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MEASURE IN GENERALIZED FUZZY SETS

MEHMET ŞAHIN and ZIYA YAPAR

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Abstract

In this article, using construction of fuzzy sets without depending on membership function, algebric properties of a family of fuzzy sets, three notions including a ring of generalized fuzzy sets $GF(X)$ of X, a complete Heyting algebra (cHa) which contains the power set $P(X)$ of X, an extension lattice $\overline{B(L)}$, where $B = P(X)$, and the set of L-fuzzy sets, where $L = \{L_x \mid x \in X\}$, definitions of fuzzy σ-algebra and fuzzy measure are generalized. We obtain some results using these definitions which include a generalization of Proposition 2 in [4].

1. Introduction

Zadeh, in his monumental paper [11], defined a fuzzy set as follows: "A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the value of $f_A(x)$ at x representing the "grade of membership of x in A". Therefore, fuzzy sets are unreal sets which exist only when identified with membership functions. For example a fuzzy set labeled old may be characterized by

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$s(x; 60, 70, 80)$ as introduced by Zadeh [12] or may be characterized by a triangular function. As shown by Nakajima [5], these two membership functions are equivalent under maxmin operations. The purpose of this paper is to generalize the definitions of fuzzy σ-algebra and fuzzy measure, called as *generalized fuzzy σ-algebra* and *generalized fuzzy measure*, respectively.

2. Preliminaries

In this section, we briefly review the well-known facts about lattice theory (Birkhoff [1], Iwamura [3]), propose an extension lattice, and investigate its properties. $(L, \vee, \wedge)$ or simply L, closed under operations $\vee$ and $\wedge$, is called a *lattice*, if it satisfies the cumulative, the associative and the absorption laws. A lattice L which satisfies, in addition, the distributive law is called *distributive*. For two lattices L and L', a bijection from L to L' which preserves lattice operations is called a *lattice-isomorphism*, or simply an *isomorphism*. If there is an isomorphism from L to L', then L is called *lattice-isomorphic* to L', and we write $L \cong L'$. We write $x \leq y$ if $x \wedge y = x$, or equivalently if $x \vee y = y$. We call a lattice L to be *complete* if every subset A of L possesses the supremum $\vee A$ and infimum $\wedge A$ with respect to the order in L. A complete lattice L has the maximum and minimum elements, which are denoted by I and O, or 1 and 0, respectively [6].

Definition 1. A complete distributive lattice L is called a *complete Heyting algebra* (*cHa*), if

$$\bigvee_{i \in I}(x_i \wedge y) = \left(\bigvee_{i \in I} x_i\right) \wedge y$$

for $x_i, i \in I$, and y in L, where I is an index set of arbitrary cardinal numbers [9].

Definition 2. Let L be a lattice and $c : L \to L$ be an operator. Then c is called a *complement* in L if the following conditions are satisfied [7]:

(i) for $a \in L$, $a^c \wedge a = 0$, $a^c \vee a = I$,

(ii) for $a, b \in L$, $a \leq b \Rightarrow a^c \geq b^c$,

(iii) for $a \in L$, $(a^c)^c = a$.

Definition 3. Let L be a bounded lattice and $c : L \to L$ be an operator. Then c is called a *generalized complement* if the following conditions are satisfied [7]:

(i) for $a, b \in L, a \leq b$, $a^c \geq b^c$,

(ii) for $a \in L$, $(a^c)^c = a$.

Definition 4. Let S be a fuzzy algebra of L^X. Then a function $m : S \to R_+$, is called a *fuzzy measure* if the following conditions are satisfied:

(i) for $A \in S$, $m(A) \geq 0$ and $m(\varnothing) = 0$,

(ii) for $A, B \in S$, if $A \leq B$, then $m(A) \leq m(B)$,

(iii) for $A, B \in S$, $m(A \vee B) + m(A \wedge B) = m(A) + m(B)$.

The pair (X, S) is called a *fuzzy measurable space* [7].

Lemma 1. *For any distributive lattice L, there exists a set E such that L is lattice-isomorphic with some sublattice of the power set $P(E)$, whose set operations are its lattice operations* [3].

Definition 5. If l is a sublattice of a lattice L, then L is said to be an *extension* of l. If L_1, a sublattice of L, is the minimal sublattice which contains the sublattice l of L and the subset M of L, then L_1 is the *extension lattice* of l obtained by adjoining M and is denoted by $L_1 = l(M)$.

