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INTERVAL NEUTROSOPHIC LOGIC

HAIBIN WANG, FLORENTIN SMARANDACHE*,
YAN-QING ZHANG and RAJSHEKHAR SUNDERRAMAN

Department of Computer Science
Georgia State University
GA 30302, U. S. A.
e-mail: hwang17@student.gsu.edu
yzhang@cs.gsu.edu
raj@cs.gsu.edu

*Mathematical Department
University of New Mexico
Gallup, NM 87301, U. S. A.
e-mail: smarand@unm.edu

Abstract

In this paper, we present a novel interval neutrosophic logic that generalizes the interval valued fuzzy logic, the intuitionistic fuzzy logic and paraconsistent logics which only consider truth-degree or falsity-degree of a proposition. In the interval neutrosophic logic, we consider not only truth-degree and falsity-degree but also indeterminacy degree which can reliably capture more information under uncertainty. We introduce mathematical definitions of an interval neutrosophic propositional calculus and an interval neutrosophic predicate calculus. We propose a general method to design an interval neutrosophic logic system which consists of neutrosophication, neutrosophic inference, a

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neutrosophic rule base, neutrosophic type reduction and deneutrosophication. A neutrosophic rule contains input neutrosophic linguistic variables and output neutrosophic linguistic variables. A neutrosophic linguistic variable has neutrosophic linguistic values which defined by interval neutrosophic sets characterized by three membership functions: truth-membership, falsity-membership and indeterminacy-membership. The interval neutrosophic logic can be applied to many potential real applications where information is imprecise, uncertain, incomplete and inconsistent such as Web intelligence, medical informatics, bioinformatics, decision making, etc.

I. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [18]. Since then fuzzy sets and fuzzy logic have been applied to many real applications to handle uncertainty. The traditional fuzzy set uses one real value $\mu_A(x) \in [0, 1]$ to represent the grade of membership of fuzzy set A defined on universe X. The corresponding fuzzy logic associates each proposition p with a real value $\mu(p) \in [0, 1]$ which represents the degree of truth. Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed [15] to capture the uncertainty of grade of membership. The interval valued fuzzy set uses an interval value $[\mu_A^L(x), \mu_A^U(x)]$ with $0 \le \mu_A^L(x) \le \mu_A^U(x) \le 1$ to represent the grade of membership of fuzzy sets. The traditional fuzzy logic can be easily extended to the interval valued fuzzy logic. There are related works such as type-2 fuzzy sets and type-2 fuzzy logic [8, 10, 12]. The family of fuzzy sets and fuzzy logic can only handle "complete" information that if a grade of truth-membership is $\mu_A(x)$, then a grade of false-membership is $1 - \mu_A(x)$ by default. In some applications such as expert systems, decision making systems and information fusion systems, the information is both uncertain and incomplete. That is beyond the scope of traditional fuzzy sets and fuzzy logic. In 1986, Atanassov introduced the intuitionistic fuzzy set [1] which is a generalization of a fuzzy set and provably equivalent to an interval valued fuzzy set. The intuitionistic fuzzy sets consider both truth-membership and false-membership. The corresponding intuitionistic fuzzy logic [2-4] associates each proposition p with two real values $\mu(p)$-truth degree and

$\nu(p)$-falsity degree, respectively, where $\mu(p), \nu(p) \in [0, 1], \mu(p) + \nu(p) \leq 1.$ So intuitionistic fuzzy sets and intuitionistic fuzzy logic can handle uncertain and incomplete information.

However, inconsistent information exists in a lot of real situations such as those mentioned above. It is obvious that the intuitionistic fuzzy logic cannot reason with inconsistency because it requires $\mu(p) + \nu(p) \leq 1.$ Generally, two basic approaches are used to solve the inconsistency problem in knowledge bases: the belief revision and paraconsistent logics. The goal of the first approach is to make an inconsistent theory consistent, either by revising it or by representing it by a consistent semantics. On the other hand, the paraconsistent approach allows reasoning in the presence of inconsistency as contradictory information can be derived or introduced without trivialization [5]. Costa's $\mathcal{C}_w$ logic [7] and Belnap's four-valued logic [6] are two well-known paraconsistent logics.

Neutrosophy was introduced by Smarandache in 1995. "Neutrosophy is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [14]. Neutrosophy includes neutrosophic probability, neutrosophic sets and neutrosophic logic. In a neutrosophic set (neutrosophic logic), indeterminacy is quantified explicitly and truth-membership (truth-degree), indeterminacy-membership (indeterminacy-degree) and false-membership (falsity-degree) are independent. The independence assumption is very important in a lot of applications such as information fusion when we try to combine different data from different sensors. A neutrosophic set (neutrosophic logic) is different from an intuitionistic fuzzy set (intuitionistic fuzzy logic), where indeterminacy membership (indeterminacy-degree) is $1 - \mu_A(x) - \nu_A(x)(1 - \mu(p) - \nu(p))$ by default.

The neutrosophic set generalizes the above mentioned sets from a philosophical point of view. From a scientific or engineering point of view, the neutrosophic set and set-theoretic operators need to be specified meaningfully. Otherwise, it will be difficult to apply to the real applications. In [16], we discussed a special neutrosophic set called an *interval neutrosophic set* and defined a set of set-theoretic operators. It is natural to define the interval neutrosophic logic based on interval neutrosophic sets. In this paper, we give mathematical definitions of an interval neutrosophic propositional calculus and a first order interval neutrosophic predicate calculus.

The rest of the paper is organized as follows. Section II gives a brief review of interval neutrosophic sets. Section III gives the mathematical definition of the interval neutrosophic propositional calculus. Section IV gives the mathematical definition of the first order interval neutrosophic predicate calculus. Section V provides one application example of interval neutrosophic logic as the foundation for the design of interval neutrosophic logic system. In Section VI, we conclude the paper and discuss the future research directions.

II. Interval Neutrosophic Sets

This section gives a brief overview of concepts of interval neutrosophic sets defined in [16]. An interval neutrosophic set is an instance of the neutrosophic set introduced in [13] which can be used in real scientific and engineering applications.

Definition 1 (Interval neutrosophic set)**.** Let X be a space of points (objects), with a generic element in X denoted by x. An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy-membership function I_A and false-membership function F_A. For each point x in X, $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$.

When X is continuous, an INS A can be written as

$$A = \int_X \langle T(x), I(x), F(x)\rangle / x, \quad x \in X.$$

When X is discrete, an INS A can be written as

$$A = \sum_{i=1}^{n} \langle T(x_i), I(x_i), F(x_i)\rangle / x_i, \quad x_i \in X.$$

Example 1. Consider parameters such as capability, trustworthiness, and price of semantic Web services. These parameters are commonly used to define quality of service of semantic Web services [17]. Assume that $X = [x_1, x_2, x_3]$. x_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are a subset of $[0, 1]$. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is an interval neutrosophic set of X defined by

$$A = \langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.5] \rangle / x_1 + \langle [0.5, 0.7], [0, 0.2], [0.2, 0.3] \rangle / x_2$$
$$+ \langle [0.6, 0.8], [0.2, 0.3], [0.2, 0.3] \rangle / x_3.$$

Definition 2. An interval neutrosophic set A is empty if and only if its $\inf T_A(x) = \sup T_A(x) = 0$, $\inf I_A(x) = \sup I_A(x) = 1$ and $\inf F_A(x) = \sup F_A(x) = 0$, for all x in X.

Let A be an interval neutrosophic set on X. Then $A(x) = \langle T_A(x), I_A(x), F_A(x) \rangle$. Let $\underline{0} = \langle 0, 0, 1 \rangle$ and $\underline{1} = \langle 1, 1, 0 \rangle$.

