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LEMMA OF MUSIELAK-ORLICZ SEQUENCE SPACE l_Φ

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Abstract

In this paper we ameliorate and prove the Lemma of Musielak-Orlicz sequence space l_Φ.

1. Preliminaries

Let X be a normed linear space. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively.

We introduce some basic facts on Musielak-Orlicz sequence space.

Suppose that a sequence of function $\Phi = (\Phi_i)_{i=1}^\infty$ for any i satisfies

(i) $\Phi_i : (-\infty, \infty) \to [0, \infty]$ is even, convex and left-continuous;

(ii) $\Phi_i(0) = 0$;

(iii) $\lim\limits_{u\to\infty} \Phi_i(u) = \infty$ and there exists $u_i > 0$ such that $\Phi_i(u_i) < \infty$.

For each real sequence $x = (x(i))_{i=1}^\infty$, we define its *modular* by

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$$\rho_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i)).$$

Then the Musielak-Orlicz space

$$l_\Phi = \{x : \rho_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

with Luxemburg norm and the Orlicz norm are defined respectively by

$$\| x \|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \le 1\right\}$$

and

$$\| x \|_\Phi = \sup\left\{\sum_{i=1}^{\infty} | x(i) y(i) | : \rho_\Psi(y) = \sum_{i=1}^{\infty} \Psi_i(| y(i) |) \le 1\right\}$$

is Banach space, where $\Psi_i(v) = \sup\{u| v | - \Phi_i(u) : u \ge 0\}$ is the complementary function of Φ_i. For short, we denote $l_{(\Phi)} = [l_\Phi, \| \cdot \|_{(\Phi)}]$ and $l_\Phi = [l_\Phi, \| \cdot \|_\Phi]$. The subspace h_Φ is defined to be the set $\{x \in l_\Phi : \| x - [x]_0^n \| \to 0,\ n \to \infty\}$, where we denote sequence $[x]_i^j = (0, 0, \ldots, x(i+1), x(i+2), \ldots, x(j), 0, \ldots)$. And denote $h_{(\Phi)} = [h_\Phi, \| \cdot \|_{(\Phi)}]$ and $h_\Phi = [h_\Phi, \| \cdot \|_\Phi]$. It is well known that $h^*_{(\Phi)} = l_\Psi$.

2. Results

Theorem. *Let* $\varepsilon \in (0, 1)$ *and* $(x_n)_1^\infty \subset l_{(\Phi)}$. *If for any* $y \in B(l_\Psi)$,

$$\limsup_{n\to\infty} \left| \sum_{i=1}^{\infty} x_n(i) y(i) \right| < \frac{\varepsilon}{8},$$

then we have

(i) $\lim\limits_{i_0\to\infty} \sup\limits_n \sum\limits_{i>i_0} | x_n(i) y(i) | < \varepsilon$ *for any* $y \in B(l_\Psi)$,

(ii) $\lim\limits_{i_0\to\infty} \lim\limits_{\xi\to 0} \sup\limits_n \sum\limits_{i>i_0} \dfrac{\Phi_i(\xi x_n(i))}{\xi} < \varepsilon.$

Proof. For $e \subset \mathbb{N}$, we denote $\min e = \min\{i : i \in e\}$, $\max e = \max\{i : i \in e\}$. We first show that

$$\lim_{i_0 \to \infty} \sup\left\{\left|\sum_{i \in e} x_n(i) y_0(i)\right| : n = 1, 2, \ldots, \min e > i_0\right\} < \frac{\varepsilon}{2}. \tag{1}$$

