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ON DOMINATION IN HAMILTONIAN CUBIC GRAPHS

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Abstract

In 1996, Reed proved that the domination number $\gamma(G)$ of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$. Also, he conjectured that $\gamma(H) \geq \left\lceil \frac{n}{3} \right\rceil$ for every connected 3-regular (cubic) n-vertex graph H. Reed's conjecture is obviously true for Hamiltonian cubic graphs. In this note, we present a sequence of Hamiltonian cubic graphs whose domination numbers are sharp. The connected domination number, independent domination number, and total domination number for these graphs are presented.

1. Introduction

Let G be a graph, with n vertices and e edges. Let $N(v)$ be the set of neighbors of a vertex v and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ be the degree of v. G is r-regular if $d(v) = r$ for all v; if $r = 3$, then G is cubic. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a *dominating set*, if each vertex of G is dominated by some vertex of S. The *domination number* $\gamma(G)$ of G is the minimum

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cardinality of a dominating set of G. A dominating set S is called a *connected dominating set* if the subgraph $G[S]$ induced by S is connected. The *connected domination number* of G denoted by $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. A dominating set S is called an *independent dominating set* if S is an independent set. The *independent domination number* of G denoted by $i(G)$ is the minimum cardinality of an independent dominating set of G. A dominating set S is a *total dominating set* of G if $G[S]$ has no isolated vertex and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G, (see [1, 2, 5, 6, 7]).

The problem of finding the domination number of a graph is NP-hard, even when restricted to cubic graphs. One simple heuristic is the greedy algorithm, (see [10]). Let d_g be the size of the dominating set returned by the greedy algorithm. In 1991, Parekh [8] showed that $d_g \le n + 1 - \sqrt{2e+1}$. Also, some bounds have been discovered on $\gamma(G)$ for cubic graphs. Reed [9] proved that $\gamma(G) \le \frac{3}{8}n$. He conjectured that $\gamma(H) \ge \left\lceil \frac{n}{3} \right\rceil$ for every connected 3-regular (cubic) n-vertex graph H. Reed's conjecture is obviously true for Hamiltonian cubic graphs. Fisher et al. [3, 4] repeated this result and showed that if G has girth at least 5, then $\gamma(G) \le \frac{5}{14}n$. In the light of these bounds on γ, in 2004, Seager considered bounds on d_g for cubic graphs and showed that:

Theorem A [10, Theorem 1]**.** *For a cubic graph G, $d_g \le \frac{4}{9}n$.*

Theorem B [10, Theorem 2]**.** *For an r-regular graph G with $r \ge 3$,*

$$d_g \le \frac{r^2 + 4r + 1}{(2r+1)^2} n.$$

The aim of this paper is to study of the domination number $\gamma(G)$, connected domination number $\gamma_c(G)$, independent domination number $i(G)$, and total domination number $\gamma_t(G)$ for Hamiltonian cubic graphs and it is given a sharp value for the domination numbers of these graphs.

The following will be useful.

Theorem C [4, Theorem 2.11]. *For any graph of order* n, $\left\lceil \frac{n}{1+\Delta G} \right\rceil \leq \gamma(G)$.

2. Domination Number

In this section we show a sharp value of domination number of some cubic graph.

Let $G = (V, E)$ be a graph denoted in Figure 1, $V = \{v_1, v_2, \ldots, v_n\}$ $(n = 2r)$ and $E = \{v_i v_j \,||\, i - j \,| = 1 \text{ or } r\}$. So G has two vertices v_1 and v_n of degree two and $n - 2$ vertices of degree three. By the graph, G is the graph described in Figure 1.

For the following we put $N_p[x] = \{z \,|\, z$ is only dominated by $x\} \cup \{x\}$.

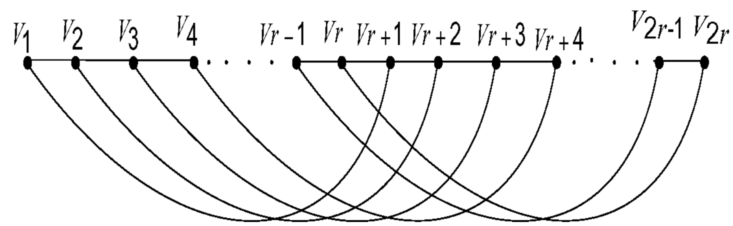

Figure 1

Lemma 1. $\gamma(G) = \begin{cases} 2\left\lfloor \frac{r}{4} \right\rfloor + 2 & \textit{if } r \equiv 3 \pmod 4 \\ 2\left\lfloor \frac{r}{4} \right\rfloor + 1 & \textit{otherwise.} \end{cases}$

Proof. Suppose that $r \equiv 3 \pmod 4$, say $r = 4k + 3$ for some positive integer k. It is easy to verify that the set of vertices $S_0 = \{v_1, v_5, v_9, \ldots, v_{r-2}, v_r, v_{r+3}, v_{r+7}, \ldots, v_{2r}\}$ is a dominating set for G. Therefore $\gamma(G) \leq 2\left\lfloor \frac{r}{4} \right\rfloor + 2 = 2k + 2$. On the other hand, Theorem A implies that $\gamma(G) \geq \left\lceil \frac{n}{1+3} \right\rceil = 2k + 2$, so $\gamma(G) = 2k + 2$. Now we suppose $r \equiv t \pmod 4$ such that $t = 0, 1$ and 2. Obviously the graph G dominated by the set $S_0 = \{v_2, v_6, v_{10}, \ldots, v_{r-t-2}, v_r, v_{r+4}, v_{r+8}, \ldots, v_{2r-t}\}$, so necessarily $\gamma(G) \leq |S_0| = 2\left\lfloor \frac{r}{4} \right\rfloor + 1 = 2k + 1$. Furthermore, Theorem A shows $\gamma(G) \geq \left\lceil \frac{n}{4} \right\rceil = 2k + \left\lceil \frac{t}{2} \right\rceil$.