Definition 3. Let A and B be two interval neutrosophic sets defined on X. $A(x) \le B(x)$ if and only if

$$\inf T_A(x) \le \inf T_B(x), \quad \sup T_A(x) \le \sup T_B(x), \tag{1}$$

$$\inf I_A(x) \le \inf I_B(x), \quad \sup I_A(x) \le \sup I_B(x), \tag{2}$$

$$\inf F_A(x) \ge \inf F_B(x), \quad \sup F_A(x) \ge \sup F_B(x). \tag{3}$$

Definition 4 (Containment)**.** An interval neutrosophic set A is contained in the other interval neutrosophic set B, $A \subseteq B$, if and only if $A(x) \le B(x)$, for all x in X.

Definition 5. Two interval neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$, and $B \subseteq A$.

Let $N = \langle [0, 1] \times [0, 1], [0, 1] \times [0, 1], [0, 1] \times [0, 1] \rangle$.

Definition 6 (Complement)**.** Let C_N denote a neutrosophic *complement* of A. Then C_N is a function

$$C_N : N \to N$$

and C_N must satisfy at least the following three axiomatic requirements:

(1) $C_N(\underline{0}) = \underline{1}$ and $C_N(\underline{1}) = \underline{0}$ (boundary conditions).

(2) Let A and B be two interval neutrosophic sets defined on X, if $A(x) \leq B(x)$, then $C_N(A(x)) \geq C_N(B(x))$, for all x in X (monotonicity).

(3) Let A be an interval neutrosophic set defined on X. Then $C_N(C_N(A(x))) = A(x)$, for all x in X (involutivity).

There are many functions which satisfy the requirement to be the complement operator of interval neutrosophic sets. Here we give one example.

Definition 7 (Complement C_{N_1})**.** The complement of an interval neutrosophic set A is denoted by $\overline{A}$ and is defined by

$$T_{\overline{A}}(x) = F_A(x), \tag{4}$$

$$\inf I_{\overline{A}}(x) = 1 - \sup I_A(x), \tag{5}$$

$$\sup I_{\overline{A}}(x) = 1 - \inf I_A(x), \tag{6}$$

$$F_{\overline{A}}(x) = T_A(x), \tag{7}$$

for all x in X.

Definition 8 (N-norm)**.** Let I_N denote a neutrosophic *intersection* of two interval neutrosophic sets A and B. Then I_N is a function

$$I_N : N \times N \to N$$

and I_N must satisfy at least the following four axiomatic requirements:

(1) $I_N(A(x), \underline{1}) = A(x)$, for all x in X (boundary condition).

(2) $B(x) \leq C(x)$ implies $I_N(A(x), B(x)) \leq I_N(A(x), C(x))$, for all x in X (monotonicity).

(3) $I_N(A(x), B(x)) = I_N(B(x), A(x))$, for all x in X (commutativity).

(4) $I_N(A(x), I_N(B(x), C(x))) = I_N(I_N(A(x), B(x)), C(x))$, for all x in X (associativity).

Here we give one example of intersection of two interval neutrosophic sets which satisfies above N-norm axiomatic requirements. Other different definitions can be given for different applications.

Definition 9 (Intersection I_{N_1}). The intersection of two interval neutrosophic sets A and B is an interval neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership, and false-membership are related to those of A and B by

$$\inf T_C(x) = \min(\inf T_A(x), \inf T_B(x)), \tag{8}$$

$$\sup T_C(x) = \min(\sup T_A(x), \sup T_B(x)), \tag{9}$$

$$\inf I_C(x) = \min(\inf I_A(x), \inf I_B(x)), \tag{10}$$

$$\sup I_C(x) = \min(\sup I_A(x), \sup I_B(x)), \tag{11}$$

$$\inf F_C(x) = \max(\inf F_A(x), \inf F_B(x)), \tag{12}$$

$$\sup F_C(x) = \max(\sup F_A(x), \sup F_B(x)), \tag{13}$$

for all x in X.

Definition 10 (N-conorm). Let U_N denote a neutrosophic *union* of two interval neutrosophic sets A and B. Then U_N is a function

$$U_N : N \times N \to N$$

and U_N must satisfy at least the following four axiomatic requirements:

(1) $U_N(A(x), \underline{0}) = A(x)$, for all x in X (boundary condition).

(2) $B(x) \le C(x)$ implies $U_N(A(x), B(x)) \le U_N(A(x), C(x))$, for all x in X (monotonicity).

(3) $U_N(A(x), B(x)) = U_N(B(x), A(x))$, for all x in X (commutativity).

(4) $U_N(A(x), U_N(B(x), C(x))) = U_N(U_N(A(x), B(x)), C(x))$, for all x in X (associativity).

Here we give one example of union of two interval neutrosophic sets which satisfies above N-conorm axiomatic requirements. Other different definitions can be given for different applications.

Definition 11 (Union U_{N_1}). The union of two interval neutrosophic sets A and B is an interval neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership, and false-membership are related to those of A and B by

$$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)), \tag{14}$$

$$\sup T_C(x) = \max(\sup T_A(x), \sup T_B(x)), \tag{15}$$

$$\inf I_C(x) = \max(\inf I_A(x), \inf I_B(x)), \tag{16}$$

$$\sup I_C(x) = \max(\sup I_A(x), \sup I_B(x)), \tag{17}$$

$$\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)), \tag{18}$$

$$\sup F_C(x) = \min(\sup F_A(x), \sup F_B(x)), \tag{19}$$

for all x in X.

Theorem 1. *Let P be the power set of all interval neutrosophic sets defined in the universe X. Then $\langle P;\, I_{N_1}, U_{N_1} \rangle$ is a distributive lattice.*

Proof. Let A, B, C be the arbitrary interval neutrosophic sets defined on X. It is easy to verify that $A \cap A = A$, $A \cup A = A$ (idempotency), $A \cap B = B \cap A$, $A \cup B = B \cup A$ (commutativity), $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$ (associativity), and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributivity).

Definition 12 (Interval neutrosophic relation). Let X and Y be two non-empty crisp sets. Then an interval neutrosophic relation $R(X, Y)$ is a subset of product space $X \times Y$, and is characterized by the truth-membership function $T_R(x, y)$, the indeterminacy membership function $I_R(x, y)$, and the falsity-membership function $F_R(x, y)$, where $x \in X$ and $y \in Y$ and $T_R(x, y)$, $I_R(x, y)$, $F_R(x, y) \subseteq [0, 1]$.

Definition 13 (Interval neutrosophic composition functions)**.** The membership functions for the composition of interval neutrosophic relations $R(X, Y)$ and $S(Y, Z)$ are given by the *interval neutrosophic sup-star composition* of R and S,

$$T_{R\circ S}(x, z) = \sup_{y\in Y} \min(T_R(x, y), T_S(y, z)), \tag{20}$$

$$I_{R\circ S}(x, z) = \sup_{y\in Y} \min(I_R(x, y), I_S(y, z)), \tag{21}$$

$$F_{R\circ S}(x, z) = \inf_{y\in Y} \max(F_R(x, y), F_S(y, z)). \tag{22}$$

If R is an interval neutrosophic set rather than an interval neutrosophic relation, then $Y = X$ and $\sup_{y\in Y} \min(T_R(x, y), T_S(y, z))$ becomes $\sup_{y\in Y} \min(T_R(x), T_S(y, z))$, which is only a function of the output variable z. It is similar for $\sup_{y\in Y} \min(I_R(x, y), I_S(y, z))$ and $\inf_{y\in Y} \max(F_R(x, y), F_S(y, z))$. Hence, the notation of $T_{R\circ S}(x, z)$ can be simplified to $T_{R\circ S}(z)$, so that in the case of R being just an interval neutrosophic set,

$$T_{R\circ S}(z) = \sup_{x\in X} \min(T_R(x), T_S(x, z)), \tag{23}$$

$$I_{R\circ S}(z) = \sup_{x\in X} \min(I_R(x), I_S(x, z)), \tag{24}$$

$$F_{R\circ S}(z) = \inf_{x\in X} \max(F_R(x), F_S(x, z)). \tag{25}$$

Definition 14 (Truth-favorite)**.** The truth-favorite of an interval neutrosophic set A is an interval neutrosophic set B, written as $B = \Delta A$, whose truth-membership and false-membership are related to those of A by

$$\inf T_B(x) = \min(\inf T_A(x) + \inf I_A(x), 1), \tag{26}$$

$$\sup T_B(x) = \min(\sup T_A(x) + \sup I_A(x), 1), \tag{27}$$

$$\inf I_B(x) = 0, \tag{28}$$

$$\sup I_B(x) = 0, \tag{29}$$

$$\inf F_B(x) = \inf F_A(x), \tag{30}$$

$$\sup F_B(x) = \sup F_A(x), \tag{31}$$

for all x in X.

