Far East J. Math. Sci. (FJMS) **24**(3) (2007), 425-437
This paper is available online at http://www.pphmj.com

DUAL SPLIT QUATERNIONS AND MOTIONS IN LORENTZIAN SPACE $\mathbb{R}_1^3$

OSMAN KEÇILIOĞLU and HALIT GÜNDOĞAN

(Received July 11, 2006)

Submitted by K. K. Azad

Abstract

In this paper, dual Lorentzian angles and dual split quaternions are defined. Then using these concepts, rotation motions, translation motions and screw motions are obtained in 3-dimensional Lorentzian space $\mathbb{R}_1^3$.

1. Introduction

For the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ the Lorentzian inner product on $\mathbb{R}^3$ is given by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The vector space on $\mathbb{R}^3$ equipped with the Lorentzian inner product is called 3-*dimensional Lorentzian space* and denoted by $\mathbb{R}_1^3$. For a vector $x \in \mathbb{R}_1^3$ the sign of $\langle x, x \rangle$ determines the type of x. If it is positive, then x is called a *space-like vector*. If it is zero, then x is called a *null vector* or

2000 Mathematics Subject Classification: 53A17, 53B30.

Keywords and phrases: Lorentzian space, split quaternion, dual number, dual quaternion, rotation, translation, screw motions.

light-like vector. If it is negative, then x is called a *time-like vector*. Moreover, if the first component x_1 of x is positive, then x is called a *positive vector*. If x_1 is negative, then x is called a *negative vector*. For $x \in \mathbb{R}_1^3$, the norm of x is defined by $\| x \| = \sqrt{\langle x, x\rangle}$. The norm $\| x \|$ is either positive or zero or positive imaginary. If $\| x \|$ is positive imaginary, then the notation $\||x\||$ used instead of $\| x \|$.

For the vectors $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$ the cross product is defined by

$$x \wedge y = (x_3y_2 - x_2y_3,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1).$$

The subspace $\vee$ of $\mathbb{R}_1^3$ is time-like space if and only if $\vee$ has a time-like vector. $\vee$ is space-like space if and only if all nonzero vectors of $\vee$ are space-like vectors. Otherwise $\vee$ is light-like space.

In Lorentzian space $\mathbb{R}_1^3$ the angle between vectors x, y is defined as follows:

(1) For the time-like vectors x, y in $\mathbb{R}_1^3$

$$\langle x, y\rangle = -\||x\|| \cdot \||y\|| \cosh \varphi,$$

$$\| x \wedge y \| = \||x\|| \cdot \||y\|| \sinh \varphi.$$

(2) For the space-like vectors x, y that span the space-like vector space in $\mathbb{R}_1^3$

$$\langle x, y\rangle = \| x \| \cdot \| y \| \cos \varphi,$$

if $x \wedge y$ is time-like vector, then

$$\||x \wedge y\|| = \| x \| \cdot \| y \| \sin \varphi.$$

(3) For the space-like vectors x, y that span the time-like vector space in $\mathbb{R}_1^3$

$$|\langle x, y\rangle| = \| x \| \cdot \| y \| \cosh \varphi,$$

$$\| x \wedge y \| = \| x \| \cdot \| y \| \sinh \varphi.$$

(4) For the space-like vector x and positive time-like vector y in $\mathbb{R}_1^3$

