Far East J. Math. Sci. (FJMS) **24**(2) (2007), 241-249
This paper is available online at http://www.pphmj.com

FREDHOLM TOEPLITZ OPERATORS ON THE HARDY SPACE OVER THE POLYDISK

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(Received July 18, 2006)

Submitted by Stephen Dilworth

Abstract

This paper studies Toeplitz operators on the Hardy space over polydisk. We show that the finite products of Toeplitz operators is a Fredholm operator only if the finite products of symbols is invertible in $L^{\infty}(T^n)$. For $f \in L^{\infty}(T^n)$, the Toeplitz operator T_f commutes essentially with all T_{z_i} $(i = 1, 2, \ldots, n)$ if and only if $f \in H^{\infty}(D^n)$.

1. Preliminaries

Let D be the open unit disk in the complex plane C. Its boundary is the unit circle T. The polydisk D^n and the torus T^n are the subsets of C^n which are Cartesian products of n copies of D and T, respectively. Let $d\sigma(z)$ be the normalized Haar measure on T^n. The Hardy space $H^2(D^n)$ is the closure of the polynomials in $L^2(T^n, d\sigma)$ (or $L^2(T^n)$). Let P be the

2000 Mathematics Subject Classification: 47B35.

Keywords and phrases: Toeplitz operators, Fredholm operators, commutes essentially, polydisk.

This work was partly supported by National Natural Science Foundation of China (10361003).

orthogonal projection from $L^2(T^n)$ onto $H^2(D^n)$. The Toeplitz operator with symbol f in $L^\infty(T^n)$ is defined by $T_f h = P(fh)$, for all $h \in H^2(D^n)$ and the Hankel operator with symbol f is defined by $H_f h = (I - P)(fh)$, for all $h \in H^2(D^n)$. $K_{z_1}(w_1) = \dfrac{1}{(1 - \bar{z}_1 w_1)}$ is the reproducing kernel of Hardy space $H^2(D)$ at the point $z_1 \in D$ and $k_{z_1}(w_1) = \dfrac{(1 - |z_1|^2)^{1/2}}{(1 - \bar{z}_1 w_1)}$ is the normalized reproducing kernel of $H^2(D)$ at the point $z_1 \in D$. It is easy to check that the reproducing kernel of $H^2(D^n)$ at the point $z \in D^n$ is the product $K_z(w) = \prod_{i=1}^n K_{z_i}(w_i)$. So the normalized reproducing kernel $k_z(w)$ of $H^2(D^n)$ at the point $z \in D^n$ is also the product $k_z(w) = \prod_{i=1}^n k_{z_i}(w_i)$. We know that k_z weakly converges to zero in $H^2(D^n)$ as z tends to the boundary of D^n. We denote by $Aut(D^n)$ the group of all biholomorphic automorphisms of D^n. The automorphisms of D^n for $n \geq 2$ are generated by the following three subgroups: rotations in each variable separately $R_\theta(z) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$, where Möbius transformations are in each variable separately $\Psi_w(z) = (\Psi_{w_1}(z_1), \ldots, \Psi_{w_n}(z_n))$, and the coordinate permutations. Here $\theta \in [0, 2\pi]^n$ and $w \in D^n$ are fixed. Möbius transformations are in the form $\Psi_w(z) = \dfrac{w - z}{1 - \bar{w} z}$ $(w \in D, z \in D)$. Thus an arbitrary $\Psi \in Aut(D^n)$ can be written in the form

$$\Psi(z) = (e^{i\theta_1} \Psi_{w_1}(z_{\sigma(1)}), \ldots, e^{i\theta_n} \Psi_{w_n}(z_{\sigma(n)}))$$

for some $w = (w_1, \ldots, w_n) \in D^n$, $\theta = (\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n$, and σ is a coordinate permutations (see [10]). Let Z denote the set of all integers, Z_+ denote the set of all nonnegative integers and Z_- denote the set of all negative integers. We recall that by using multiple Fourier series,

$$L^2(T^n) = \left\{ f : f = \sum_{\alpha \in Z^n} \hat{f}(\alpha)\zeta^{\alpha}, \ \sum_{\alpha \in Z^n} |f(\alpha)|^2 < \infty \right\}.$$

We note that for every $\zeta = (\zeta_1, \ldots, \zeta_n) \in T^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n$, $\zeta^{\alpha} = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$, $\zeta_j^{-\alpha_j} = \overline{\zeta}_j^{\alpha_j}$, $\zeta_j \overline{\zeta}_j = 1$. So we can also write $f \in L^2(T^n)$ as

$$f = f(\zeta, \overline{\zeta}) = \sum_{\alpha \in Z_+^n} \hat{f}(\alpha) \widetilde{\zeta}^{\alpha},$$

where $\widetilde{\zeta}_j = \zeta_j$ or $\widetilde{\zeta}_j = \overline{\zeta}_j$.