Definition 15 (False-favorite)**.** The false-favorite of an interval neutrosophic set A is an interval neutrosophic set B, written as $B = \nabla A$, whose truth-membership and false-membership are related to those of A by

$$\inf T_B(x) = \inf T_A(x), \tag{32}$$

$$\sup T_B(x) = \sup T_A(x), \tag{33}$$

$$\inf I_B(x) = 0, \tag{34}$$

$$\sup I_B(x) = 0, \tag{35}$$

$$\inf F_B(x) = \min(\inf F_A(x) + \inf I_A(x), 1), \tag{36}$$

$$\sup F_B(x) = \min(\sup F_A(x) + \sup I_A(x), 1), \tag{37}$$

for all x in X.

III. Interval Neutrosophic Propositional Calculus

In this section, we introduce the elements of an interval neutrosophic propositional calculus based on the definition of the interval neutrosophic sets by using the notations from the theory of classical propositional calculus [11].

A. *Syntax of the interval neutrosophic propositional calculus*

Here we give the formalization of syntax of the interval neutrosophic propositional calculus.

Definition 16. An *alphabet* of the interval neutrosophic propositional calculus consists of three classes of symbols:

(1) A set of *interval neutrosophic propositional variables*, denoted by lower-case letters, sometimes indexed;

(2) Five *connectives* $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\leftrightarrow$ which are called *conjunction*, *disjunction*, *negation*, *implication* and *biimplication* symbols respectively;

(3) The parentheses (and).

The alphabet of the interval neutrosophic propositional calculus has combinations obtained by assembling connectives and interval neutrosophic propositional variables in strings. The purpose of the construction rules is to have the specification of distinguished combinations, called formulas.

Definition 17. The set of formulas (well-formed formulas) of the interval neutrosophic propositional calculus is defined as follows:

(1) Every *interval neutrosophic propositional variable* is a formula;

(2) If p is a formula, then so is $(\neg p)$;

(3) If p and q are formulas, then so are

(a) $(p \wedge q)$,

(b) $(p \vee q)$,

(c) $(p \to q)$, and

(d) $(p \leftrightarrow q)$.

(4) No sequence of symbols is a formula which is not required to be (1), (2) and (3).

To avoid having formulas cluttered with parentheses, we adopt the following precedence hierarchy, with the highest precedence at the top:

$$\neg,$$

$$\wedge, \vee,$$

$$\to, \leftrightarrow .$$

Here is an example of the interval neutrosophic propositional calculus formula:

$$\neg p_1 \wedge p_2 \vee (p_1 \to p_3) \to p_2 \wedge \neg p_3.$$

Definition 18. The *language of the interval neutrosophic propositional calculus* given by an alphabet consists of the set of all formulas constructed from the symbols of the alphabet.

B. *Semantics of the interval neutrosophic propositional calculus*

The study of the interval neutrosophic propositional calculus comprises, among others, a *syntax*, which has the distinction of well-formed formulas, and a *semantics*, the purpose of which is the assignment of a meaning to well-formed formulas.

To each interval neutrosophic proposition p, we associate it with an ordered triple components $\langle t(p), i(p), f(p)\rangle$, where $t(p), i(p), f(p) \subseteq [0, 1]$. $t(p), i(p), f(p)$ are called *truth-degree*, *indeterminacy-degree* and *falsity-degree* of p, respectively. Let this assignment be provided by an *interpretation function* or *interpretation INL* defined over a set of propositions P in such a way that

$$INL(p) = \langle t(p), i(p), f(p)\rangle.$$

Hence, the function $INL : P \to N$ gives the truth, indeterminacy and falsity degrees of all propositions in P. We assume that the interpretation function INL assigns to the logical truth $T : INL(T) = \langle 1, 1, 0\rangle$, and to $F : INL(F) = \langle 0, 0, 1\rangle$.

An interpretation which makes a formula true is a *model* of the formula.

Let i, l be the subinterval of $[0, 1]$. Then $i + l = [\inf i + \inf l, \sup i + \sup l]$, $i - l = [\inf i - \sup l, \sup i - \inf l]$, $\max(i, l) = [\max(\inf i, \inf l), \max(\sup i, \sup l)]$, $\min(i, l) = [\min(\inf i, \inf l), \min(\sup i, \sup l)]$.

The semantics of five interval neutrosophic propositional connectives is given in Table I. Note that $p \leftrightarrow q$ if and only if $p \to q$ and $q \to p$.

Example 2. $INL(p) = \langle 0.5, 0.4, 0.7\rangle$ and $INL(q) = \langle 1, 0.7, 0.2\rangle$. Then $INL(\neg p) = \langle 0.7, 0.6, 0.5\rangle$, $INL(p \wedge \neg p) = \langle 0.5, 0.4, 0.7\rangle$, $INL(p \vee q) = \langle 1, 0.7, 0.2\rangle$, $INL(p \to q) = \langle 1, 1, 0\rangle$.

Table I

Semantics of five connectives in the interval neutrosophic propositional logic

Connectives	Semantics
$INL(\neg p)$	$\langle f(p), 1 - i(p), t(p)\rangle$
$INL(p \wedge q)$	$\langle \min(t(p), t(q)), \min(i(p), i(q)), \max(f(p), f(q))\rangle$
$INL(p \vee q)$	$\langle \max(t(p), t(q)), \max(i(p), i(q)), \min(f(p), f(q))\rangle$
$INL(p \to q)$	$\langle \min(1, 1 - t(p) + t(q)), \min(1, 1 - i(p) + i(q)),$ $\max(0, f(q) - f(p))\rangle$
$INL(p \leftrightarrow q)$	$\langle \min(1 - t(p) + t(q), 1 - t(q) + t(p)),$ $\min(1 - i(p) + i(q), 1 - i(q) + i(p)),$ $\max(f(p) - f(q), f(q) - f(p))\rangle$

A given well-formed interval neutrosophic propositional formula will be called a *tautology* (valid) if $INL(A) = \langle 1, 1, 0\rangle$, for all interpretation functions INL. It will be called a *contradiction* (inconsistent) if $INL(A) = \langle 0, 0, 1\rangle$, for all interpretation functions INL.

Definition 19. Two formulas p and q are said to be *equivalent*, denoted $p = q$, if and only if the $INL(p) = INL(q)$ for every interpretation function INL.

Theorem 2. *Let F be the set of formulas, $\wedge$ be the meet and $\vee$ be the join. Then $\langle F; \wedge, \vee\rangle$ is a distributive lattice.*

Proof. It is analogous to the proof of Theorem 1.