Definition 6. A family $GF(X)$, which is closed under operations $\vee$ and $\wedge$, is called a *ring of generalized fuzzy subsets* of X, if it satisfies:

(i) $GF(X)$ is complete Heyting algebra with respect to $\vee$ and $\wedge$,

(ii) $GF(X)$ contains $P(X)$, the power set of X, as a sublattice of $GF(X)$,

(iii) the operations $\vee$ and $\wedge$ coincide with set operations $\bigcup$ (sum) and $\bigcap$ (intersection), respectively, in $P(X)$,

(iv) for any element A in $GF(X)$, $A \vee X = X$ and $A \wedge \varnothing = \varnothing$.

Note that because of (iii), the lattice operations may also be denoted by $\bigcup$ and $\bigcap$, in place of $\vee$ and $\wedge$, respectively.

Next we generalize the notions of L-fuzzy sets [2] to $\mathcal{L}$-fuzzy sets. Suppose that a lattice L_x is assigned to each x in X. Let $\mathcal{L} = \{L_x \mid x \in X\}$ [6].

Definition 7. An *$\mathcal{L}$-fuzzy set A* is characterized by a membership function μ_A which associates with each point x in X, an element $\mu_A(x)$ in L_x. The family of all $\mathcal{L}$-fuzzy sets is denoted by $LF(X)$. Set operations in $LF(X)$ are defined as usual: that is, for A and B in $LF(X)$, $\mu_{A\bigcup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\mu_{A\bigcap B}(x) = \mu_A(x) \wedge \mu_B(x)$. When each L_x is equal to L, $\mathcal{L}$-fuzzy sets reduce to Goguen's [2] L-fuzzy sets. An L-fuzzy set is called *simple*, if its membership function is a simple function: that is, if there is a finite division $\{E_1, ..., E_n\}$ of X and a finite set $\{A_1, ..., A_n\}$ of L such that $\mu_A(x) = A_i$ for $x \in E_i$, $i = 1, ..., n.$ The set of all simple L-fuzzy sets on X is denoted by $LF(X)$ [8].

Lemma 2. *Let L be a distributive lattice such that* $\max L = 1$ *and* $\min L = 0,$ *and let l and M be sublattices of L such that* $\max l = \max M = 1,$ *and* $\min l = \min M = 0.$ *Then $l(M)$ equals*

$$L_0 = \left\{\bigvee_{i=1}^{n}(a_i \wedge x_i) \mid a_i \in l,\ x_i \in M,\ n = 1, 2, ...\right\}.$$

Proof. An element of L_0 can be denoted by $\bigvee_{i=1}^{n}(a_i \wedge x_i)$ with a_i not being zero for a finite number of i's. For $y = \bigvee_{i=1}^{n}(a_i \wedge x_i)$ and $z = \bigvee_{j=1}^{m}(b_j \wedge y_j)$, from the fact that L is a distributive, it follows that

$$y \vee z = \left(\left[\bigvee_{i=1}^{n} (a_i \wedge x_i) \right] \vee \left[\bigvee_{j=1}^{m} (b_j \wedge y_j) \right] \right) \in L_0.$$

Also,

$$y \wedge z = \left[\bigvee_{i=1}^{n} (a_i \wedge x_i) \right] \wedge \left[\bigvee_{j=1}^{m} (b_j \wedge y_j) \right]$$

$$= \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (a_i \wedge x_i) \wedge (b_j \wedge y_j)$$

$$= \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (a_i \wedge b_j) \wedge (x_i \wedge y_j).$$

Noting that, $a_i \wedge b_j \in l$ and $x_i \wedge y_j \in M$, we obtain

$$y \wedge z \in L_0.$$

Depending on the same operations in L, it is seen that L_0 is a sublattice of L. L_0 contains l and M, because for any a in l and any x in M, $a = a \wedge 1$ and $x = 1 \wedge x$ lie in L_0. L' is a lattice in L, if $M \subseteq L'$ and $l \subseteq L'$, for $\bigvee a_i \in l$ and $x_i \in M$, $n \in N/\{0\}$,

$$a_i \wedge x_i \in l \Rightarrow \bigvee_{i=1}^{n} (a_i \wedge x_i) \in L' \Rightarrow L_0 \subseteq L' \Rightarrow L_0 = l(M).$$

So $l(M)$ is a minimal sublattice which contains M [8].