$$|\langle x,\ y\rangle| = \| x \| \cdot \| y \| \sinh \varphi,$$

$$\| x \wedge y \| = \| x \| \cdot \| y \| \cosh \varphi \qquad [1, 2].$$

A split quaternion is defined by the base $\{1, \vec{e_1}, \vec{e_2}, \vec{e_3}\}$, where, $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ satisfy the equalities $\vec{e}_1^{\,2} = -1$, $\vec{e}_2^{\,2} = +1$, $\vec{e}_3^{\,2} = +1$, $\vec{e_1} \cdot \vec{e_2} = -\vec{e_2} \cdot \vec{e_1} = \vec{e_3}$, $\vec{e_2} \cdot \vec{e_3} = -\vec{e_3} \cdot \vec{e_2} = -\vec{e_1}$, $\vec{e_3} \cdot \vec{e_1} = -\vec{e_1} \cdot \vec{e_3} = -\vec{e_2}$. So a split quaternion can be expressed as $q = d + a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$, where, a, b, c, d are real scalars. The set of split quaternions is represented by H. If we take $S_q = d$ and $V_q = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$, then the split quaternion $q = d + a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$ can be re-written as $q = S_q + V_q$. The split quaternion addition is defined as

$$q_1 + q_2 = S_{q_1} + S_{q_2} + V_{q_1} + V_{q_2}$$

for every $q_1, q_2 \in H$. Note that $S_{q_1+q_2} = S_{q_1} + S_{q_2}$ and $V_{q_1+q_2} = V_{q_1} + V_{q_2}$. The scalar product of split quaternion is

$$\lambda q = \lambda S_q + \lambda V_q,$$

where λ is real scalar.

The split quaternion product denoted by $\times$, is defined in the table below

$\times$	1	$\vec{e_1}$	$\vec{e_2}$	$\vec{e_3}$
1	1	$\vec{e_1}$	$\vec{e_2}$	$\vec{e_3}$
$\vec{e_1}$	$\vec{e_1}$	-1	$\vec{e_3}$	$-\vec{e_2}$
$\vec{e_2}$	$\vec{e_2}$	$-\vec{e_3}$	1	$-\vec{e_1}$
$\vec{e_3}$	$\vec{e_3}$	$\vec{e_2}$	$\vec{e_1}$	1

Thus H is a real algebra.

The conjugate $K(q)$ of the split quaternion $q = S_q + V_q$ is defined as $K(q) = S_q - V_q$. The norm of the split quaternion $q = d + a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$ denoted by $N(q)$ is $N(q) = \sqrt{K(q) \times q} = \sqrt{q \times K(q)}$. Observe that

$$N(q) = \sqrt{d^2 + a^2 - b^2 - c^2} \quad [3].$$

The set $\{a + \varepsilon a_0 \,|\, a, a_0 \in \mathbb{R}, \varepsilon^2 = 0\}$ is called the *set of dual numbers* and represented by D. The set $D^3 = \{\vec{a} + \varepsilon \overrightarrow{a_0} \,|\, \vec{a}, \overrightarrow{a_0} \in \mathbb{R}^3, \varepsilon^2 = 0\}$, with the inner product of $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon \overrightarrow{b_0}$ in D^3 defined by

$$\langle \vec{A}, \vec{B} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon(\langle \vec{a}, \overrightarrow{b_0} \rangle + \langle \overrightarrow{a_0}, \vec{b} \rangle) \tag{1.1}$$

forms a space on D^3 that is called *dual Lorentzian space* and denoted by D_1^3. Here the inner products on the right side are Lorentzian inner products in $\mathbb{R}^3$.

For all $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon \overrightarrow{b_0}$ in D_1^3 the cross product $\vec{A} \wedge \vec{B}$ is defined as

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon(\vec{a} \wedge \overrightarrow{b_0} + \overrightarrow{a_0} \wedge \vec{b}),$$

where, the cross product in the right side of equality are the cross products on $\mathbb{R}_1^3$.

Let $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a_0} \in D_1^3$. If the vector $\vec{a}$ is space-like vector, then $\vec{A}$ is said to be *space-like dual vector*, if the vector $\vec{a}$ is time-like vector, then $\vec{A}$ is said to be *time-like dual vector*, and if the vector $\vec{a}$ is light-like (null) vector, then $\vec{A}$ is said to be *light-like dual vector* or *null dual vector*.