Theorem 3. *If p and $p \to q$ are tautologies, then q is also a tautology.*

Proof. Since p and $p \to q$ are tautologies, for every INL, $INL(p) = INL(p \to q) = \langle 1, 1, 0\rangle$, that is, $t(p) = i(p) = 1$, $f(p) = 0$, $t(p \to q) = \min(1, 1 - t(p) + t(q)) = 1$, $i(p \to q) = \min(1, 1 - i(p) + i(q)) = 1$, $f(p \to q) = \max(0, f(q) - f(p)) = 0$. Hence $t(q) = i(q) = 1$, $f(q) = 0$. So q is a tautology.

C. *Proof theory of interval neutrosophic propositional calculus*

Here we give the proof theory for interval neutrosophic propositional logic to complement the semantics part.

Definition 20. The interval neutrosophic propositional logic is defined by the following axiom schema:

$$p \to (q \to p)$$

$$p_1 \wedge \ldots \wedge p_m \to q_1 \vee \ldots \vee q_n \text{ provided some } p_i \text{ is some } q_j$$

$$p \to (q \to p \wedge q)$$

$$(p \to r) \to ((q \to r) \to (p \vee q \to r))$$

$$(p \vee q) \to r \text{ if and only if } p \to r \text{ and } q \to r$$

$$p \to q \text{ if and only if } \neg q \to \neg p$$

$$p \to q \text{ and } q \to r \text{ imply } p \to r$$

$$p \to q \text{ if and only if } p \leftrightarrow (p \wedge q) \text{ if and only if } q \to (p \vee q).$$

The concept of (formal) deduction of a formula from a set of formulas, that is, using the standard notation, $\Gamma \vdash p$, is defined as usual; in this case, we say that p is a *syntactical consequence* of the formulas in T.

Theorem 4. *For interval neutrosophic propositional logic, we have*

(1) $\{p\} \vdash p$,

(2) $\Gamma \vdash p$ *entails* $\Gamma \bigcup \Delta \vdash p$,

(3) *if* $\Gamma \vdash p$ *for any* $p \in \Delta$ *and* $\Delta \vdash q$, *then* $\Gamma \vdash q$.

Proof. It is immediate from the standard definition of the syntactical consequence $(\vdash)$.

Theorem 5. *In interval neutrosophic proposition logic, we have*

(1) $\neg\neg p \leftrightarrow p$,

(2) $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$,

(3) $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$.

Proof. Proof is straightforward by following the semantics of interval neutrosophic propositional logic.

Theorem 6. *In interval neutrosophic propositional logic, the following schemas do not hold*:

(1) $p \vee \neg p$

(2) $\neg(p \wedge \neg p)$

(3) $p \wedge \neg p \rightarrow q$

(4) $p \wedge \neg p \rightarrow \neg q$

(5) $\{p,\ p \rightarrow q\} \vdash q$

(6) $\{p \rightarrow q,\ \neg q\} \vdash \neg p$

(7) $\{p \vee q,\ \neg q\} \vdash p$

(8) $\neg p \vee q \leftrightarrow p \rightarrow q$.

Proof. Immediate from the semantics of interval neutrosophic propositional logic.

Example 3. To illustrate the use of the interval neutrosophic propositional consequence relation, let us consider the following example:

$$p \rightarrow (q \wedge r)$$

$$r \rightarrow s$$

$$q \rightarrow \neg s$$

$$a.$$

From $p \rightarrow (q \wedge r)$, we get $p \rightarrow q$ and $p \rightarrow r$. From $p \rightarrow q$ and $q \rightarrow \neg s$, we get $p \rightarrow \neg s$. From $p \rightarrow r$ and $r \rightarrow s$, we get $p \rightarrow s$. Hence, p is equivalent to $p \wedge s$ and $p \wedge \neg s$. However, we cannot detach s from p nor $\neg s$ from p. This is in part due to interval neutrosophic propositional logic incorporating neither modus ponens nor elimination.

IV. Interval Neutrosophic Predicate Calculus

In this section, we will extend our consideration to the full language of the first order interval neutrosophic predicate logic. First we give the formalization of syntax of the first order interval neutrosophic predicate logic as in classical first-order predicate logic.

A. *Syntax of interval neutrosophic predicate calculus*

Definition 21. An *alphabet* of the first order interval neutrosophic predicate calculus consists of seven classes of symbols:

(1) *variables*, denoted by lower-case letters, sometimes indexed;

(2) *constants*, denoted by lower-case letters;

(3) *function symbols*, denoted by lower-case letters, sometimes indexed;

(4) *predicate symbols*, denoted by lower-case letters, sometimes indexed;

(5) Five *connectives* $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ which are called the *conjunction*, *disjunction*, *negation*, *implication* and *biimplication* symbols respectively;

(6) Two *quantifiers*, the *universal quantifier* $\forall$ (for all) and the *existential quantifier* $\exists$ (there exists);

(7) The parentheses (and).

To avoid having formulas cluttered with brackets, we adopt the following precedence hierarchy, with the highest precedence at the top:

$$\neg, \forall, \exists$$
$$\wedge, \vee$$
$$\rightarrow, \leftrightarrow$$

Next we turn to the definition of the first order interval neutrosophic language given by an alphabet.

Definition 22. A *term* is defined as follows:

(1) A variable is a term.

(2) A constant is a term.

(3) If f is an n-ary function symbol and $t_1, ..., t_n$ are terms, then $f(t_1, ..., t_n)$ is a term.

Definition 23. A (*well-formed*) *formula* is defined inductively as follows:

(1) If p is an n-ary predicate symbol and $t_1, ..., t_n$ are terms, then $p(t_1, ..., t_n)$ is a formula (called an *atomic formula* or, more simply, an *atom*).

(2) If F and G are formulas, then so are $(\neg F)$, $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$ and $(F \leftrightarrow G)$.

(3) If F is a formula and x is a variable, then $(\forall x F)$ and $(\exists x F)$ are formulas.

Definition 24. The *first order interval neutrosophic language* given by an alphabet consists of the set of all formulas constructed from the symbols of the alphabet.

Example 4. $\forall x \exists y (p(x, y) \rightarrow q(x))$, $\neg \exists x (p(x, a) \wedge q(x))$ are formulas.

Definition 25. The *scope* of $\forall x$ (resp. $\exists x$) in $\forall x F$ (resp. $\exists x F$) is F. A *bound occurrence* of a variable in a formula is an occurrence immediately following a quantifier or an occurrence within the scope of a quantifier, which has the same variable immediately after the quantifier. Any other occurrence of a variable is *free*.

Example 5. In the formula $\forall x p(x, y) \vee q(x)$, the first two occurrences of x are bound, while the third occurrence is free, since the scope of $\forall x$ is $p(x, y)$ and y is free.

B. *Semantics of interval neutrosophic predicate calculus*

In this section, we study the semantics of interval neutrosophic predicate calculus, the purpose of which is the assignment of a meaning to well-formed formulas. In the interval neutrosophic propositional logic, an interpretation is an assignment of truth values (ordered triple component) to propositions. In the first order interval neutrosophic predicate logic, since there are variables involved, we have to do more than that. To define an interpretation for a well-formed formula in this logic, we have to specify two things, the domain and an assignment to constants and predicate symbols occurring in the formula. The following is the formal definition of an interpretation of a formula in the first order interval neutrosophic predicate logic.

Definition 26. An *interpretation function* (or *interpretation*) of a formula F in the first order interval neutrosophic predicate logic consists of a non-empty domain D, and an assignment of "values" to each constant and predicate symbol occurring in F as follows:

(1) To each constant, we assign an element in D.

(2) To each n-ary function symbol, we assign a mapping from D^n to D. (Note that $D^n = \{(x_1, \ldots, x_n) | x_1 \in D, \ldots, x_n \in D\}$.)

