holds*:

(i) $A' = (+\varepsilon)A$, $b' = (+\delta)b$ *and* $\varepsilon \geq \max(b - a_i)$;

(ii) $A' = (-\varepsilon)A$, $b' = (-\delta)b$ *and* $\delta \leq \max(a_i - b)$;

(iii) $A' = (+\varepsilon)A$, $b' = (-\delta)b$ *and* $\varepsilon + \delta \geq \max(b - a_i)$, *where* $\forall i \in N$.

Remark. (1) Suppose the solution of (3) exists, since the right $(*)$ is not negative forever, the solution of (4) does not exist anyway.

(2) Obviously $G(b) \subseteq G(b')$ holds.

3. On the Relation between the Maximal Solutions of (3) and (4)

We suppose that the solutions of (3) and (4) exist. For writing conveniently, the maximal solutions of (3) and (4) are denoted by X^* and X'^* respectively, while the minimal solutions of (3) and (4) are denoted by X_* and X'_*, respectively. In addition, x_i^* and $x_i'^*$ denote as X^* i-th coordinate of X^* and X'^* respectively, while x_{*i} and x'_{*i} denote as i-th coordinate of X_* and X'_*, respectively.

Lemma 3.1 [6]. *If the solution of* (3) *exists, then* $X^* = (a_1 \alpha b, a_2 \alpha b, \ldots, a_n \alpha b)^T$ *is its maximal solution, where* $a \alpha b = \vee\{x \mid a * x \leq b\}$ *and* $*$ *is a continuous t-norm on* $[0, 1]$.

Theorem 3.2. *Let*

$$\Delta_i^+ = \begin{cases} ((a_i - \varepsilon) \vee 0)\alpha((b + \delta) \wedge 1) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'), \end{cases}$$

$$\Delta_i^- = \begin{cases} ((a_i + \varepsilon) \wedge 1)\alpha((b - \delta) \vee 0) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'). \end{cases}$$

If $A' = (\varepsilon)A$ *and* $b' = (\delta)b$, *then* $\inf(x_i'^* - x_i^*) = \Delta_i^-$, $x_i'^* - x_i^* \leq \Delta_i^+$.

Proof. We will prove the case of $i \in G(b)$. The proof of the rest is similar.

Obviously $x_i'^* - x_i^* = a_i' \alpha b' - a_i \alpha b$. Since the operator α is non-increasing to first variable and non-decreasing to second variable, $-\varepsilon \leq a_i' - a_i \leq \varepsilon$ and $-\delta \leq b' - b \leq \delta$ hold, hence $((a_i + \varepsilon) \wedge 1)\alpha((b - \delta) \vee 0) - a_i \alpha b \leq x_i'^* - x_i^* \leq ((a_i - \varepsilon) \vee 0)\alpha((b + \delta) \wedge 1) - a_i \alpha b$. Especially, if $a_i' = a_i + \varepsilon$, $b' = b - \delta$, we have $\inf(x_i'^* - x_i^*) = \Delta_i^-$.

Remark. Δ_i^+ is not always the minimal upper bound. However, if $a_i - b > \varepsilon + \delta$ holds, then $a_i' = a_i - \varepsilon$, $b' = b + \delta$, therefore Δ_i^+ is the minimal upper bound.

Similarly we have the following:

Corollary 3.3. *Let*

$$\Delta_i^+ = \begin{cases} a_i \alpha((b + \delta) \wedge 1) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'), \end{cases}$$

$$\Delta_i^- = \begin{cases} ((a_i + \varepsilon) \wedge 1)\alpha b - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'). \end{cases}$$

If $A' = (+\varepsilon)A$ *and* $b' = (+\delta)b$, *then* $\inf(x_i'^* - x_i^*) = \Delta_i^-$, $x_i'^* - x_i^* \leq \Delta_i^+$.

Corollary 3.4. *Let*

$$\Delta_i^+ = \begin{cases} ((a_i - \varepsilon) \vee 0)\alpha b - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'), \end{cases}$$

$$\Delta_i^- = \begin{cases} a_i \alpha((b - \delta) \vee 0) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'). \end{cases}$$

If $A' = (-\varepsilon)A$ *and* $b' = (-\delta)b$, *then* $\inf(x_i'^* - x_i^*) = \Delta_i^-$, $x_i'^* - x_i^* \leq \Delta_i^+$.

Corollary 3.5. *Let*

$$\Delta_i^+ = 0,$$

$$\Delta_i^- = \begin{cases} ((a_i + \varepsilon) \wedge 1)\alpha((b - \delta) \vee 0) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'). \end{cases}$$

If $A' = (+\varepsilon)A$ *and* $b' = (-\delta)b$, *then* $\inf(x_i'^* - x_i^*) = \Delta_i^-$, $x_i'^* - x_i^* \leq 0$.

Corollary 3.6. *Let*

$$\Delta_i^- = 0,$$

$$\Delta_i^+ = \begin{cases} ((a_i - \varepsilon) \vee 0)\alpha((b + \delta) \wedge 1) - a_i \alpha b, & i \in G(b), \\ 0, & i \notin G(b'). \end{cases}$$

If $A' = (-\varepsilon)A$ *and* $b' = (+\delta)b$, *then* $\inf(x_i'^* - x_i^*) = 0$, $x_i'^* - x_i^* \le \Delta_i^+$.