Lemma 1.1. *Let*

$$f = f(\zeta, \overline{\zeta}) = \sum_{\alpha \in Z_+^n} \hat{f}(\alpha) \widetilde{\zeta}^{\alpha}.$$

Then

$$T_f K_z(w) = f(w, \overline{z}) K_z(w) \in H^2(D^n)$$

for every $z \in D^n$ *(see* [4]*).*

Lemma 1.2. *Let* f *and* g *be in* $L^2(T^n)$ $(n \geq 2)$*. Then for any* $z_1 \in D$, $\mu_1 \in T$, *we have*

$$\lim_{z_1 \to \mu_1} \int_T \langle T_f T_g k_{z_1 e^{i\theta}} k_{z'}, k_{z_1 e^{i\theta}} k_{z'} \rangle e^{im\theta} d\theta$$

$$= \int_T \langle T_{f(\mu_1 e^{i\theta} \cdot)} T_{g(\mu_1 e^{i\theta} \cdot)} k_{z'}, k_{z'} \rangle e^{im\theta} d\theta,$$

where $\theta \in [0, 2\pi]$, *for all* $m \in Z$ *and* $z' \in D^{n-1}$ *are fixed. For fixed* $\mu_1 \in T$, *where* $T_{f(\mu_1 e^{i\theta} \cdot)}$ *is the Toeplitz operator on the* $H^2(D^{n-1})$ *(see* [4]*).*

2. Fredholm Toeplitz Operators

The object of the present section is to study the properties of Toeplitz algebra. We write $L(H^2(D^n))$ for all bounded linear operators on Hardy space $H^2(D^n)$. For the subset $N \subseteq L^{\infty}(T^n)$, denote by $F(N)$ the closed algebra generated by the set $\{T_f \mid f \in N\}$. In the Hardy space $H^2(D)$ of the unit disk, Douglas [6] gave the following exact sequence:

$$(0) \to com\, F(L^\infty(T)) \to F(L^\infty(T)) \to L^\infty(T) \to (0),$$

where $com\, F(L^\infty(T^n))$ is the commutator ideal in $F(L^\infty(T^n))$, $\rho : F(L^\infty(T)) \to L^\infty(T)$ is symbol map. In 2003, Zhang get the following exact sequence:

$$(0) \to semi\, F(L^\infty(T^n)) \to F(L^\infty(T^n)) \to L^\infty(T^n) \to (0),$$

where $semi\, F(L^\infty(T^n))$ denotes the semi-commutator ideal in $F(L^\infty(T^n))$ (see [13]). We have the following results.

Theorem 2.1. *There exists a* $*$*-homomorphism* ξ *from the quotient algebra* $F(L^\infty(T^n))/K$ *onto* $L^\infty(T^n)$ *such that* $\rho = \xi \circ \pi$, *where* K *is compact operator ideal of* $L(H^2(D^n))$, π *is the canonical homomorphism from* $L(H^2(D^n))$ *onto the corresponding Calkin algebra and* $\rho : F(L^\infty(T^n)) \to L^\infty(T^n)$ *is symbol map.*

Proof. By Zhang's Proposition 1 in [13], $K \subseteq com\, F(L^\infty(T^n)) \subseteq semi\, F(L^\infty)$, so the theorem holds immediately from Zhang's exact sequence.

Theorem 2.2. *Let* f_i $(i = 1, 2, \ldots, m)$ *be all in* $L^\infty(T^n)$ *such that product* $T_{f_1}T_{f_2} \cdots T_{f_m}$ *is a Fredholm operator. Then product* $f_1 f_2 \cdots f_m$ *is invertible in* $L^\infty(T^n)$.

Proof. If $T_{f_1}T_{f_2} \cdots T_{f_m}$ is a Fredholm operator, then $\pi(T_{f_1}T_{f_2} \cdots T_{f_m})$ is invertible in $L(H^2(D^n))/K$. Since $F(L^\infty(T^n))/K$ is a closed self-adjoint subalgebra of $L(H^2(D^n))/K$, $\pi(T_{f_1}T_{f_2} \cdots T_{f_m})$ is invertible in $F(L^\infty(T^n))/K$ by Douglas's Theorem 4.28 in [6]. Hence there exists a $B \in F(L^\infty)(T^n)$ such that $\pi(T_{f_1} \cdots T_{f_m})\pi(B) = \pi(1)$. Since ξ is a

$*$-homomorphism from $F(L^\infty(T^n))/K$ onto $L^\infty(T^n)$, $\xi \circ \pi(T_{f_1} \cdots T_{f_m})\xi \circ \pi(B) = \xi \circ \pi(1) = 1$. It follows that $\rho(T_{f_1} \cdots T_{f_m}) = \xi \circ \pi(T_{f_1} \cdots T_{f_m}) = f_1 \cdots f_m$ is invertible in $L^\infty(T^n)$. This completes the proof.

