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THE ITERATIVE APPROXIMATION METHOD FOR FIXED POINTS OF Φ-HEMICONTRACTIVE MAPPING AND APPLICATIONS

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Abstract

The objective of this paper is to introduce the Φ-hemicontractive mapping and to study iterative approximation method for the fixed points of the mapping by Mann iterative sequence with random errors $\{x_n\}$. Let X be a real Banach space and $T : X \to X$ be Φ-hemicontractive. The results show that $\{x_n\}$ converges strongly to an unique fixed point if T is uniformly continuous, and if X is uniformly smooth, then any continuity of T is unnecessary. As application, the approximation method for the solution of nonlinear equation with Φ-accretive mapping is obtained.

Throughout this paper, X is assumed a real Banach space with dual X^*, $(\cdot, \cdot)$ denotes the generalized duality pairing of X and X^*. The mapping $J : X \to 2^{X^*}$ defined by

$$Jx = \{j \in X^* : (x, j) = \| x \| \| j \|, \| j \| = \| x \|\} \quad \forall x \in X \tag{0.1}$$

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is called the *normalized duality mapping*. In particular, X is a uniformly smooth (equivalently, X^* is uniformly convex) Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of X (see, Browder [2]).

To set the framework, we recall some basic notations as follows.

Definition 1 [9]**.** Let $T : X \to X$ be a mapping. For any given $x_0 \in X$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + \gamma_n u_n \quad (n \geq 0) \tag{0.2}$$

is called *Mann iteration sequence* with random errors. Here $\{u_n\}$ is a bounded sequence in X; $\{\alpha_n\}$ and $\{\gamma_n\}$ are two sequences in $[0, 1]$.

Definition 2. Let T be a mapping with domain $D(T) \subset X$ and range $R(T) \subset X$. T is called ϕ-*hemicontractive* if for all $x \in D(T)$ and $q \in F(T) := \{x \in D(T) : Tx = x\}$ there exist $j(x - q) \in J(x - q)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$(Tx - q,\ j(x - q)) \leq \| x - q \|^2 - \phi(\| x - q \|)\| x - q \|. \tag{0.3}$$

T is called Φ-*hemicontractive* if for all $x \in D(T)$ and $q \in F(T)$ there exists $j(x - q) \in J(x - q)$ such that

$$(Tx - q,\ j(x - q)) \leq \| x - q \|^2 - \Phi(\| x - q \|). \tag{0.4}$$

T is called Φ-*accretive* if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$(Tx - Ty,\ j(x - y)) \geq \Phi(\| x - y \|). \tag{0.5}$$

Remark 1. The ϕ-hemicontractive mapping was introduced and studied by Osilike [6] in 1996. Obvious, every ϕ-hemicontractive mapping must be a Φ-hemicontractive mapping defined by $\Phi(s) = \phi(s)s$, and the class of ϕ-hemicontractive mappings is a proper subset of the class of Φ-hemicontractive mappings. For example, let $E = R$ (the reals with the usual norm) and let $K = [0, +\infty)$. Define $T : K \to K$ by $Tx = x - \dfrac{x}{1 + x^2}$. It is easy to verify that T is Φ-hemicontractive with a fixed point $x = 0$ and $\Phi : [0, +\infty) \to [0, +\infty)$ defined by $\Phi(s) = s^2/(1 + s^2)$, and T is not ϕ-hemicontractive.

Suppose that $A : X \to X$ is a Φ-accretive mapping and $S : X \to X$ is defined by $Sx = f + x - Ax$ for all $x \in X$ and any given $f \in X$, it is easy to verify that q is a solution of $Ax = f$ if and only if q is a fixed point of S. Hence, the solution of $Ax = f$ is intimately connected with the fixed point of the mapping.

The following lemma plays a crucial role in the proofs of our main results.

Lemma 1 [1]. *If X is a real Banach space, then there exists a $j(x + y) \in J(x + y)$ such that*

$$\| x + y \|^2 \leq \| x \|^2 + 2(y, j(x + y)) \quad \forall x, y \in X. \tag{0.6}$$

Now we prove the following approximative theorems.

Theorem 1. *Suppose that $T : X \to X$ is a uniformly continuous Φ-hemicontractive operator with bounded range. If the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by* (0.2) *satisfying*

(1.1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{+\infty} \alpha_n = +\infty$;

(1.2) $\sum_{n=0}^{+\infty} \gamma_n < +\infty$,

then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. From (0.4), we know that $F(T) = \{q\}$. Putting $c = \sup\{\| Tx - q \| : x \in X\} + \| x_0 - q \|$ and $d = \sup\{\| u_n \| : n \geq 0\}$. For any $n \geq 0$, using induction, we obtain $\| x_n - q \| \leq c + d\sum_{i=0}^{n-1} \gamma_i \leq c + d\sum_{i=0}^{+\infty} \gamma_i$. Hence, we set $M = c + d\sum_{i=0}^{+\infty} \gamma_i$. Since $\lim_{n\to\infty}\| x_n - x_{n+1} \| =$

$\lim_{n\to\infty}\| \alpha_n x_n - \alpha_n Tx_n - \gamma_n u_n \| = 0,$ therefore

$$e_n := \| Tx_n - Tx_{n+1} \| \to 0 \quad (\text{as } n \to \infty) \tag{0.7}$$

by the uniformly continuity of T.