The norm of the dual vector $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a_0}$ in D_1^3 is defined by

$$\|\vec{A}\| = \sqrt{\langle \vec{A}, \vec{A} \rangle} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \overrightarrow{a_0} \rangle}{\|\overrightarrow{a_0}\|}, \quad \vec{a} \neq 0.$$

If $\|\vec{A}\| = 1$, then the dual vector $\vec{A}$ is called *unit dual vector*.

Let d be a directed line in $\mathbb{R}_1^3$ whose direction is given by the vector $\vec{a}$. Then the type of the vector $\vec{a}$ determines the type of d. Namely, if the vector $\vec{a}$ is time-like vector, then the line d is time-like line, if the vector $\vec{a}$ is space-like vector, then the line d is space-like line and if the vector $\vec{a}$ is null vector, then the line d is null line.

There exists one to one correspondence between directed lines in $\mathbb{R}_1^3$ and unit dual vectors in D_1^3 [4, 5].

2. Dual Split Quaternions

Let $q,\ q_0$ be split quaternions. Then a dual split quaternion Q is defined by $Q = q + \varepsilon q_0$. The set of dual split quaternions is denoted by $\mathcal{D}$. By taking dual numbers $D,\ A,\ B,\ C$ the dual split quaternion $Q = D + A\overrightarrow{e_1} + B\overrightarrow{e_2} + C\overrightarrow{e_3}$ can be re-written as $Q = S_Q + V_Q$, where $S_Q = D,\ V_Q = A\overrightarrow{e_1} + B\overrightarrow{e_2} + C\overrightarrow{e_3}$.

The sum of dual split quaternions $Q_1,\ Q_2$ is defined as

$$Q_1 + Q_2 = S_{Q_1} + S_{Q_2} + V_{Q_1} + V_{Q_2}.$$

The product of dual split quaternion Q by the real scalar λ is given by

$$\lambda Q = \lambda S_Q + \lambda V_Q.$$

For all $Q_1 = q_1 + \varepsilon q_{1_0},\quad Q_2 = q_2 + \varepsilon q_{2_0}$ in $\mathcal{D}$, their dual split quaternionic product is given by

$$Q_1 \times Q_2 = q_1 \times q_2 + \varepsilon(q_1 \times q_{2_0} + q_{1_0} \times q_2).$$

As a result $\mathcal{D}$ forms a real algebra.

The conjugate of the dual split quaternion $Q = q + \varepsilon q_0$ is denoted by $K(Q)$ and is defined as

$$K(Q) = K(q) + \varepsilon K(q_0).$$

The norm $N(Q)$ of Q is given by

$$N(Q) = K(Q) \times Q = Q \times K(Q).$$

The inverse of Q with $N(Q) \neq 0$ is defined as

$$Q^{-1} = \frac{K(Q)}{N(Q)}.$$

For all Q_1, Q_2 in $\mathcal{D}$

$$N(Q_1 \times Q_2) = N(Q_1) \cdot N(Q_2),$$

$$(Q_1 \times Q_2)^{-1} = Q_2^{-1} \times Q_1^{-1}.$$

3. Dual Lorentzian Angles

Using the inner product (1.1), the following theorems can be proven.

Theorem 1. *Let $\vec{A}$ and $\vec{B}$ be time-like unit dual vectors. Then*

$$\langle \vec{A}, \vec{B} \rangle = -\cosh(\varphi + \varepsilon\varphi_0),$$

$$\vec{A} \wedge \vec{B} = \vec{N} \sinh(\varphi + \varepsilon\varphi_0).$$

Here, $\vec{N}$ is unit dual vector corresponding to the line which is perpendicular to both lines corresponding to the vectors $\vec{A}$ and $\vec{B}$.