Corollary 2.3 (Spectrum Inclusion Theorem)**.** *Let* $f_1, f_2, \ldots, f_m$ *be in* $L^\infty(T^n)$. *Then*

$$\sigma(M_{f_1} \cdots M_{f_m}) \subseteq \sigma_e(T_{f_1} \cdots T_{f_m}),$$

where M_{f_i} *is a multiplication operators on* $H^2(D^n)$, $\sigma(M_{f_1} \cdots M_{f_m})$ *denotes the spectrum of operator* $M_{f_1} \cdots M_{f_m}$, *and* $\sigma_e(T_{f_1} \cdots T_{f_m})$ *denotes the essential spectrum of operator* $T_{f_1} \cdots T_{f_m}$.

3. The Essential Commutant of Analytic Toeplitz Algebra

The main purpose of this section is the description of the essential commutant of analytic Toeplitz algebra. For $A, B \in L(H^2(D^n))$, if $AB - BA \in K$, then we say that A *essentially commutes* with B. The set of operators which essentially commutes with all operators in $F(N)$ is called the *essential commutant* of $F(N)$, denoted by $E_cF(N)$, where $N \subseteq L^\infty(T^n)$. Let $A_c = \{f \in L^\infty(T^n) : T_f \in E_cF(H^\infty(D^n))\}$, $A = \{f \in L^\infty(T^n) : H_f$ is compact$\}$. When $n = 1$, Davidson [2] showed $E_cF(H^\infty(D^n)) = F(A_c) + K = F((A)) + K$, $A_c = A = H^\infty + C$ in 1977. When $n > 1$, $A_c \supset A \supseteq H^\infty(S_n) + C(S_n)$, but $A \neq H^\infty(S_n) + C(S_n)$ (see [3]), where S_n is the unit sphere in C^n. Guo and Sun have obtained $E_cF(H^\infty(S_n)) = F(A_c) + K$ in [8], and Ding and Sun prove that $A_c = A$, $E_cF(H^\infty) = E_cF(H^\infty + C) = E_cF(A)$, on Hardy space over unit sphere (see [5]). We know that the function theory on the polydisk D^n is quite different from the function theory on the unit disk and unit ball (see [11], [12]). Naturally, Sun put forward the problem: $E_cF(H^\infty(D^n)) = ?$ The object of the section is to discuss which operators commute essentially with all analytic Toeplitz operators on Hardy space over polydisk. We shall see many differences in Toeplitz operator theory between on the polydisk and the disk or the sphere.

Theorem 3.1. *Let A be a bounded operator on* $H^2(D^n)$ *such that*

$$T_{\bar{z}_i} A T_{z_i} = A, \quad i = 1, 2, \ldots, n.$$

Then there exists a function $f \in L^\infty(T^n)$ *such that*

$$A = T_f.$$

Proof. Let $A(D^n)$ be the polydisk algebra. For any $f \in A(D^n)$, we have $f \in H^2(D^n)$ and $\| f \|_2 \leq \| f \|_\infty$. Hence $A : (A(D^n), \| \cdot \|_\infty) \to H^2(D^n)$ is continuous, since A is a bounded linear operator on $H^2(D^n)$. Also $T_{\bar{z}_i} A T_{z_i} = A$, $i = 1, 2, \ldots, n$ implies

$$T_{\bar{z}_1^{m_1}} \cdots T_{\bar{z}_n^{m_n}} A T_{z_1^{m_1}} \cdots T_{z_n^{m_n}} = A$$

for every $(m_1, \ldots, m_n) \in Z_+^n$. By Guo and Chen's Lemma 2.2 in [7], there exists a function $f \in L^2(T^n)$ such that $A = T_f$. Since A is bounded on $H^2(D^n)$, $f \in L^\infty(T^n)$. This completes the proof of Theorem 3.1.

Theorem 3.2. *Let f be in* $L^\infty(T^n)$. *If* T_f *commutes essentially with all* T_{z_i} $(i = 1, 2, \ldots, n)$, *then* $f \in H^\infty(D^n)$ $(n \geq 2)$.

