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SOME PROPERTIES OF THE SOLUTION SET FOR INTEGRAL NONCOMPACT EQUATIONS

GIUSEPPE ANICHINI and GIUSEPPE CONTI

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Abstract

In this paper we study the existence of continuous solutions over noncompact intervals of some nonlinear integral equations. A special interest is devoted to the study of the solution sets of nonlinear Volterra equations.

1. Introduction and Notations

Many problems (for instance in applied mechanics character) are reduced to the solution of a nonlinear integral equation. This mainly consists of the nonlinear generalization for the linear (possibly singular) integral equations to nonlinear integral equations: these methods have been used by many authors in the literature; see the books [14], [17] for typical references.

One of the first beginning is the study of the existence of continuous solutions of the following nonlinear Urison integral equation,

$$x(t) = h(t) + \int_0^t g(t, s, x(s))ds, \quad t \in (-\infty, +\infty). \tag{1}$$

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In general the specific character of a given integral equation is expressed only in terms of the properties of the corresponding integral operators which are contained in the equation we are dealing with. Usually a proof showing the existence of a solution starting with some conditions on the function $g(t, s, x)$ as well as the limits of integration and the function h is found. Besides the continuity property of the kernel of the operator, the property of compactness is often requested as an essential condition. This happens to be true, for instance, when the state variable $I \subset \mathbb{R}$ is a compact set or a measurable set in a suitable space. In this case the study of the nonlinear operator, associated to the integral equation, is achieved when a couple of Banach spaces such that the nonlinear operator mapping the first of them into the other is shown to have such good properties (see, e.g., [17]). So, based on these conditions, a Banach space and a topology are chosen in such a way that the existence problem is converted in showing the existence of a very good property of the set of fixed point of our integral operator, i.e., the fact that the latter in an R_δ-set (see, for instance, [1], [2], [5]).

So, in this paper, we are concerned with the solution sets for Volterra integral equations like:

$$\begin{cases} x(t) = h(t) + \int_0^t k(t, s) g(s, x(s)) ds \\ x(0) = x_0, \end{cases} \tag{2}$$

where $h : I = [0, T) \to \mathbb{R}^n$, $k : I \times I \to \mathbb{R}^n$, $g : I \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions, x_0 is a given vector of $\mathbb{R}^n$ and I is a (possible unbounded) interval of $\mathbb{R}$.

So we are trying to find conditions under which a given integral operator defines a mapping from (suitably chosen) Banach spaces and shares the property of continuity and, also in case of unboundedness of the state set I, it turns out to be a condensing operator.

Remark. The use of a measure of noncompactness and the reduction of the problem to the search for a fixed point of a condensing operator (or to the study of good properties of it) is a tool previously used by the authors (see [1]-[6]).

In the following $B(x_0, r)$ denotes an r-ball (in a metric space (S, d)), i.e., the set $\{x \in S : d(x, x_0) < r\}$, centered in x_0, where x_0 is any point in S.

Further, let us consider the (Hilbert) space $L^2(I, \mathbb{R}^n)$ normed, as usually, by $\| x \|_2 = \left(\int_I x^2(t) dt \right)^{1/2}$ and its (affine) subspace $E = \{x \in L^2(I, \mathbb{R}^n) : x(0) = x_0\}$.

Let X be some Banach space. If $V \subset X$ is a subset, then $\overline{V}$ denotes its (topological) closure and V^c denotes the complement of V. Finally $B_d(X)$ denotes the set of all nonempty and bounded subsets of X.

Definition 1. Let X be a Banach space and $V \subset X$ be a subset. A measure $\mu : B_d(X) \to \mathbb{R}^+$ defined by

$$\mu(V) = \inf\{\varepsilon > 0 : V \in B_d(X) \text{ admits a finite cover by sets of diameter } \leq \varepsilon\},$$

where the diameter of V is the $\sup\{\| x - y \| : x \in V, y \in V\}$, is called the (*Kuratowski*) *measure of noncompactness*.

A measure like μ has interesting properties, some of which are listed in the sequel:

(a) $\mu(V) = 0$ if and only if $\overline{V}$ is compact;

(b) $\mu(V) = \mu(\overline{V})$; $\mu(conv(V)) = \mu(V)$; ($conv(V) =$ convex hull of V);

(c) $\mu(\alpha(V_1) + (1 - \alpha)V_2) \leq \alpha\mu(V_1) + (1 - \alpha)\mu(V_2)$, $\alpha \in [0, 1]$;

(d) if $V_1 \subset V_2$, then $\mu(V_1) \leq \mu(V_2)$;

(e) if $\{V_n\}$ is a nested sequence of closed sets of $B_d(X)$ and if $\lim_{n \to +\infty} \mu(V_n) = 0$, then $\bigcap_{n=1}^{\infty} V_n \neq \varnothing$.