Theorem 2. *Let $\vec{a}$, $\vec{b}$ be space-like unit vectors that span space-like vector space. Then $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ are space-like unit dual vectors such that*

$$\langle \vec{A}, \vec{B} \rangle = \cos(\varphi + \varepsilon\varphi_0),$$

$$\vec{A} \wedge \vec{B} = \vec{N} \sin(\varphi + \varepsilon\varphi_0).$$

Theorem 3. *Let $\vec{a}$, $\vec{b}$ be space-like unit vectors that span time-like vector space. Then $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ are time-like unit dual vectors such that*

$$\langle \vec{A}, \vec{B} \rangle = \cosh(\varphi + \varepsilon\varphi_0),$$

$$\vec{A} \wedge \vec{B} = \vec{N} \sinh(\varphi + \varepsilon\varphi_0).$$

Theorem 4. *Let* $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ *be space-like unit dual vector and let* $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ *be time-like unit dual vector. Then,*

$$\langle \vec{A}, \vec{B} \rangle = \sinh(\varphi + \varepsilon\varphi_0),$$

$$\vec{A} \wedge \vec{B} = \vec{N} \cosh(\varphi + \varepsilon\varphi_0).$$

4. Motions in Lorentzian Space $\mathbb{R}_1^3$

4.1. Motions in between time-like lines

Theorem 5. *Let* $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ *be time-like unit dual vectors and let* $\vec{N} = \dfrac{\vec{A} \wedge \vec{B}}{\|\vec{A} \wedge \vec{B}\|}$. *Then*

$$\vec{B} \times \vec{A} = -(\cosh(\varphi + \varepsilon\varphi_0) + \vec{N} \sinh(\varphi + \varepsilon\varphi_0)).$$

Proof. Consider the equality

$$\vec{A} \times \vec{B} = \langle \vec{A}, \vec{B} \rangle + \vec{A} \wedge \vec{B}.$$

Then the proof follows from Theorem 1. □

Corollary 1. *Let* $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ *be time-like unit dual vectors and* $\overrightarrow{P_0} = \cosh(\varphi + \varepsilon\varphi_0) + \vec{N} \sinh(\varphi + \varepsilon\varphi_0)$. *Then* $\vec{A} = \vec{B} \times \overrightarrow{P_0}$, $\vec{B} = \overrightarrow{P_0} \times \vec{A}$.

Corollary 2 (Rotation Operator)**.** *If the lines corresponding to the time-like unit dual vectors* $\vec{A}$, $\vec{B}$ *intersect, then the dual angle between these lines is* $\varphi + \varepsilon 0 = \varphi$. *In this case,*

$$\overrightarrow{P_0} = \cosh\varphi + \vec{N} \sinh\varphi.$$

Since $\vec{A} = \vec{B} \times \overrightarrow{P_0}$ *and* $\vec{B} = \overrightarrow{P_0} \times \vec{A}$, *multiplying* $\vec{A}$ *by* $\overrightarrow{P_0}$ *from left* *means that rotating the line corresponding to* $\vec{A}$ *around* $\vec{N}$*-axis in positive direction by* φ *angle. Similarly, multiplying* $\vec{B}$ *by* $\overrightarrow{P_0}$ *from right means that rotating the line corresponding to* $\vec{B}$ *around* $\vec{N}$*-axis in negative direction by* φ *angle. Here,* $\overrightarrow{P_0}$ *is called a rotation operator.*

Corollary 3 (Translation Operator)**.** *If the lines corresponding to the time-like unit dual vectors* $\vec{A}$, $\vec{B}$ *are parallel, then the dual angle between these lines is* $0 + \varepsilon\varphi_0 = \varepsilon\varphi_0$. *In this case,*

$$\overrightarrow{P_0} = 1 + \varepsilon\varphi_0\vec{N}.$$

Since $\vec{A} = \vec{B} \times \overrightarrow{P_0}$ *and* $\vec{B} = \overrightarrow{P_0} \times \vec{A}$, *multiplying* $\vec{A}$ *by* $\overrightarrow{P_0}$ *from left means that sliding the line corresponding to* $\vec{A}$ *in the direction of* $\vec{N}$*-axis by* φ_0. *Similarly, multiplying* $\vec{B}$ *by* $\overrightarrow{P_0}$ *from right means that sliding the line corresponding to* $\vec{B}$ *in the direction of* $-\vec{N}$ *by* φ_0. *Here,* $\overrightarrow{P_0}$ *is called a translation operator.*