Lemma 3. *Suppose that L is a complete Heyting algebra such that* $\max L = 1$ *and* $\min L = 0$. *If l and M are complete Heyting subalgebras of L such that* $\max l = \max M = 1$ *and* $\min l = \min M = 0$, *then, with I being any index set of arbitrary cardinal number,*

$$\overline{l(M)} = \left\{ \bigvee_{i \in I} (a_i \wedge x_i) \mid a_i \in l, \, x_i \in M \right\}$$

is the minimal complete Heyting subalgebra of L which contains l and M.

Proof. See [8].

3. Generalized Fuzzy Measures

In this section, we investigate the algebraic structure of a family of fuzzy sets and give the definition of a σ-algebra of generalized fuzzy sets and also of generalized fuzzy measure. Furthermore, a lemma and a theorem are given depending on these definitions.

Definition 8. A family $S \subset GF(X)$ is called a *generalized fuzzy σ-algebra* of $GF(X)$, if the following conditions are satisfied:

(i) $\varnothing \in S$ and $X \in S$,

(ii) for $A \in S$, $A \wedge A^c = 0$, $A \vee A^c = 1$, when $(A^c)^c = A$, $A^c \in S$,

(iii) $\{A_i\}_{i \in N} \subset S \Rightarrow \bigcup_{i=1}^{\infty} A_i \in S$.

Since $L = \{L_x : x \in X\}$, L_x is a complete Heyting algebra. For $A \in S \subset GF(X)$ and for $x \in X$, we have $\mu_A(x) \in L_{(x)}$. In other words, A is a L-fuzzy set. $A \wedge \varnothing = \varnothing$, $A \vee X = X$ and $S \subset GF(X)$ are closed under operations $\wedge$ and $\vee$.

Definition 9. A function $m : S \to R_+^* = R_+ \cup \{+\infty\}$ is called a *generalized fuzzy measure* on S if the following conditions are satisfied:

(i) for $A \in S$, $m(A) \geq 0$, $m(\varnothing) = 0$,

(ii) for $A, B \in S$, if $m(A) \leq m(B)$, $A \leq B$,

(iii) for $A, B \in S$, $m(A \vee B) + m(A \wedge B) = m(A) + m(B)$,

(iv) $(A_n)_{n \in N} \subset S^N$, $A \in S : (A_n) \uparrow A \Rightarrow m(A_n) \uparrow m(A)$.

The pair (X, S) is called *generalized fuzzy measurable space*.

Lemma 4. *Let $S = P(X)$. Then a function $m : S \to R_+^*$, defined by*

$$m(A) = \begin{cases} Number\ of\ elements\ in\ A,\ if\ A\ is\ finite, \\ \infty,\ if\ A\ is\ infinite, \end{cases}$$

is a generalized fuzzy measure.

Proof. Write $\bigcap$ *and* $\bigcup$ in place of $\wedge$ and $\vee$, respectively.

(i) $m(A) \geq 0$ for $A \in P(X)$. $m(\varnothing) = 0$ for $\varnothing \in P(X)$.

(ii) For $A, B \in S = P(X)$, if $A \subseteq B$, then the number of elements in A cannot exceed that of in B. Thus $m(A) \leq m(B)$.

(iii) When $A, B \in S \Rightarrow m(A \vee B) + m(A \wedge B) = m(A) + m(B)$, $A \vee B = A \bigcup B$, $A \wedge B = A \bigcap B$ are obtained with m's reducing to classical measure.

$$\left.\begin{aligned} A \setminus B &= A \setminus (A \bigcap B) \\ A \setminus B &= (A \bigcup B) \setminus B \end{aligned}\right\} \Rightarrow A \setminus (A \bigcap B) = (A \bigcup B) \setminus B. \tag{1}$$