The corresponding measure of noncompactness for an operator is defined by $\mu(F) = \inf\{k > 0 : \mu(F(V)) \leq k\mu(V)\}$ for all bounded subsets $V \subset X$.

Definition 2. Let X be a complete metric space and $f : X \to X$ be a continuous mapping. Then f is called a *k-set contraction* if there exists $k \in [0, 1)$ such that, for all bounded noncompact subsets V of X, the following relation holds: $\mu(f(V)) \leq k\mu(V)$ [15, p. 160].

Definition 3. A continuous operator $F : X \to X$ such that $\mu(F(V)) < \mu(V)$, for any bounded $V \subset X$, is called *condensing* or *densifying*.

(The concept of measure of noncompactness is considerably dealt with in [8] and [11].)

Definition 4. Let S and S_1 be topological spaces and let $f : S \to S_1$. Then f is said to be *proper* if, whenever K_1 is a compact subset of S_1, $f^{-1}(K_1)$ is a compact set in S.

It is also known [15, p. 160] that if X is a Banach space and $f : X \to X$ is a continuous k-set contraction, then $I - f$ is a proper mapping.

The following considerations are contained in a result, due to Juberg [16]; they will be useful in the proof of our main result:

Let (a, b) be any real (possible unbounded) interval and let $L^p(a, c)$, $1 \leq p \leq +\infty$ be the Lebesgue's space of (the power p) summable functions over (a, c) for every $c \in (a, b)$. For $u \in L^p(c, b)$, $v \in L^q(a, c)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we set

$$\rho = \lim_{\varepsilon \to 0} \sup\left\{\left[\int_x^{a+\varepsilon} |u(y)|^p dy\right]^{\frac{1}{p}} \left[\int_a^x |v(y)|^q dy\right]^{\frac{1}{q}},\ a < x \leq a + \varepsilon\right\}$$

$$+ \lim_{\delta \to 0} \sup\left\{\left[\int_x^b |u(y)|^p dy\right]^{\frac{1}{p}} \left[\int_{b-\delta}^x |v(y)|^q dy\right]^{\frac{1}{q}},\ b - \delta \leq x < b\right\}.$$

Let D be the linear operator defined by:

$$D(f)(x) = \int_0^x u(x)v(y)f(y)dy;$$

in the sequel we shall assume that D is a bounded operator in the space $L^p(a, b)$. We want to recall that the operator D is bounded (in the $L^p(a, b)$ space) if and only if the function

$$\psi(x) = \left[\int_x^b |u(y)|^p dy\right]^{\frac{1}{p}} \left[\int_a^x |v(y)|^q dy\right]^{\frac{1}{q}}$$

is bounded on (a, b).

As matter of fact (see, for instance, [7]) let I be the real interval $(0, 1]$, $p = 1$, $q = +\infty$, $u(y) = \frac{1}{y}$, $v(y) = y$. A simple computation allows us to see that $\frac{1}{4e} < \mu(D) < \frac{1}{e}$: so that D is a condensing noncompact operator from $L^1((0, 1]) \to L^\infty((0, 1])$.

Furthermore the measure of noncompactness of D, i.e., $\mu(D)$ satisfies (see [16]) $\left(\frac{1}{2}\right)^{1+\frac{1}{p}} \rho \leq \mu(D) \leq p^{\frac{1}{q}} q^{\frac{1}{p}} \rho$.

In the special case when $p = q = 2$, i.e., when the (Lebesgue) space L^p is a Hilbert space L^2, we obtain $\rho\sqrt{\frac{1}{8}} \leq \mu(D) \leq 2\rho$.

Definition 5. An R_δ-set is the intersection of a decreasing sequence $\{A_n\}$ of compact AR (metric absolute retracts; see [13] or [20], for a reference).

Moreover it is known (see, for instance, [9]) that an R_δ-set is an acyclic set in the Čech homology.

The following characterization of an R_δ-set will be used in the sequel:

Proposition 1 [13, p. 159]. *Let S be a topological space and let X be a Banach space with norm $\|\cdot\|$; let $f : S \to X$ be a proper mapping. Assume further that for each positive $n \in \mathbb{N}$ a proper mapping $f_n : S \to X$ is given and the couple of conditions is satisfied*:

- *for any* $\varepsilon_n > 0$, $\| f_n(x) - f(x) \| < \varepsilon_n$, $\forall x \in S$;
- *for any* $y \in X$ *such that* $\| y \| \leq \varepsilon_n$, *the equation* $f_n(x) = y$ *has exactly one solution*.