(3) Predicate symbols get their meaning through evaluation functions E which assign to each variable x an element $E(x) \in D$. To each n-ary predicate symbol p, there is a function $INP(p) : D^n \to N$. So $INP(p(x_1, \ldots, x_n)) = INP(p)(E(x_1), \ldots, E(x_n))$.

That is, $INP(p)(a_1, \ldots, a_n) = \langle t(p(a_1, \ldots, a_n)), i(p(a_1, \ldots, a_n)), f(p(a_1, \ldots, a_n))\rangle$, where $t(p(a_1, \ldots, a_n)), i(p(a_1, \ldots, a_n)), f(p(a_1, \ldots, a_n)) \subseteq [0, 1]$. They are called *truth-degree*, *indeterminacy-degree* and *falsity-degree* of $p(a_1, \ldots, a_n)$, respectively. We assume that the interpretation function *INP* assigns to the logical truth $T : INP(T) = \langle 1, 1, 0\rangle$, and to $F : INP(F) = \langle 0, 0, 1\rangle$.

The semantics of five interval neutrosophic predicate connectives and two quantifiers are given in Table II. For simplification of notation, we use p to denote $p(a_1, \ldots, a_i)$. Note that $p \leftrightarrow q$ if and only if $p \to q$ and $q \to p$.

Table II

Semantics of five connectives and two quantifiers in interval neutrosophic predicate logic

Connectives	Semantics
$INP(\neg p)$	$\langle f(p), 1 - i(p), t(p)\rangle$
$INP(p \wedge q)$	$\langle \min(t(p), t(q)), \min(i(p), i(q)), \max(f(p), f(q))\rangle$
$INP(p \vee q)$	$\langle \max(t(p), t(q)), \max(i(p), i(q)), \min(f(p), f(q))\rangle$
$INP(p \rightarrow q)$	$\langle \min(1, 1 - t(p) + t(q)), \min(1, 1 - i(p) + i(q)),$ $\max(0, f(q) - f(p))\rangle$
$INP(p \leftrightarrow q)$	$\langle \min(1 - t(p) + t(q), 1 - t(q) + t(p)),$ $\min(1 - i(p) + i(q), 1 - i(q) + i(p)),$ $\max(f(p) - f(q), f(q) - f(p))\rangle$
$INP(\forall xF)$	$\langle \min t(F(E(x))), \min i(F(E(x))), \max f(F(E(x)))\rangle,$ $E(x) \in D$
$INP(\exists xF)$	$\langle \max t(F(E(x))), \max i(F(E(x))), \min f(F(E(x)))\rangle,$ $E(x) \in D$

Example 6. Let $D = 1, 2, 3$ and $p(1) = \langle 0.5, 1, 0.4\rangle$, $p(2) = \langle 1, 0.2, 0\rangle$, $p(3) = \langle 0.7, 0.4, 0.7\rangle$. Then $INP(\forall xp(x)) = \langle 0.5, 0.2, 0.7\rangle$ and $INP(\exists xp(x)) = \langle 1, 1, 0\rangle$.

Definition 27. A formula F is *consistent* (*satisfiable*) if and only if there exists an interpretation I such that F is evaluated to $\langle 1, 1, 0\rangle$ in I. If a formula F is T in an interpretation I, we say that I is a *model* of F and I *satisfies* F.

Definition 28. A formula F is *inconsistent* (*unsatisfiable*) if and only if there exists no interpretation that satisfies F.

Definition 29. A formula F is *valid* if and only if every interpretation of F satisfies F.

Definition 30. A formula F is a *logical consequence* of formulas

$F_1, \ldots, F_n$ if and only if for every interpretation I, if $F_1 \wedge \ldots \wedge F_n$ is true in I, then F is also true in I.

Example 7. $(\forall x)(p(x) \to (\exists y)p(y))$ is valid, $(\forall x)p(x) \wedge (\exists y)\neg p(y)$ is consistent.

Theorem 7. *There is no inconsistent formula in the first order interval neutrosophic predicate logic.*

Proof. It is direct from the definition of semantics of interval neutrosophic predicate logic.

Note that the first order interval neutrosophic predicate logic can be considered as an extension of the interval neutrosophic propositional logic. When a formula in the first order logic contains no variables and quantifiers, it can be treated just as a formula in the propositional logic.

C. *Proof theory of interval neutrosophic predicate calculus*

In this part, we give the proof theory for the first order interval neutrosophic predicate logic to complement the semantics part.

Definition 31. The first order interval neutrosophic predicate logic is defined by the following axiom schema:

$$(p \to q(x)) \to (p \to \forall x q(x))$$

$$\forall x p(x) \to p(a)$$

$$p(x) \to \exists x p(x)$$

$$(p(x) \to q) \to (\exists x p(x) \to q).$$

Theorem 8. *In the first order interval neutrosophic predicate logic, we have*

(1) $p(x) \vdash \forall x p(x)$

(2) $p(a) \vdash \exists x p(x)$

(3) $\forall x p(x) \vdash p(y)$

(4) $\Gamma \cup \{p(x)\} \vdash q$, *then* $\Gamma \cup \{\exists x p(x)\} \vdash q$.

Proof. Directly from the definition of the semantics of the first order interval neutrosophic predicate logic.

Theorem 9. *In the first order interval neutrosophic predicate logic, the following schemes are valid, where r is a formula in which x does not appear free*:

(1) $\forall xr \leftrightarrow r$

(2) $\exists xr \leftrightarrow r$

(3) $\forall x \forall yp(x, y) \leftrightarrow \forall y \forall xp(x, y)$

(4) $\exists x \exists yp(x, y) \leftrightarrow \exists y \exists xp(x, y)$

(5) $\forall x \forall yp(x, y) \to \forall xp(x, x)$

(6) $\exists xp(x, x) \to \exists x \exists yp(x, y)$

(7) $\forall xp(x) \to \exists xp(x)$

(8) $\exists x \forall yp(x, y) \to \forall y \exists xp(x, y)$

(9) $\forall x(p(x) \wedge q(x)) \leftrightarrow \forall xp(x) \wedge \forall xq(x)$

(10) $\exists x(p(x) \vee q(x)) \leftrightarrow \exists xp(x) \vee \exists xq(x)$

(11) $p \wedge \forall xq(x) \leftrightarrow \forall x(p \wedge q(x))$

(12) $p \vee \forall xq(x) \leftrightarrow \forall x(p \vee q(x))$

(13) $p \wedge \exists xq(x) \leftrightarrow \exists x(p \wedge q(x))$

(14) $p \vee \exists xq(x) \leftrightarrow \exists x(p \vee q(x))$

(15) $\forall x(p(x) \to q(x)) \to (\forall xp(x) \to \forall xq(x))$

(16) $\forall x(p(x) \to q(x)) \to (\exists xp(x) \to \exists xq(x))$

(17) $\exists x(p(x) \wedge q(x)) \to \exists xp(x) \wedge \exists xq(x)$

(18) $\forall xp(x) \vee \forall xq(x) \to \forall x(p(x) \vee q(x))$

(19) $\neg \exists x \neg p(x) \leftrightarrow \forall xp(x)$

(20) $\neg \forall x \neg p(x) \leftrightarrow \exists p(x)$

(21) $\neg \exists xp(x) \leftrightarrow \forall x \neg p(x)$

(22) $\exists x \neg p(x) \leftrightarrow \neg \forall xp(x)$.

Proof. It is straightforward from the definition of the semantics and axiomatic schema of the first order interval neutrosophic predicate logic.