Corollary 4 (Screw Operator)**.** *If the dual angle between the lines corresponding to the time-like unit dual vectors* $\vec{A}$, $\vec{B}$ *is* $\varphi + \varepsilon\varphi_0$ *and*

$$\overrightarrow{P_0} = \cosh(\varphi + \varepsilon\varphi_0) + \vec{N}\sinh(\varphi + \varepsilon\varphi_0),$$

then since $\vec{A} = \vec{B} \times \overrightarrow{P_0}$ *and* $\vec{B} = \overrightarrow{P_0} \times \vec{A}$, *multiplying* $\vec{A}$ *by* $\overrightarrow{P_0}$ *from left means that first, rotating the line corresponding to* $\vec{A}$ *around* $\vec{N}$*-axis in positive direction by* φ *angle, then sliding this line in the direction of* $\vec{N}$ *by* φ_0. *This is the screw motion. Similarly, multiplying* $\vec{B}$ *by* $\overrightarrow{P_0}$ *from right means that, first, rotating the line corresponding to* $\vec{B}$ *around* $\vec{N}$*-axis in negative direction by* φ *angle, then sliding this line in the direction of* $-\vec{N}$ *by* φ_0. *Here,* $\overrightarrow{P_0}$ *is called a screw operator. By taking* $\varphi_0 = 0$ *in screw operator, a rotation operator is obtained. By taking* $\varphi = 0$ *in screw operator, a translation operator is obtained.*

Example 1. For $t \in \mathbb{R}$, the unit dual vectors corresponding the lines $\alpha(t) = (0, 0, 0) + t(3, 2, 1)$ and, $\beta(t) = (1, 0, 0) + t(3, 1, 2)$ are $\vec{A} = \left(\frac{3}{2}, 1, \frac{1}{2}\right) + \varepsilon(0, 0, 0)$ and $\vec{B} = \left(\frac{3}{2}, \frac{1}{2}, 1\right) + \varepsilon\left(0, -1, \frac{1}{2}\right)$, respectively. Then the corresponding screw operator is

$$\overrightarrow{P_0} = \frac{5}{4} + \left(-\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}\right) + \varepsilon\left(\frac{3}{4} + \left(-1, -\frac{3}{4}, \frac{3}{2}\right)\right).$$

4.2. Motions between space-like lines in space-like vector space

Theorem 6. *Let $\vec{a}$, $\vec{b}$ be space-like vectors that span space-like vector space and $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ be space-like unit dual vectors. Then*

$$\vec{B} \times \vec{A} = \cos(\varphi + \varepsilon\varphi_0) - \vec{N}\sin(\varphi + \varepsilon\varphi_0),$$

where

$$\vec{N} = \frac{\vec{A} \wedge \vec{B}}{\|\vec{A} \wedge \vec{B}\|}.$$

Corollary 5. *Let* $\overrightarrow{P_0} = \cos(\varphi + \varepsilon\varphi_0) - \vec{N}\sin(\varphi + \varepsilon\varphi_0)$. *Then*

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \quad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

In Corollaries 6, 7 and 8 below, the vectors $\vec{a}$ and $\vec{b}$ are space-like vectors that span space-like vector space. Also, $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ are space-like unit dual vectors.