If (1) does not hold, then put $i_1 = 0$, there exists e_1 such that $\min e_1 > i_1$ and there exists n_1 such that

$$\left|\sum_{i \in e_1} x_{n_1}(i) y_0(i)\right| \geq \frac{\varepsilon}{2}.$$

Since $\sum_{i=1}^{\infty} |x_{n_1}(i) y_0(i)| < \infty$, we can choose θ_1 such that

$$\sum_{i > \theta_1} |x_{n_1}(i) y_0(i)| < \frac{\varepsilon}{8}.$$

Define $E_1 = \{i \in e_1 : i \leq \theta_1\}$, we obtain

$$\left|\sum_{i \in E_1} x_{n_1}(i) y_0(i)\right| > \frac{3\varepsilon}{8}$$

and

$$\sum_{i > \max E_1} |x_{n_1}(i) y_0(i)| < \frac{\varepsilon}{8}.$$

Since $y_0 \chi_{E_1} \in B(l_\Psi)$, for $n \in \mathbb{N}$ large enough we have

$$\left|\sum_{i \in E_1} x_n(i) y_0(i)\right| = \left|\sum_{i=1}^{\infty} x_n(i) y_0(i) \chi_{E_1}(i)\right| < \frac{\varepsilon}{8}.$$

Next we shall show that there exist infinitely many n such that there is $e \subset \mathbb{N}$,

$$\min e > \max E_1 \quad \text{and} \quad \left|\sum_{i \in e} x_n(i) y_0(i)\right| \geq \frac{\varepsilon}{2}. \tag{2}$$

Otherwise, there exist finitely many $x_{s_1}, x_{s_2}, \ldots, x_{s_m}$ $(s_m \in \mathbb{N})$ satisfying (2). We choose θ_m large enough such that

$$\sum_{i>\theta_m} |x_{s_j}(i) y_0(i)| < \frac{\varepsilon}{2}, \quad (j = 1, 2, \ldots, m),$$

then for e satisfying $\min e > \max\{\max E_1, \theta_m\}$ and for all n we have

$$\left| \sum_{i \in e} x_n(i) y_0(i) \right| < \frac{\varepsilon}{2},$$

a contradiction.

Hence we may choose n_2 and e_2, $\min e_2 > \max E_1$ such that

$$\left| \sum_{i \in E_1} x_{n_2}(i) y_0(i) \right| < \frac{\varepsilon}{8} \quad \text{and} \quad \left| \sum_{i \in e_2} x_{n_2}(i) y_0(i) \right| \geq \frac{\varepsilon}{2}.$$

Next, we take $E_2 \subset e_2$ such that

$$\left| \sum_{i \in E_2} x_{n_2}(i) y_0(i) \right| \geq \frac{3\varepsilon}{8} \quad \text{and} \quad \sum_{i > \max E_2} |x_{n_2}(i) y_0(i)| < \frac{\varepsilon}{8}.$$

There exist n_3 and e_3, $\min e_3 > \max E_2$ such that

$$\left| \sum_{i \in E_1 \cup E_2} x_{n_3}(i) y_0(i) \right| < \frac{\varepsilon}{8} \quad \text{and} \quad \left| \sum_{i \in e_3} x_{n_3}(i) y_0(i) \right| \geq \frac{\varepsilon}{2}.$$

In such a way, we can find sequences $(E_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ such that $\min E_{k+1} > \max E_k$ and

$$\left| \sum_{i \in E_1 \cup E_2 \cup \cdots \cup E_{k-1}} x_{n_k}(i) y_0(i) \right| < \frac{\varepsilon}{8},$$

$$\left| \sum_{i \in E_k} x_{n_k}(i) y_0(i) \right| \geq \frac{3\varepsilon}{8},$$

$$\sum_{i > \max E_k} |x_{n_k}(i) y_0(i)| < \frac{\varepsilon}{8},$$

where $k = 1, 2, \ldots$ and without loss of generality, we may assume $E_0 = \varnothing$. Put $E = \bigcup_{k=1}^{\infty} E_k$, we have that

$$\begin{aligned}
&\left| \sum_{i \in E} x_{n_k}(i) y_0(i) \right| \\
&\geq \left| \sum_{i \in E_k} x_{n_k}(i) y_0(i) \right| - \left| \sum_{i \in \bigcup_{j=1}^{k-1} E_j} x_{n_k}(i) y_0(i) \right| - \left| \sum_{i \in E_{k+1} \cup E_{k+2} \cup \cdots} x_{n_k}(i) y_0(i) \right| \\
&\geq \frac{3\varepsilon}{8} - \frac{\varepsilon}{8} - \sum_{i > \max E_k} |x_{n_k}(i) y_0(i)| \\
&\geq \frac{\varepsilon}{8}.
\end{aligned}$$

This is a contradiction. Hence (1) is true.