V. An Application of Interval Neutrosophic Logic

In this section, we provide a practical application of the interval neutrosophic logic – Interval Neutrosophic Logic System (INLS). INLS can handle rule uncertainty as same as type-2 FLS [10], besides, it can handle rule inconsistency without the danger of trivialization. Like the classical FLS, INLS is also characterized by IF-THEN rules. INLS consists of neutrosophication, neutrosophic inference, a neutrosophic rule base, neutrosophic type reduction and deneutrosophication. Given an input vector $x = (x_1, ..., x_n)$, where $x_1, ..., x_n$ can be crisp inputs or neutrosophic sets, the INLS will generate a crisp output y. The general scheme of INLS is shown in Figure 1.

Suppose the neutrosophic rule base consists of M rules in which each rule has n antecedents and one consequent. Let the kth rule be denoted by R^k such that IF x_1 is A_1^k, x_2 is A_2^k, ..., and x_n is A_n^k, THEN y is B^k. A_i^k is an interval neutrosophic set defined on universe X_i with truth-membership function $T_{A_i^k}(x_i)$, indeterminacy-membership function $I_{A_i^k}(x_i)$ and falsity-membership function $F_{A_i^k}(x_i)$, where $T_{A_i^k}(x_i)$, $I_{A_i^k}(x_i)$, $F_{A_i^k}(x_i) \subseteq [0, 1]$, $1 \le i \le n$. B^k is an interval neutrosophic set defined on universe Y with truth-membership function $T_{B^k}(y)$, indeterminacy-membership function $I_{B^k}(y)$ and falsity-membership function $F_{B^k}(y)$, where $T_{B^k}(y), I_{B^k}(y), F_{B^k}(y) \subseteq [0, 1]$. Given fact x_1 is $\widetilde{A}_1^k$, x_2 is $\widetilde{A}_2^k$, ..., and x_n is $\widetilde{A}_n^k$, then consequence y is $\widetilde{B}^k$. $\widetilde{A}_i^k$ is an interval neutrosophic set defined on universe X_i with truth-membership function $T_{\widetilde{A}_i^k}(x_i)$, indeterminacy-membership function $I_{\widetilde{A}_i^k}(x_i)$ and falsity-membership function $F_{\widetilde{A}_i^k}(x_i)$, where $T_{\widetilde{A}_i^k}(x_i), I_{\widetilde{A}_i^k}(x_i), F_{\widetilde{A}_i^k}(x_i) \subseteq [0, 1]$, $1 \le i \le n$. $\widetilde{B}^k$ is an interval neutrosophic set defined on universe

Y with truth-membership function $T_{\tilde{B}^k}(y)$, indeterminacy-membership function $I_{\tilde{B}^k}(y)$ and falsity-membership function $F_{\tilde{B}^k}(y)$, where $T_{\tilde{B}^k}(y)$, $I_{\tilde{B}^k}(y)$, $F_{\tilde{B}^k}(y) \subseteq [0, 1]$. In this paper, we consider $a_i \leq x_i \leq b_i$ and $\alpha \leq y \leq \beta$.

An unconditional neutrosophic proposition is expressed by the phrase: "Z is C", where Z is a variable that receives values z from a universal set U, and C is an interval neutrosophic set defined on U that represents a neutrosophic predicate. Each neutrosophic proposition p is associated with $\langle t(p), i(p), f(p)\rangle$ with $t(p), i(p), f(p) \subseteq [0, 1]$. In general, for any value z of Z, $\langle t(p), i(p), f(p)\rangle = \langle T_C(z), I_C(z), F_C(z)\rangle$.

For implication $p \to q$, we define the semantics as:

$$\sup t_{p\to q} = \min(\sup t(p), \sup t(q)), \tag{38}$$

$$\inf t_{p\to q} = \min(\inf t(p), \inf t(q)), \tag{39}$$

$$\sup i_{p\to q} = \min(\sup i(p), \sup i(q)), \tag{40}$$

$$\inf i_{p\to q} = \min(\inf i(p), \inf i(q)), \tag{41}$$

$$\sup f_{p\to q} = \max(\sup f(p), \sup f(q)), \tag{42}$$

$$\inf f_{p\to q} = \max(\inf f(p), \inf f(q)), \tag{43}$$

where $t(p), i(p), f(p), t(q), i(q), f(q) \subseteq [0, 1]$.

Let $X = X_1 \times \cdots \times X_n$. The truth-membership function, indeterminacy-membership function and falsity-membership function $T_{\tilde{B}^k}(y)$, $I_{\tilde{B}^k}(y)$, $F_{\tilde{B}^k}(y)$ of a fired kth rule can be represented using the definition of interval neutrosophic composition functions (23)-(25) and the semantics of conjunction and disjunction defined in Table II and equations (38)-(43) as:

$$\sup T_{\tilde{B}^k}(y) = \sup_{x\in X} \min(\sup T_{\tilde{A}_1^k}(x_1), \sup T_{A_1^k}(x_1), \ldots, \sup T_{\tilde{A}_n^k}(x_n), \sup T_{A_n^k}(x_n), \sup T_{B^k}(y)), \tag{44}$$

$$\inf T_{\widetilde{B}^k}(y) = \sup_{x \in X} \min(\inf T_{\widetilde{A}_1^k}(x_1), \inf T_{A_1^k}(x_1), \ldots, \inf T_{\widetilde{A}_n^k}(x_n),$$
$$\inf T_{A_n^k}(x_n), \inf T_{B^k}(y)), \tag{45}$$

$$\sup I_{\widetilde{B}^k}(y) = \sup_{x \in X} \min(\sup I_{\widetilde{A}_1^k}(x_1), \sup I_{A_1^k}(x_1), \ldots, \sup I_{\widetilde{A}_n^k}(x_n),$$
$$\sup I_{A_n^k}(x_n), \sup I_{B^k}(y)), \tag{46}$$

$$\inf I_{\widetilde{B}^k}(y) = \sup_{x \in X} \min(\inf I_{\widetilde{A}_1^k}(x_1), \inf I_{A_1^k}(x_1), \ldots, \inf I_{\widetilde{A}_n^k}(x_n),$$
$$\inf I_{A_n^k}(x_n), \inf I_{B^k}(y)), \tag{47}$$

$$\sup F_{\widetilde{B}^k}(y) = \inf_{x \in X} \max(\sup F_{\widetilde{A}_1^k}(x_1), \sup F_{A_1^k}(x_1), \ldots, \sup F_{\widetilde{A}_n^k}(x_n),$$
$$\sup F_{A_n^k}(x_n), \sup F_{B^k}(y)), \tag{48}$$

$$\inf F_{\widetilde{B}^k}(y) = \inf_{x \in X} \max(\inf F_{\widetilde{A}_1^k}(x_1), \inf F_{A_1^k}(x_1), \ldots, \inf F_{\widetilde{A}_n^k}(x_n),$$
$$\inf F_{A_n^k}(x_n), \inf F_{B^k}(y)), \tag{49}$$

where $y \in Y$.

Now, we give the algorithmic description of INLS.

Step 1. Neutrosophication

The purpose of neutrosophication is to map inputs into interval neutrosophic input sets. Let G_i^k be an interval neutrosophic input set to represent the result of neutrosophication of ith input variable of kth rule. Then

$$\sup T_{G_i^k}(x_i) = \sup_{x_i \in X_i} \min(\sup T_{\widetilde{A}_i^k}(x_i), \sup T_{A_i^k}(x_i)), \tag{50}$$

$$\inf T_{G_i^k}(x_i) = \sup_{x_i \in X_i} \min(\inf T_{\widetilde{A}_i^k}(x_i), \inf T_{A_i^k}(x_i)), \tag{51}$$

$$\sup I_{G_i^k}(x_i) = \sup_{x_i \in X_i} \min(\sup I_{\widetilde{A}_i^k}(x_i), \sup I_{A_i^k}(x_i)), \tag{52}$$

$$\inf I_{G_i^k}(x_i) = \sup_{x_i \in X_i} \min(\inf I_{\tilde{A}_i^k}(x_i), \inf I_{A_i^k}(x_i)), \tag{53}$$

$$\sup F_{G_i^k}(x_i) = \inf_{x_i \in X_i} \max(\sup F_{\tilde{A}_i^k}(x_i), \sup F_{A_i^k}(x_i)), \tag{54}$$

$$\inf F_{G_i^k}(x_i) = \inf_{x_i \in X_i} \max(\inf F_{\tilde{A}_i^k}(x_i), \inf F_{A_i^k}(x_i)), \tag{55}$$

where $x_i \in X_i$.