Since $m(A \setminus B) = m(A) - m(B)$ for $A, B \in S$, from (1), it follows that

$$m(A) - m(A \bigcap B) = m(A \bigcup B) - m(B). \tag{2}$$

Let $F = A \bigcap B$ and $F \subset E$. Then from (2), $E = A \bigcup B$ and $F = B$. For $E = A$, we have

$$m(A \vee B) + m(A \wedge B) = m(A) + m(B).$$

(iv) For $A \in S$, $(A_n) \uparrow A \Rightarrow \lim_{n \to \infty} A_n = A$, that is,

$$\bigcup_{n=1}^{\infty} A_n = A \Rightarrow m(\lim_{n \to \infty} A_n) = m(A) \Leftrightarrow m(A_n) \uparrow m(A).$$

Notation 1. Define μ^0 as $\mu_A^0 = \{(x, \alpha_x) \in X_X L_x : \alpha_x < \mu_A(x)\}$. Thus μ_A^0 means area under $\mu_A(x)$. For operators $\wedge$, $\vee$ used in the lattice L_x, $\mu_A(x) \wedge \mu_B(x)$ and $\mu_A(x) \vee \mu_B(x)$, have their usual meaning. If $\wedge$, $\vee$ are operators used in $GF(X)$, then $A \wedge B$ means $\mu_{A \bigcap B}$, where $\bigcap$, $\bigcup$ are used in $X \times L$. For $A, B \in S$, consider

$$\begin{aligned} \mu_{A \bigcap B} &= \{(x, \alpha_x) : \alpha_x < \mu_{A \bigcap B}(x)\} \\ &= \{(x, \alpha_x) : \alpha_x < \mu_{A \bigcap B}(x) \leq \mu_A(x) \leq \mu_B(x), \\ &\qquad \alpha_x < \mu_{A \wedge B}(x) \leq \mu_B(x)\}. \end{aligned}$$

Then

$$\alpha_x < \mu_A(x) \text{ and } \alpha_x < \mu_B(x) \Rightarrow \alpha_x < \min\{\mu_A(x), \mu_B(x)\}.$$

Thus $\mu_A(x) \wedge \mu_B(x)$ is valid for L_x. Here $\mu_A^0, \mu_B^0 \in X \times L$.

$$(x, \alpha_x) \in (\mu_A \wedge \mu_B)^0 \Rightarrow (x, \alpha_x) \in \mu_A^0, (x, \alpha_x) \in \mu_B^0,$$

where

$$(\mu_A \wedge \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\},$$

$$(\mu_A \wedge \mu_B)^0 \subset \mu_A^0 \cap \mu_B^0, \quad \mu_A^0 \cap \mu_B^0 \subset X \times L.$$

Since

$$(x, \alpha_x) \in (\mu_A^0 \cap \mu_B^0) \Rightarrow \alpha_x < \mu_A(x) \text{ and } \alpha_x < \mu_B(x),$$

$$\alpha_x < \min(\mu_A(x), \mu_B(x)) \Rightarrow (x, \alpha_x) \in (\mu_A \wedge \mu_B)^0,$$

$$(\mu_A \wedge \mu_B)^0 = \mu_A^0 \cap \mu_B^0.$$

Next,

$$(x, \alpha_x) \in (\mu_A \vee \mu_B)(x) \Rightarrow \alpha_x < (\mu_A \vee \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}.$$

If $(\mu_A \vee \mu_B)(x) = \mu_B(x)$, then $\alpha_x < \mu_B(x)$, $(x, \alpha_x) \in \mu_B^0$, and hence $(x, \alpha_x) \in \mu_A^0 \cup \mu_B^0$. Similarly, if

$$(\mu_A \vee \mu_B)(x) = \mu_A(x) \Rightarrow (x, \alpha_x) \in \mu_A^0 \cup \mu_B^0,$$

then

$$(\mu_A \vee \mu_B)^0 \subset \mu_A^0 \cup \mu_B^0 \in X \times L \Rightarrow \mu_{A \vee B} \subset \mu_A^0 \cup \mu_B^0.$$

Conversely, $\forall (x, \alpha_x) \in \mu_A^0 \cup \mu_B^0 \Rightarrow (x, \alpha_x) \in \mu_A^0$ or $(x, \alpha_x) \in \mu_B^0$. Let, for example, $(x, \alpha_x) \in \mu_A^0$. Then

$$\begin{aligned} \alpha_x < \mu_A(x) &\Rightarrow \alpha_x < \max\{\mu_A(x), \mu_B(x)\} \\ &\Rightarrow \alpha_x \in (\mu_A \vee \mu_B)(x) \\ &\Rightarrow (x, \alpha_x) \in (\mu_A \vee \mu_B)^0 \\ &\Rightarrow (\mu_A \vee \mu_B)^0 = \mu_A^0 \cup \mu_B^0. \end{aligned}$$