Put $E_n' = \{i > i_0 : x_n(i) y_0(i) > 0\}$, $E_n'' = \{i > i_0 : x_n(i) y_0(i) < 0\}$. Then when i_0 large enough we have

$$\begin{aligned}
\sum_{i > i_0} |x_n(i) y_0(i)| &= \left| \sum_{i \in E_n'} x_n(i) y_0(i) \right| + \left| \sum_{i \in E_n''} x_n(i) y_0(i) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus we obtain (i).

By the condition of the Theorem, it is easy to see $\{x_n\}$ is bounded in norm, hence we may assume $\rho_\Phi(x_n) \leq 1$ $(n = 1, 2, \ldots)$. If (ii) is not true, then there exists $i_k \nearrow \infty$ such that

$$\lim_{\xi \to 0} \sup_n \sum_{i > i_k} \frac{1}{\xi} \Phi_i(\xi x_n(i)) \geq \varepsilon.$$

Hence we may take $\xi_k \searrow 0$, $\sum_k \xi_k \leq \frac{1}{2}$ and $n_k \nearrow \infty$ such that

$$\frac{1}{\xi_k}\sum_{i>i_k}\Phi_i(\xi_k x_{n_k}(i)) \geq \varepsilon.$$

Put $y_0(i) = \sup_k p_i(\xi_k|x_{n_k}(i)|)$, $(i = 1, 2, \ldots)$, $y = (y(i))_{i=1}^{\infty}$, then

$$\begin{aligned}
\rho_\Psi(y_0) &= \sum_{i=1}^{\infty}\Psi_i(\sup_k p_i(\xi_k|x_{n_k}(i)|)) \\
&\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\Psi_i(p_i(\xi_k|x_{n_k}(i)|)) \\
&\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\xi_k|x_{n_k}(i)|p_i(\xi_k|x_{n_k}(i)|) \\
&\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\Phi_i(2\xi_k x_{n_k}(i)) \\
&\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}2\xi_k\Phi_i(x_{n_k}(i)) \\
&= 2\sum_{k=1}^{\infty}\xi_k\sum_{i=1}^{\infty}\Phi_i(x_{n_k}(i)) \\
&\leq 2\sum_{k=1}^{\infty}\xi_k \leq 1.
\end{aligned}$$

This means $y_0 \in B(l_\Psi)$. But

$$\begin{aligned}
\sum_{i>i_k}|x_{n_k}(i)y_0(i)| &\geq \sum_{i>i_k}|x_{n_k}(i)|p_i(\xi_k|x_{n_k}(i)|) \\
&\geq \frac{1}{\xi_k}\sum_{i>i_k}\Phi_i(\xi_k x_{n_k}(i)) \geq \varepsilon \quad (k = 1, 2, \ldots).
\end{aligned}$$

Thus we have reached a contradiction to (i). Therefore (ii) is true.

References

[1] T. Andô, Weakly compact sets in Orlicz spaces, Canad. J. Math. 14 (1962), 170-176.

[2] S. Banach and S. Saks, Sur la convergence forte dans les champs L^p, Studia Math. 2 (1930), 51-57.

[3] B. Beauzamy and J.-T. Lapresté, Modèles Étalés des Espaces de Banach, Hermann, Paris, 1983.

[4] S. Chen, Geometry of Orlicz spaces, Dissertationes Math. 356 (1996), 204.

[5] M. M. Day, Normed Linear Spaces, Springer-Verlag, Berlin, 1962.

[6] J. Diestel, Sequence and Series in Banach Spaces, Springer-Verlag, New York, 1984.

[7] M. I. Ostrovskii, A comparison between the weak and the continuous Banach-Saks properties, Teor. Funktsii Funktsional. Anal. i Prilozhen. 48 (1987), 130-134 (in Russian).

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