If x_i are crisp inputs, then equations (50)-(55) are simplified to

$$\sup T_{G_i^k}(x_i) = \sup T_{A_i^k}(x_i), \tag{56}$$

$$\inf T_{G_i^k}(x_i) = \inf T_{A_i^k}(x_i), \tag{57}$$

$$\sup I_{G_i^k}(x_i) = \sup I_{A_i^k}(x_i), \tag{58}$$

$$\inf I_{G_i^k}(x_i) = \inf I_{A_i^k}(x_i), \tag{59}$$

$$\sup F_{G_i^k}(x_i) = \sup F_{A_i^k}(x_i), \tag{60}$$

$$\inf F_{G_i^k}(x_i) = \inf F_{A_i^k}(x_i), \tag{61}$$

where $x_i \in X_i$.

Figure 2 shows the conceptual diagram for neutrosophication of a crisp input x_1.

Step 2. Neutrosophic inference

The core of INLS is the neutrosophic inference, the principle of which has already been explained above. Suppose the kth rule is fired. Let G^k be an interval neutrosophic set to represent the result of the input and antecedent operation for kth rule. Then

$$\sup T_{G^k}(x) = \sup_{x \in X} \min(\sup T_{\tilde{A}_1^k}(x_1), \sup T_{A_1^k}(x_1), \ldots,$$
$$\sup T_{\tilde{A}_n^k}(x_n), \sup T_{A_n^k}(x_n)), \tag{62}$$

$$\inf T_{G^k}(x) = \sup_{x \in X} \min(\inf T_{\tilde{A}_1^k}(x_1), \inf T_{A_1^k}(x_1), \ldots,$$

$$\inf T_{\tilde{A}_n^k}(x_n), \inf T_{A_n^k}(x_n)), \tag{63}$$

$$\sup I_{G^k}(x) = \sup_{x \in X} \min(\sup I_{\tilde{A}_1^k}(x_1), \sup I_{A_1^k}(x_1), \ldots,$$

$$\sup I_{\tilde{A}_n^k}(x_n), \sup I_{A_n^k}(x_n)), \tag{64}$$

$$\inf I_{G^k}(x) = \sup_{x \in X} \min(\inf I_{\tilde{A}_1^k}(x_1), \inf I_{A_1^k}(x_1), \ldots,$$

$$\inf I_{\tilde{A}_n^k}(x_n), \inf I_{A_n^k}(x_n)), \tag{65}$$

$$\sup F_{G^k}(x) = \inf_{x \in X} \max(\sup F_{\tilde{A}_1^k}(x_1), \sup F_{A_1^k}(x_1), \ldots,$$

$$\sup F_{\tilde{A}_n^k}(x_n), \sup F_{A_n^k}(x_n)), \tag{66}$$

$$\inf F_{G^k}(x) = \inf_{x \in X} \max(\inf F_{\tilde{A}_1^k}(x_1), \inf F_{A_1^k}(x_1), \ldots,$$

$$\inf F_{\tilde{A}_n^k}(x_n), \inf F_{A_n^k}(x_n)), \tag{67}$$

where $x_i \in X_i$.

Here we restate the result of neutrosophic inference:

$$\sup T_{\tilde{B}^k}(y) = \min(\sup T_{G_1^k}(x), \ldots, \sup T_{B^k}(y)), \tag{68}$$

$$\inf T_{\tilde{B}^k}(y) = \min(\inf T_{G^k}(x), \inf T_{B^k}(y)), \tag{69}$$

$$\sup I_{\tilde{B}^k}(y) = \min(\sup I_{G^k}(x), \sup I_{B^k}(y)), \tag{70}$$

$$\inf I_{\tilde{B}^k}(y) = \min(\inf I_{G^k}(x), \inf I_{B^k}(y)), \tag{71}$$

$$\sup F_{\tilde{B}^k}(y) = \max(\sup F_{G^k}(x), \sup F_{B^k}(y)), \tag{72}$$

$$\inf F_{\tilde{B}^k}(y) = \max(\inf F_{G^k}(x), \inf F_{B^k}(y)), \tag{73}$$

where $x \in X$, $y \in Y$.

Suppose that N rules in the neutrosophic rule base are fired, where $N \leq M$, then, the output interval neutrosophic set $\widetilde{B}$ is

$$\sup T_{\widetilde{B}}(y) = \max_{k=1}^{N} \sup T_{\widetilde{B}^k}(y), \tag{74}$$

$$\inf T_{\widetilde{B}}(y) = \max_{k=1}^{N} \inf T_{\widetilde{B}^k}(y), \tag{75}$$

$$\sup I_{\widetilde{B}}(y) = \max_{k=1}^{N} \sup I_{\widetilde{B}^k}(y), \tag{76}$$

$$\inf I_{\widetilde{B}}(y) = \max_{k=1}^{N} \inf I_{\widetilde{B}^k}(y), \tag{77}$$

$$\sup F_{\widetilde{B}}(y) = \min_{k=1}^{N} \sup F_{\widetilde{B}^k}(y), \tag{78}$$

$$\inf F_{\widetilde{B}}(y) = \min_{k=1}^{N} \inf F_{\widetilde{B}^k}(y), \tag{79}$$

where $y \in Y$.

Step 3. Neutrosophic type reduction

After neutrosophic inference, we will get an interval neutrosophic set $\widetilde{B}$ with $T_{\widetilde{B}}(y), I_{\widetilde{B}}(y), F_{\widetilde{B}}(y) \subseteq [0, 1]$. Then, we do the neutrosophic type reduction to transform each interval into one number. There are many ways to do it, here, we give one method:

$$T'_{\widetilde{B}}(y) = (\inf T_{\widetilde{B}}(y) + \sup T_{\widetilde{B}}(y))/2, \tag{80}$$

$$I'_{\widetilde{B}}(y) = (\inf I_{\widetilde{B}}(y) + \sup I_{\widetilde{B}}(y))/2, \tag{81}$$

$$F'_{\widetilde{B}}(y) = (\inf F_{\widetilde{B}}(y) + \sup F_{\widetilde{B}}(y))/2, \tag{82}$$

where $y \in Y$.

So, after neutrosophic type reduction, we will get an ordinary neutrosophic set (a type-1 neutrosophic set) $\widetilde{B}$. Then we need to do the deneutrosophication to get a crisp output.

Step 4. Deneutrosophication

The purpose of deneutrosophication is to convert an ordinary neutrosophic set (a type-1 neutrosophic set) obtained by neutrosophic type reduction to a single real number which represents the real output. Similar to defuzzification [9], there are many deneutrosophication methods according to different applications. Here we give one method. The deneutrosophication process consists of two steps.

Step 4.1. *Synthesization*. It is the process to transform an ordinary neutrosophic set (a type-1 neutrosophic set) $\widetilde{B}$ into a fuzzy set $\widetilde{B}$. It can be expressed using the following function:

$$f(T'_{\widetilde{B}}(y),\ I'_{\widetilde{B}}(y),\ F'_{\widetilde{B}}(y)) : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]. \tag{83}$$

Here we give one definition of f:

$$T_{\widetilde{B}}(y) = a * T'_{\widetilde{B}}(y) + b * (1 - F'_{\widetilde{B}}(y)) + c * I'_{\widetilde{B}}(y)/2 + d * (1 - I'_{\widetilde{B}}(y)/2), \tag{84}$$

where $0 \le a, b, c, d \le 1$, $a + b + c + d = 1$.