Corollary 6. *If the lines corresponding to the unit dual vectors $\vec{A}$ and $\vec{B}$ intersect, then the dual angle between these lines is $\varphi + \varepsilon 0 = \varphi$. The rotation operator for this case is*

$$\overrightarrow{P_0} = \cos\varphi - \vec{N}\sin\varphi.$$

And also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \quad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Corollary 7. *If the lines corresponding to the unit dual vectors* $\vec{A}$ *and* $\vec{B}$ *are parallel, then the dual angle between these lines is* $0 + \varepsilon\varphi_0 = \varphi_0$. *The translation operator for this case is*

$$\overrightarrow{P_0} = 1 - \varepsilon\varphi_0 \vec{N}.$$

Also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Corollary 8. *If the dual angle between the lines corresponding to the unit dual vectors is* $\varphi + \varepsilon\varphi_0$, *then screw operator is*

$$\overrightarrow{P_0} = \cos(\varphi + \varepsilon\varphi_0) - \vec{N}\sin(\varphi + \varepsilon\varphi_0).$$

Also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Example 2. The lines $\alpha(t) = (0, 0, 0) + t(1, 2, 1)$ and $\beta(t) = (1, 0, 0) + t(1, -1, 3)$ corresponding to the unit dual vectors $\vec{A} = \left(\frac{1}{2}, 1, \frac{1}{2}\right) + \varepsilon(0, 0, 0)$ and $\vec{B} = \left(\frac{1}{3}, -\frac{1}{3}, 1\right) + \varepsilon\left(0, -1, -\frac{1}{3}\right)$, respectively. Then the corresponding screw operator is

$$\overrightarrow{P_0} = 0 + \left(\frac{7}{6}, \frac{1}{3}, \frac{1}{2}\right) + \varepsilon\left(-\frac{7}{6} + \left(\frac{1}{6}, -\frac{1}{6}, \frac{1}{2}\right)\right).$$

4.3. Motions between space-like lines in time-like vector space

Theorem 7. *Let* $\vec{a}$, $\vec{b}$ *be space-like vectors that span time-like vector space and* $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$, $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ *be space-like unit dual vectors. Then*

$$\vec{B} \times \vec{A} = \cosh(\varphi + \varepsilon\varphi_0) - \vec{N}\sinh(\varphi + \varepsilon\varphi_0),$$

where

$$\vec{N} = \frac{\vec{A} \wedge \vec{B}}{\|\vec{A} \wedge \vec{B}\|}.$$

Corollary 9. *Let* $\overrightarrow{P_0} = \cosh(\varphi + \varepsilon\varphi_0) - \vec{N}\sinh(\varphi + \varepsilon\varphi_0)$. *Then*

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

In Corollaries 10, 11 and 12 below, the vectors $\vec{a}$ and $\vec{b}$ are space-like vectors spanning time-like vector space. Moreover, $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ are space-like unit dual vectors.

Corollary 10. *If the lines corresponding to the unit dual vectors* $\vec{A}$ *and* $\vec{B}$ *intersect, then the dual angle between these lines is* $\varphi + \varepsilon 0 = \varphi$. *Thus the rotation operator is*

$$\overrightarrow{P_0} = \cosh\varphi - \vec{N}\sinh\varphi.$$

Also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Corollary 11. *If the lines corresponding to the unit dual vectors* $\vec{A}$ *and* $\vec{B}$ *are parallel, then the dual angle between these lines is* $0 + \varepsilon\varphi_0 = \varphi_0$. *In this case, the translation operator is*

$$\overrightarrow{P_0} = 1 - \varepsilon\varphi_0\vec{N}.$$

Also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Corollary 12. *If the dual angle between the lines corresponding to the unit dual vectors is* $\varphi + \varepsilon\varphi_0$, *then the screw operator is*

$$\overrightarrow{P_0} = \cosh(\varphi + \varepsilon\varphi_0) - \vec{N}\sinh(\varphi + \varepsilon\varphi_0).$$