Theorem 1. *Let* $S(\in GF(X))$ *be a* σ*-algebra of generalized fuzzy sets. Let* $A(S)$ *be classical* σ*-algebra determined by* $\{\mu_A^0 : A \in S\}$ *on* $X \times L_x$. *If* $Q : (X \times L_x, A(S)) \to R_+^*$ *is a (finite) measure where* $m : S \to R_+^*$ *is such that* $m(\mu_A) = Q(\mu_A^0)$, *then m is a generalized fuzzy measure.*

Proof. (i) We have $m(\varnothing) = Q(\mu_\varnothing^0) = 0$, where $\mu_\varnothing^0 = \{(x, \alpha_x) \in X \times L_x : \alpha_x < \mu_\varnothing(x) = 0\}$. As $\alpha_x \in L_x$ cannot be found so, $\mu_\varnothing^0$ is a $\varnothing$ set. Since Q is a measure, $Q(\mu_\varnothing^0) = Q(\varnothing) = 0$. Therefore, $m(\mu_\varnothing) = Q(\mu_\varnothing^0)$.

(ii) Suppose $\mu_A, \mu_B \in S$. Let $\mu_A \leq \mu_B$. Then

$$\mu_A^0 = \{(x, \alpha_x) \in X \times L_x : \alpha_x < \mu_A(x)\}$$

and

$$\mu_A(x) \leq \mu_B(x) \Rightarrow \alpha_x < \mu_B(x).$$

Thus $(x, \alpha_x) \in \mu_B^0$. Since Q is a measure, $Q(\mu_A^0) \leq Q(\mu_B^0)$. Further, since $m(\mu_A) = Q(\mu_A^0)$ and $m(\mu_B) = Q(\mu_B^0)$,

$$m(\mu_A) = Q(\mu_A^0) \leq Q(\mu_B^0) = m(\mu_B) \Rightarrow m(\mu_A) \leq m(\mu_B).$$

(iii) For $A, B \in S$, consider $\mu_A, \mu_B \in S$. Then

$$m(\mu_A \wedge \mu_B) + m(\mu_A \vee \mu_B) = Q((\mu_A \wedge \mu_B)^0) + Q((\mu_A \vee \mu_B)^0). \tag{3}$$

Since $(\mu_A \wedge \mu_B)^0 = \mu_A^0 \cap \mu_B^0$ and $(\mu_A \vee \mu_B)^0 = \mu_A^0 \cup \mu_B^0$,

$$Q((\mu_A \vee \mu_B)^0) = Q(\mu_A^0) + Q(\mu_B^0) - Q(\mu_A^0 \cap \mu_B^0).$$

From (3)

$$\begin{aligned} & Q((\mu_A \wedge \mu_B)^0) + Q((\mu_A \vee \mu_B)^0) \\ &= Q((\mu_A \wedge \mu_B)^0) + Q(\mu_A^0) + Q(\mu_B^0) - Q(\mu_A^0 \cap \mu_B^0) \\ &= Q((\mu_A \wedge \mu_B)^0) + Q(\mu_A^0) + Q(\mu_B^0) - Q((\mu_A \wedge \mu_B)^0) \\ &= Q(\mu_A^0) + Q(\mu_B^0) \\ &= m(\mu_A) + m(\mu_B). \end{aligned}$$

Hence

$$m(\mu_A \vee \mu_B) + m(\mu_A \wedge \mu_B) = m(\mu_A) + m(\mu_B).$$

(iv) Let $(\mu_{A_n})_{n\in N} \in S^N$. Suppose that $(\mu_{A_n}) \uparrow \mu_A$. This requires $(\mu^0_{A_n}) \uparrow \mu^0_A$.

Since Q is a measure, $Q(\mu^0_{A_n}) \uparrow Q(\mu^0_A)$ is valid. From this it follows that $m(\mu_{A_n}) = Q(\mu^0_{A_n}) \uparrow Q(\mu^0_A) = m(\mu_A)$.

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Department of Mathematics
Faculty of Arts and Sciences
University of Gaziantep
Gaziantep, Turkey
e-mail: mesahin@gantep.edu.tr