Let $\sigma = \inf\{\| x_{n+1} - q \| : n \geq 0\}$. If $\sigma > 0$, then $\Phi(\| x_{n+1} - q \|) > \Phi(\sigma/2) > 0$ for all $n \geq 0$. Thus, there exists a natural number $N \in \mathcal{N}$ such that

$$\alpha_n \leq \frac{1}{6} \text{ and } 3M^2\alpha_n^2 + 3M\alpha_n e_n + 3M^2\gamma_n = o(\alpha_n)\alpha_n \leq \alpha_n\Phi\left(\frac{\sigma}{2}\right) \tag{0.8}$$

for all $n \geq N$, respectively. By (0.2), (0.6), (0.4) and (0.8), we have

$$\begin{aligned}
\| x_{n+1} - q \|^2 &= \| (1 - \alpha_n)(x_n - q) + \alpha_n(Tx_n - q) + \gamma_n u_n \|^2 \\
&\leq \| (1 - \alpha_n)(x_n - q) \|^2 + 2\alpha_n(Tx_n - Tx_{n+1},\ j(x_{n+1} - q)) \\
&\quad + 2\alpha_n(Tx_{n+1} - q,\ j(x_{n+1} - q)) + 2M^2\gamma_n \\
&\leq (1 - \alpha_n)^2\| x_n - q \|^2 + 2M\alpha_n e_n + 2\alpha_n\| x_{n+1} - q \|^2 \\
&\quad - 2\alpha_n\Phi(\| x_{n+1} - q \|) + 2M^2\gamma_n \\
&\leq \| x_n - q \|^2 + 3M^2\alpha_n^2 + 3M\alpha_n e_n \\
&\quad + 3M^2\gamma_n - 2\alpha_n\Phi(\| x_{n+1} - q \|) \\
&= \| x_n - q \|^2 + o(\alpha_n) - 2\alpha_n\Phi(\| x_{n+1} - q \|)
\end{aligned} \tag{0.9}$$

for all $n \geq N$. It follows from (0.8) and (0.9) that

$$\| x_{n+1} - q \|^2 \leq \| x_n - q \|^2 + o(\alpha_n) - 2\alpha_n\Phi\left(\frac{\sigma}{2}\right) \leq \| x_n - q \|^2 - \alpha_n\Phi\left(\frac{\sigma}{2}\right)$$

for all $n \geq N$. By induction, we obtain

$$\Phi\left(\frac{\sigma}{2}\right)\sum_{j=N}^{+\infty}\alpha_j \leq \| x_N - q \|^2 \leq M^2. \tag{0.10}$$

(0.10) is in contradiction with $\sum_{j=0}^{+\infty} \alpha_j = +\infty$. From this contradiction, we get $\sigma = 0$. Therefore, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. For any given $\varepsilon > 0$ there exists an integer $j_0 \geq N$ such that $\| x_{n_j} - q \| < \varepsilon$ and $o(\alpha_{n_j}) \leq 2\alpha_{n_j}\Phi(\varepsilon)$ for all $j \geq j_0$. If j_0 is fixed, then we shall prove that $\| x_{n_{j_0}+k} - q \| < \varepsilon$ for all integers $k \geq 1$.

The proof is by induction. For $k = 1$, suppose $\| x_{n_{j_0}+1} - q \| \geq \varepsilon$. It follows from (0.9) and $\Phi(\| x_{n_{j_0}+1} - q \|) \geq \Phi(\varepsilon)$ that

$$\begin{aligned}
\varepsilon^2 &\leq \| x_{n_{j_0}+1} - q \|^2 \\
&\leq \| x_{n_{j_0}} - q \|^2 + o(\alpha_{n_{j_0}}) - 2\alpha_{n_{j_0}}\Phi(\varepsilon) \\
&\leq \| x_{n_{j_0}} - q \|^2 < \varepsilon^2.
\end{aligned}$$

It is a contradiction. Hence, $\| x_{n_{j_0}+1} - q \| < \varepsilon$ holds for $k = 1$. Assume now that $\| x_{n_{j_0}+p} - q \| < \varepsilon$ for some integer $p > 1$. We prove $\| x_{n_{j_0}+p+1} - q \| < \varepsilon$. Again, assuming the contrary, $\Phi(\| x_{n_{j_0}+p+1} - q \|) \geq \Phi(\varepsilon)$, as above, it leads to a contradiction as follows

$$\varepsilon^2 \leq \| x_{n_{j_0}+p+1} - q \|^2 \leq \| x_{n_{j_0}+p} - q \|^2 < \varepsilon^2,$$

where $n_{j_0} + p > n_{j_0} \geq j_0 \geq N$. Therefore, $\| x_{n_{j_0}+k} - q \| < \varepsilon$ holds for all integers $k \geq 1$, so that $x_{n_{j_0}+k} \to q$ as $k \to \infty$. The proof is completed.