Proof. Let $T_f T_{z_i} = T_{z_i} T_f + K_i$ $(i = 1, 2, \ldots, n)$, where K_i are all compact operators on $H^2(D^n)$. Thus we have

$$\langle T_f T_{z_i} k_{w'} k_{w_i}, k_{w'} k_{w_i} \rangle = \langle T_{z_i} T_f k_{w'} k_{w_i} \rangle + \langle K_i k_{w'} k_{w_i}, k_{w'} k_{w_i} \rangle.$$

As $w' \to \mu' \in T^{n-1}$, we obtain that

$$\langle T_{f(\mu' \cdot)} T_{z_i} k_{w_i}, k_{w_i} \rangle = \langle T_{z_i} T_{f(\mu' \cdot)} k_{w_i}, k_{w_i} \rangle$$

by Lemma 1.2. Hence

$$T_{f(\mu'\cdot)}T_{z_i} = T_{z_i}T_{f(\mu'\cdot)},$$

where $T_{f(\mu'\cdot)}$ and T_{z_i} are Toeplitz operators on $H^2(T)$, for almost all fix $\mu' \in T^{n-1}$. It follows that f is analytic in variable z_i by Halmos theorem in [9]. So $f \in H^{\infty}(D^n)$. This completes the proof of Theorem 3.2.

Corollary 3.3. $A_c = \{f \in L^{\infty}(T^n) : T_f \in E_c F(H^{\infty}(D^n))\} = H^{\infty}(D^n)$ $(n \geq 2)$.

Theorem 3.4. *Let* $f \in L^{\infty}(T^n)$ *and* H_f *be Hankel operator on* $H^2(D^n)$ $(n \geq 2)$. *Then the following are equivalent*:

(1) $H_f = 0$.

(2) H_f *is compact*.

(3) $\| H_f k_z \| \to 0$ *as* $z \to \partial D^n$.

(4) $\langle (T_{|f|^2} - T_{\bar{f}}T_f)k_z, k_z\rangle \to 0$ *as* $z \to \partial D^n$.

(5) $f \in H^{\infty}$.

Proof. We have only to prove that (4) implies (5). Suppose that condition (4) holds, then by using Theorem 2.1, we have

$$\lim_{r \to 1^-} \langle (T_{|f|^2} - T_{\bar{f}}T_f)k_{r\mu'}k_{z_i}, k_{r\mu'}k_{z_i}\rangle = \langle (T_{|f(\mu'\cdot)|^2} - T_{\bar{f}(\mu'\cdot)}T_{f(\mu'\cdot)})k_{z_i}, k_{z_i}\rangle = 0$$

for almost all fix $\mu' \in T^{n-1}$. This implies that

$$T_{\bar{f}(\mu'\cdot)}T_{f(\mu'\cdot)} = T_{|f(\mu'\cdot)|^2}.$$

Hence $f(\mu'\cdot)$ is analytic in variable z_i due to Brown and Halmos's theorem in [1]. Thus $f \in H^{\infty}(D^n)$. This completes the proof of Theorem 3.4.

Corollary 3.5. $A_c = A = H^\infty(D^n)$, *when* $n \geq 2$.

Theorem 3.6. *Let* $S \in E_c F(H^\infty(D^n))$. *Then there is an* $f \in L^\infty(T^n)$ *and bounded operator* $\sigma \in L(H^2(D^n))$ *such that* $S = T_f + \sigma$.

Proof. Set $\sigma_k = S - T_{\bar{z}_1^k} S T_{z_1^k}$ for $k \in Z_+$. Since $\{\sigma_k : k \in Z_+\}$ is norm bounded, there exists a subsequence which is $w*-$ convergent. We may assume that, without loss of generality, the subsequence is $\{\sigma_k : k \in Z_+\}$. Let

$$\sigma = w*-\lim_{k\to\infty} \sigma_k.$$

Let

$$ST_{z_i} = T_{z_i} S + K_i,$$

where K_i are all compact. Then

$$\begin{aligned} T_{\bar{z}_i}(S-\sigma)T_{z_i} &= w*-\lim_{m\to\infty}(T_{\bar{z}_i} T_{\bar{z}_1^m} S T_{z_1^m} T_{z_i}) \\ &= w*-\lim_{m\to\infty}(T_{\bar{z}_1^m} S T_{z_1^m} + T_{\bar{z}_1^m} T_{\bar{z}_i} K_i T_{z_1^m}) \\ &= w*-\lim_{m\to\infty} T_{\bar{z}_1^m} S T_{z_1}^m = S - \sigma, \end{aligned}$$

since K_i is compact implies $w*-\lim_{m\to\infty} T_{\bar{z}_1^m} T_{\bar{z}_i} K_i T_{z_1^m} = 0$. Theorem 3.1 implies that $S - \sigma = T_f$ for some $f \in L^\infty(T^n)$. This gives the desired result.

Note. If we can prove operator σ is compact, then $E_c F(H^\infty(D^n)) = F(H^\infty(D^n)) + K$. Although we cannot prove that in now, but we still have the conjecture: $E_c F(H^\infty(D^n)) = F(H^\infty(D^n)) + K$ when $n \geq 2$.

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