The purpose of synthesization is to calculate the overall truth-degree according to three components: truth-membership function, indeterminacy-membership function and falsity-membership function. The component-truth-membership function gives the direct information about the truth-degree, so we use it directly in the formula. The component-falsity-membership function gives the indirect information about the truth-degree, so we use $(1 - F)$ in the formula. To understand the meaning of indeterminacy-membership function I, we give an example: a statement is "The quality of service is good", now firstly a person has to select a decision among $\{T, I, F\}$, secondly he or she has to answer the degree of the decision in $[0, 1]$. If he or she chooses $I = 1$, it means 100% "not sure" about the statement, i.e., 50% true and 50% false for the statement (100% balanced), in this sense, $I = 1$ contains the potential truth value 0.5. If he or she chooses $I = 0$, it means 100% "sure" about the statement, i.e., either 100% true or 100% false for the statement (0% balanced), in this sense, $I = 0$ is related to two extreme cases, but we do not know which one is in his or her mind. So we have to consider both at the same time: $I = 0$ contains the potential truth value that is either 0 or 1. If I decreases from 1 to 0, then the potential truth value changes from one value 0.5 to two different possible values gradually to the final possible ones 0 and 1 (i.e., from 100% balanced to 0% balanced), since he or she does not choose either T or F but I, we do not know his or her final truth value. Therefore, the formula has to consider two potential truth values implicitly represented by I with different weights (c and d) because of lack of his or her final decision information after he or she has chosen I. Generally, $a > b > c$, d; c and d could be decided subjectively or objectively as long as enough information is available. The parameters a, b, c and d can be tuned using learning algorithms such as neural networks and genetic algorithms in the development of application to improve the performance of the INLS.

Step 4.2. *Calculation of a typical neutrosophic value*. Here we introduce one method of calculation of center of area. The method is sometimes called the *center of gravity method* or *centroid method*, the deneutrosophicated value, $dn(T_{\widetilde{B}}(y))$ is calculated by the formula

$$dn(T_{\widetilde{B}}(y)) = \frac{\int_{\alpha}^{\beta} T_{\widetilde{B}}(y)\, y dy}{\int_{\alpha}^{\beta} T_{\widetilde{B}} dy}. \tag{85}$$

VI. Conclusions

In this paper, we give the formal definitions of interval neutrosophic logic which are extension of many other classical logics such as fuzzy logic, intuitionistic fuzzy logic and paraconsistent logics, etc. Interval neutrosophic logic includes the interval neutrosophic propositional logic and the first order interval neutrosophic predicate logic. We call them *classical* (*standard*) *neutrosophic logic*. In the future, we will also discuss and explore the non-classical (nonstandard) neutrosophic logic such as modal interval neutrosophic logic, temporal interval neutrosophic logic, etc. Interval neutrosophic logic can handle not only imprecise, fuzzy and incomplete propositions but also inconsistent propositions without the danger of trivialization. The paper also gives an application based on the semantic notion of interval neutrosophic logic – the Interval Neutrosophic

Logic System (INLS) which is the generalization of classical FLS and interval valued fuzzy FLS. Interval neutrosophic logic will have a lot of potential applications in computational Web intelligence [19]. For example, current fuzzy Web intelligence techniques can be improved by using more reliable interval neutrosophic logic methods because T, I and F are all used in decision making. In large, such robust interval neutrosophic logic methods can also be used in other applications such as medical informatics, bioinformatics and human-oriented decision-making under uncertainty. In fact, interval neutrosophic sets and interval neutrosophic logic could be applied in fields where fuzzy sets and fuzzy logic are suitable, also fields where paraconsistent logics are suitable.

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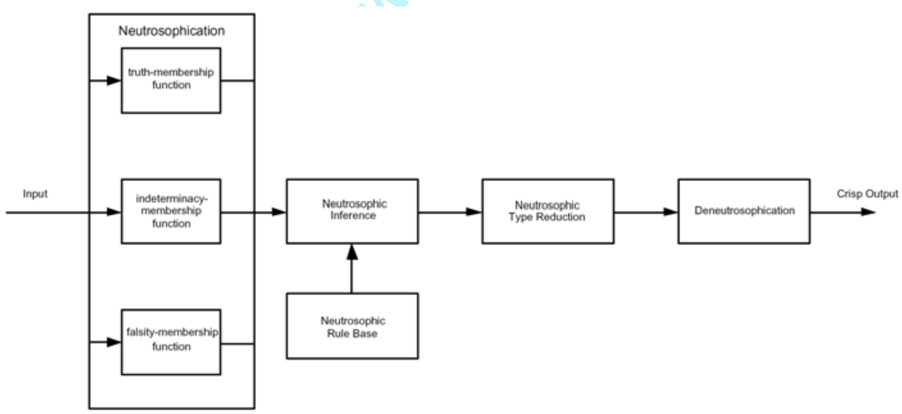

Figure 1. General scheme of an INLS.

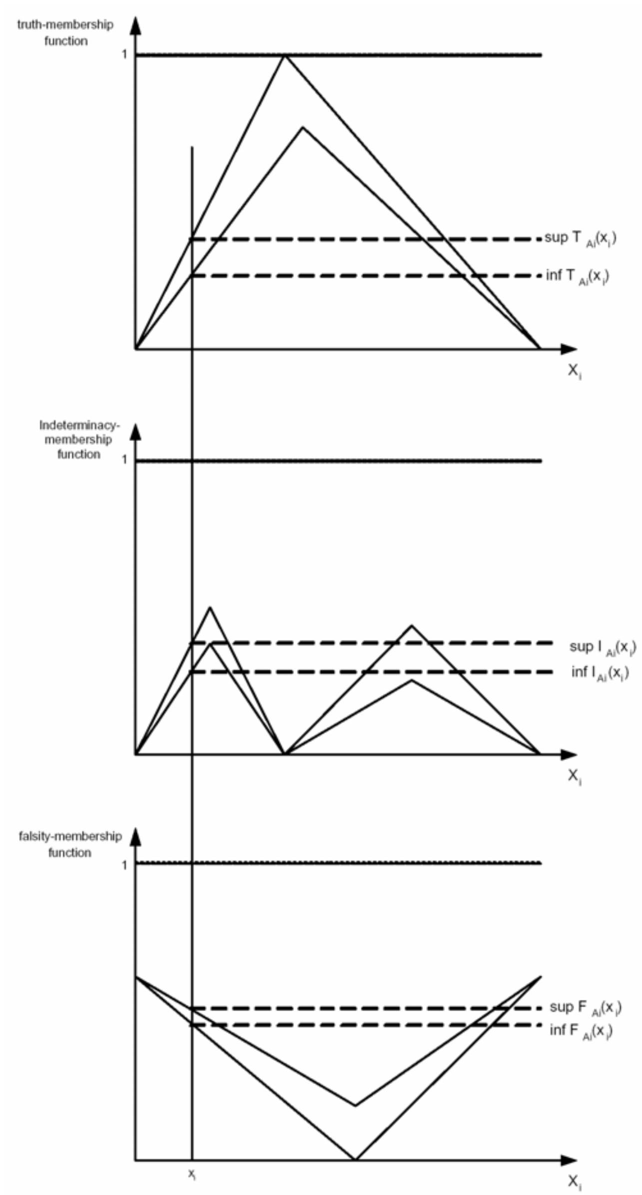

Figure 2. Conceptual diagram for neutrosophication of crisp input.

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