Theorem 2. *Let $T : X \to X$ be a Φ-hemicontractive mapping with bounded range and X be uniformly smooth. Suppose that the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by* (0.2) *satisfying the conditions* (1.1) *and* (1.2) *in Theorem* 1, *then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. From (0.4), we know that the fixed point of T is unique. Let q be the fixed point of T in X. By similar arguments as in the proof of Theorem 1, we set $M = c + d\sum_{i=0}^{+\infty}\gamma_i$. From the uniformly continuity of J, we have

$$e_n := \| J(x_{n+1} - q) - J(x_n - q) \| \to 0 \quad (\text{as } n \to \infty).$$

Using (0.2), (0.6) and (0.4), we have

$$\begin{aligned} \| x_{n+1} - q \|^2 &= \| (1 - \alpha_n)(x_n - q) + \alpha_n(Tx_n - q) + \gamma_n u_n \|^2 \\ &\le \| (1 - \alpha_n)(x_n - q) \|^2 + 2\alpha_n(Tx_n - q, J(x_{n+1} - q)) \\ &\quad + 2\gamma_n(u_n, J(x_{n+1} - q)) \\ &\le \| (1 - \alpha_n)(x_n - q) \|^2 + 2\alpha_n(Tx_n - q, J(x_n - q)) \\ &\quad + 2\alpha_n(Tx_n - q, J(x_{n+1} - q) - J(x_n - q)) + 2M^2\gamma_n \\ &\le \| x_n - q \|^2 + 2M^2\gamma_n + M^2\alpha_n^2 + 2M\alpha_n e_n \\ &\quad - 2\alpha_n\Phi(\| x_{n+1} - q \|) \\ &= \| x_n - q \|^2 + o(\alpha_n) - 2\alpha_n\Phi(\| x_{n+1} - q \|). \end{aligned} \tag{0.11}$$

By similar arguments as in the proof of Theorem 1, we have that $\{x_n\}$ converges strongly to the unique fixed point q of T. The proof is completed.

Corollary 1. *Suppose that* $A : X \to X$ *is a uniformly continuous* Φ*-accretive mapping and the range of* $(I - A)$ *is bounded. If the equation* $Ax = f$ *has a solution and the Mann iteration sequence with random errors* $\{x_n\}_{n=0}^{\infty}$ *defined by* (0.2) *satisfying the conditions* (1.1) *and* (1.2) *in Theorem* 1, *then for arbitrary* $x_0 \in X$, $\{x_n\}$ *converges strongly to the unique solution of* $Ax = f$.

Proof. Putting $T : X \to X$ by $Tx = f + x - Ax$ for all $x \in X$. Obvious, if $q \in X$ is a solution of $Ax = f$, then q is a fixed point of T and T is Φ-hemicontractive. Thus, Corollary 1 follows from Theorem 1.

Similarly, we obtain

Corollary 2. *Let* $A : X \to X$ *be a* Φ*-accretive mapping and the range of* $(I - A)$ *be bounded and X be uniformly smooth. Suppose that the Mann iteration sequence with random errors* $\{x_n\}_{n=0}^{\infty}$ *defined by* (0.2) *satisfying the conditions* (1.1) *and* (1.2) *in Theorem* 1. *For any given* $f \in X$, *if* $Ax = f$ *has a solution in X, then for arbitrary* $x_0 \in X$, $\{x_n\}$ *converges strongly to the unique solution of* $Ax = f$.

Remark 2. The corresponding results (see, for example, Theorems 4.1 and 4.2 in [3], Theorems 3 and 4 in [8], Corollary 4 in [7], Corollary 3.3 in [5], Corollaries 3.2 and 3.4 in [9], Theorem 2 in [10] and Corollary 3.2 in [4]) are improved in the following senses:

(i) For the convergence of $\{x_n\}_{n=0}^{\infty}$, if X is arbitrary Banach space, then the mapping may not be Lipschitz; if X is uniformly smooth, then the mapping may not be continuous or demicontinuous.

(ii) The mappings are Φ-hemicontractive or Φ-accretive, they may not be ϕ-hemicontractive or ϕ-strongly accretive.

(iii) The random errors of iterative process have been considered.

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