Remark. (1) Δ_i^+ may be negative, for example, in case of $i \in G(b') - G(b)$. This means that there is a shrink at i coordinate of the maximal solution.

(2) If $\varepsilon + \delta < \min(b - a_i)(i \notin G(b))$, then $G(b') = G(b)$ holds. Only when $G(b') = G(b)$ the total maximal solution of (4), increases or decreases, otherwise the change occurs at some coordinates of the maximal solution.

(3) Let $\Delta^+ = \max \Delta_i^+$, $\Delta^- = \min \Delta_i^-$. If $G(b) = G(b')$, then $\inf(X'^* - X^*) = \Delta^-$, $X'^* - X^* \le \Delta^+$. Especially, if $\Delta^+ = \sup(X'^* - X^*)$, then $X'^* = (\Delta)X^*$, where $\Delta = \max\{\Delta^+, -\Delta^-\}$.

4. On the Relation between the Minimal Solutions of Fuzzy Relational Equations (3) and (4)

Lemma 4.1 [3]. *The minimal solutions of* (3) *exist if and only if* $G(b) \neq \varnothing$.

Lemma 4.2 [3]. *The minimal solutions of* (3) *are formed by* $X_* = (x_{*j})_{j \in N}$, *where* $x_{*j} = \begin{cases} a_i \sigma b, & j = i \\ 0, & j \neq i \end{cases}$ $(i \in G(b))$, $a\sigma b = \wedge\{x \mid a * x \ge b\}$, *and* $*$ *is continuous t-norm on* $[0, 1]$.

Lemma 4.3 [3]. *The number of minimal solutions of* (3) *is* $|G(b)|$.

According to Lemma 4.3, we obtain the following lemma.

Lemma 4.4. (i) *If* $G(b) \subset G(b')$, *then the difference of the number of minimal solutions between equations* (3) *and* (4) *is* $|G(b')| - |G(b)|$.

(ii) *If* $G(b') \subset G(b)$, *then the difference of the number of minimal solutions between* (3) *and* (4) *is* $|G(b)| - |G(b')|$.

(iii) *If* $G(b) = G(b')$, *then the number of minimal solutions of* (4) *is equivalent to that of* (3).

The same as above, after being perturbed the minimal solution of fuzzy relation equations is either increased or vanished. So it is sufficient to discuss the situation that the corresponding coordinates of minimal solutions of (3) and (4) are both nonzero.

Theorem 4.5. *Let*

$$\Delta_j^+ = \begin{cases} ((a_i - \varepsilon) \vee 0)\sigma((b + \delta) \wedge 1) - a_i\sigma b, & \text{if } A' = (\varepsilon)A,\ b' = (\delta)b, \\ a_j\sigma((b + \delta) \wedge 1) - a_j\sigma b, & \text{if } A' = (+\varepsilon)A,\ b' = (+\delta)b, \\ ((a_i - \varepsilon) \vee 0)\sigma b - a_i\sigma b, & \text{if } A' = (-\varepsilon)A,\ b' = (-\delta)b, \\ 0, & \text{if } A' = (+\varepsilon)A,\ b' = (-\delta)b, \\ ((a_i - \varepsilon) \vee 0)\sigma((b + \delta) \wedge 1) - a_i\sigma b, & \text{if } A' = (-\varepsilon)A,\ b' = (+\delta)b, \end{cases}$$

$$\Delta_j^- = \begin{cases} ((a_i + \varepsilon) \wedge 1)\sigma((b - \delta) \vee 0) - a_i\sigma b, & \text{if } A' = (\varepsilon)A,\ b' = (\delta)b, \\ ((a_i + \varepsilon) \wedge 1)\sigma b - a_j\sigma b, & \text{if } A' = (+\varepsilon)A,\ b' = (+\delta)b, \\ a_i\sigma((b - \delta) \vee 0) - a_j\sigma b, & \text{if } A' = (-\varepsilon)A,\ b' = (-\delta)b, \\ ((a_i + \varepsilon) \wedge 1)\sigma((b - \delta) \vee 0) - a_i\sigma b, & \text{if } A' = (+\varepsilon)A,\ b' = (-\delta)b, \\ 0, & \text{if } A' = (-\varepsilon)A,\ b' = (+\delta)b, \end{cases}$$

we have $\inf(X'_{*j} - X_{*j}) = \Delta_j^-$, $X'_{*j} - X_{*j} \le \Delta_j^+$, *especially if* $\sup(X'_{*j} - X_{*j}) = \Delta_j^+$, *then* $X'_* = (\Delta)X_*$, *where* $\Delta = \max\{\Delta_j^+, -\Delta_j^-\}$.

Proof. The proof is similar to that of Theorem 3.2.

In summary we first show the condition that perturbational fuzzy relational equation possesses solutions, further we investigate the relation between the maximal solutions of fuzzy relational equations (3) and (4) as well as the relation between the minimal solutions of (3) and (4).

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