Then the set $Q = f^{-1}(0)$ *is an* R_δ*-set*.

Remark. Such a sequence f_n is called an ε_n *approximation* (of the function f).

2. Main Result

We are ready to establish our (main) existence result for the (initial value problem for) integral equations of the type here introduced.

First of all let $F : \overline{B(0, r)} \to E$, where r is a real number (suitably defined later on in the proof), be defined as follows:

$$F(y)(t) = h(t) + \int_0^t k(t, s) g(s, y(s)) ds; \text{ put also } m_0 = \| F(0) \|_2.$$

Theorem 1. *Let ρ be the number previously defined. Then we assume that*:

1. *there are functions* $\alpha, \phi : I \to \mathbb{R}^n$ *belonging to* $L^2(I)$ *such that* $k(t, s) = \alpha(t)\phi(s)$ *for every* $(t, s) \in I \times I$; *moreover we assume that* $\| k \|_2 < 2\rho$;
2. $\| g(t, x) \| \leq \frac{1}{2\rho} \| x \| + b(t)$, *for* $(t, x) \in I \times \mathbb{R}^n$, $b \in L^2(I)$, $b(t) \geq 0$;
3. *there is a ball* $B(0, r)$ *such that* $r > \dfrac{2m_0\rho}{2\rho - \| k \|_2}$.

Then the set of solutions of the integral problem (2) *is an* R_δ*-set*.

Remark. The first part of the assumption (1) is satisfied in many cases: for instance when $k(t, s)$ is a Green function; see also [14] for similar cases.

Proof. Clearly the above operator F is a single value mapping and a possible fixed point of F is a solution of the integral problem (2).

In order to prove the theorem the following steps in the proof have to be established:

(i) F has a closed graph;

(ii) F is a condensing mapping;

(iii) The set of fixed points of F is an R_δ-set.

Proof of Step (i). In fact, let $y_n \to y_0$ and put $G(y)(t) = g(t, y(t))$. Now, from assumption (2), it follows that the superposition operator G mapping the space $L^2([0, T))$ into $L^2([0, T))$ is continuous (see [12]); thus we have $\lim_n \| G(y_n) - G(y_0) \|_2 = 0$. By using the Hölder inequality, we get:

$$\begin{aligned} \| F(y_n) - F(y_0) \|_2 &= \left[\int_I | F(y_n)(s) - F(y_0)(s) |^2 ds \right]^{\frac{1}{2}} \\ &= \left[\int_I \left[\int_0^t k(t, s) g(s, y_n(s)) - k(t, s) g(s, y_0(s)) ds \right]^2 dt \right]^{\frac{1}{2}} \\ &\le \| k \|_2 \| G(y_n) - G(y_0) \|_2 \end{aligned}$$

and this quantity is going to zero whenever $n \to +\infty$.

Proof of Step (ii). Always working from $\overline{B(0, r)}$ into E, we have $F(y) = (H \circ G)(y)$, where

$$H(y)(t) = \int_0^t \phi(s) \alpha(t) y(s) ds + h(t).$$

Now, by assumptions (1) and (2), we have (see [12]) $\mu(G(V)) \le \frac{1}{2\rho} \mu(V)$, for any bounded set $V \subset L^2(I \times \mathbb{R}^n)$ and also, from (1), $\mu(H) < 2\rho$; so (see [4]) $\mu(F) = \mu(H \circ G)(y) \le \mu(H)\mu(G) < 1$.

Proof of Step (iii). Finally we have to prove that the set of fixed points of the operator F is an R_δ-set.

Let us consider the mappings $F_n : L^2([0, T)) \to L^2([0, T))$ defined as:

$$F_n(x)(t) = \begin{cases} h(t) & \text{if } 0 \le t \le \frac{T}{n}; \\ h(t) + \int_0^{t-\frac{T}{n}} \phi(t)\alpha(s)g(s, y(s))ds & \text{if } \frac{T}{n} \le t < T. \end{cases}$$

The mappings F_n are continuous mappings; by assumptions (1) and (2) we have that they are also condensing.

The intervals $\left[0, \frac{T}{n}\right], \left[\frac{T}{n}, \frac{2T}{n}\right], \ldots, \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right], \ldots, \left[\frac{(n-1)T}{n}, T\right)$ are now coming in one after the other: each time the mappings F_n are bijective and their inverses F_n^{-1} are continuous. Moreover we have $\| F_n - F \|_2 \to 0$ as $n \to +\infty$. The latter fact allows us to say that the mappings $I - F_n$ and $I - F$ are proper maps. Finally we can conclude that the set of fixed points of F is an R_δ-set.

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Facoltà di Ingegneria
Università di Firenze, Italy

Facoltà di Architettura
Università di Firenze, Italy