Also,

$$\vec{A} = \vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Example 3. The lines $\alpha(t) = (0, 0, 0) + t(1, 2, 1)$ and $\beta(t) = (1, 0, 0) + t(3, 3, 1)$ corresponding to the unit dual vectors $\vec{A} = \left(\frac{1}{2}, 1, \frac{1}{2}\right) + \varepsilon(0, 0, 0)$

and $\vec{B} = (3, 3, 1) + \varepsilon(0, -1, 3)$, respectively. Then the corresponding screw operator is

$$\overrightarrow{P_0} = 2 + \left(-\frac{1}{2}, -1, \frac{3}{2}\right) + \varepsilon\left(\frac{1}{2} + \left(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}\right)\right).$$

4.4. Motions between time-like lines and space-like lines

Theorem 8. *Let* $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ *be space-like unit dual vector and,* $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ *be time-like unit dual vector. And let* $\vec{N} = \frac{\vec{A} \wedge \vec{B}}{\|\vec{A} \wedge \vec{B}\|}$. *Then*

$$\vec{B} \times \vec{A} = \sinh(\varphi + \varepsilon\varphi_0) - \vec{N}\cosh(\varphi + \varepsilon\varphi_0).$$

Corollary 13. *Let* $\overrightarrow{P_0} = \sinh(\varphi + \varepsilon\varphi_0) - \vec{N}\cosh(\varphi + \varepsilon\varphi_0)$. *Then*

$$\vec{A} = -\vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

In Corollaries 14 and 15, $\vec{A} = \vec{a} + \varepsilon\overrightarrow{a_0}$ is space-like unit dual vector and $\vec{B} = \vec{b} + \varepsilon\overrightarrow{b_0}$ is time-like unit dual vector.

Corollary 14. *If the lines corresponding to the unit dual vectors* $\vec{A}$ *and* $\vec{B}$ *intersect, then the dual angle between these lines is* $\varphi + \varepsilon 0 = \varphi$. *In this case, the rotation operator is*

$$\overrightarrow{P_0} = \sinh\varphi - \vec{N}\cosh\varphi.$$

Also,

$$\vec{A} = -\vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = \overrightarrow{P_0} \times \vec{A}.$$

Corollary 15. *Let the dual angle between the lines corresponding to the unit dual vectors* $\vec{A}$, $\vec{B}$ *be* $\varphi + \varepsilon\varphi_0$. *Then screw operator is*

$$\overrightarrow{P_0} = \sinh(\varphi + \varepsilon\varphi_0) - \vec{N}\cosh(\varphi + \varepsilon\varphi_0).$$

Also,

$$\vec{A} = -\vec{B} \times \overrightarrow{P_0}, \qquad \vec{B} = -\overrightarrow{P_0} \times \vec{A}.$$

Example 4. The lines $\alpha(t) = (0, 0, 0) + t(2, 1, 2)$ and $\beta(t) = (1, 0, 0) + t(3, 2, 1)$ corresponding to the unit dual vectors $\vec{A} = (2, 1, 2) + \varepsilon(0, 0, 0)$ and $\vec{B} = \left(\frac{3}{2}, 1, \frac{1}{2}\right) + \varepsilon\left(0, -\frac{1}{2}, 1\right)$, respectively. The screw operator corresponding to the these lines is

$$\overrightarrow{P_0} = -1 + \left(-\frac{3}{2}, -2, -\frac{1}{2}\right) + \varepsilon\left(\frac{3}{2} + (2, 2, 1)\right).$$

References

[1] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.

[2] J. G. Ratcliffe, Foundation of Hyperbolic Manifolds, Springer-Verlag, New York, 1994.

[3] B. Rosenfeld, Geometry of Lie Groups, Kluwer Academic Publishers, 1997.

[4] E. Study, Geometrie der Dynamen, Leibzig, Teubner, 1903.

[5] H. H. Uğurlu and A. Çalişkan, The study mapping for directed space-like and time-like lines in Minkowsky 3-space $\mathbb{R}_1^3$, Math. Comput. Appl. l(2) (1996), 142-148.

Department of Mathematics
Kirikkale University
Kirikkale, Turkey
e-mail: kecilioglu@kku.edu.tr
hagundogan@hotmail.com