Now, if $t = 1$ or 2, then $\gamma(G) \geq 2k + 1$, so $\gamma(G) = 2k + 1$ in this case.

Finally, assume $t = 0$, so $n = 4k$. We assume that S is an arbitrary dominating set for G. If $\{v_1, v_n\} \cap S \neq \emptyset$, then $\gamma(G) > 2k$. So we suppose that $\{v_1, v_n\} \cap S = \emptyset$. But $\{v_2, v_{r+1}\} \cap S \neq \emptyset$ and $\{v_r, v_{2r-1}\} \cap S \neq \emptyset$. Thus we consider four cases:

Case 1. $\{v_r, v_{r+1}\} \subset S$. Since $N[v_r] \cap N[v_{r+1}] \neq \emptyset$, so $\gamma(G) > 2k$.

Case 2. $\{v_2, v_{2r-1}\} \subset S$. If $v_r \in S$, then $\gamma(G) > 2k$, since $N[v_r] \cap N[v_{2r-1}] \neq \emptyset$. Now we suppose that $v_r \notin S$, so $v_{r-1} \in S$ or $v_{r+1} \in S$ for example $v_{r-1} \in S$, since $N[v_{r-1}] \cap N[v_{2r-1}] \neq \emptyset$, so $\gamma(G) > 2k$.

Case 3. $\{v_2, v_r\} \subset S$. But $\{v_4, v_{r+5}\} \cap S = \emptyset$, so $v_6 \in S$. By the same description we have $\{v_{10}, v_{14}, \ldots, v_{r-2}\} \subset S$ and this is impossible, because $N[v_r] \cap N[v_{r-2}] \neq \emptyset$, so $\gamma(G) > 2k$.

Case 4. $\{v_{r+1}, v_{2r-1}\} \subset S$. The same argument which described in Case 3 can be used this case.

Suppose that the graphs G' and G'' are two induced subgraphs of G such that $V(G') = V(G) - \{v_1, v_n\}$ and $V(G'') = V(G) - \{v_1\}$ (or $V(G'') = V(G) - \{v_{2r}\}$).

Lemma 2. *If* $r \equiv 2$ *or* $3 \pmod 4$, *then* $\gamma(G') = \gamma(G)$.

Proof. First, we suppose $r \equiv 2 \pmod 4$, so $r = 4k + 2$ for some positive integer k.

By Theorem A, $\gamma(G') \geq \left\lceil \frac{n(G')}{1 + \Delta(G')} \right\rceil = 2k + 1$.

Now we attend to S_0 (Lemma 1, in the case $r \equiv 2 \pmod 4$). It is a dominating set for G', so $\gamma(G') = 2k + 1$.

Suppose that $r \equiv 3 \pmod 4$. If $\gamma(G') = \gamma(G) - 1 = 2\left\lfloor \frac{r}{4} \right\rfloor + 1$, then we suppose that S is a dominating set for G', such that $|S| = 2\left\lfloor \frac{r}{4} \right\rfloor + 1$, so for each $v \in S$, $|N_p[v]| = 4$. By this description we have $\{v_{r-1}, v_{r+2}\} \subset S$, obviously the vertex v_3 does not dominate by v_{r+3} or v_2, so $v_4 \in S$. Similarly $v_{r+6} \in S$ and finally the vertices v_{r-3}, v_{r-4}, v_{2r-2} and v_{2r-3} must be dominate by one vertex and this is impossible. So $\gamma(G') = 2\left\lfloor \frac{r}{4} \right\rfloor + 2 = \gamma(G)$.

Lemma 3. *If* $r \equiv 0 \pmod 4$, *then* $\gamma(G'') = \gamma(G) - 1$.

Proof. We suppose $r = 4k$, where $k \in N$. It is easy to verify that $S_0' = \{v_4, v_8, v_{12}, \ldots, v_{r-4}, v_r, v_{r+2}, v_{r+6}, \ldots, v_{2r-6}, v_{2r-2}\}$ is a dominating set for G', consequently $\gamma(G') \leq |S_0| = 2k$. But by Theorem A, $\gamma(G') \geq \left\lceil \frac{8k-2}{4} \right\rceil = 2k$, so $\gamma(G') = \gamma(G) - 1$.

Lemma 4. *If* $r \equiv 1 \pmod 4$, *then* $\gamma(G') = \gamma(G) - 1$.

Proof. We suppose $r = 4k + 1$, where $k \in N$, by Theorem A, $\gamma(G') \geq 2k$. On the other hand, the set $S_0 = \{v_4, v_8, \ldots, v_{r-1}, v_{r+2}, v_{r+6}, \ldots, v_{2r-3}\}$ is a dominating set for G, so $\gamma(G') \leq |S_0| = 2k$. Therefore $\gamma(G') = 2k = \gamma(G) - 1$.

Let G_0 be a graph of order mn $(n = 2r)$, $V(G_0) = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}\}$ and $E = \{\{v_{ij}, v_{il}\} \mid |j - l| = 1 \text{ or } n\} \cup \{\{v_{in}, v_{(i+1)1}\} \mid i = 1, 2, \ldots, m-1\} \cup \{v_{11}, v_{mn}\}$. By this definition of G_0 the graph G_0 is 3-regular graph. Suppose that the graph G_i' is an induced subgraph of G_0 with the vertices $v_{i1}, v_{i2}, \ldots, v_{in}$.

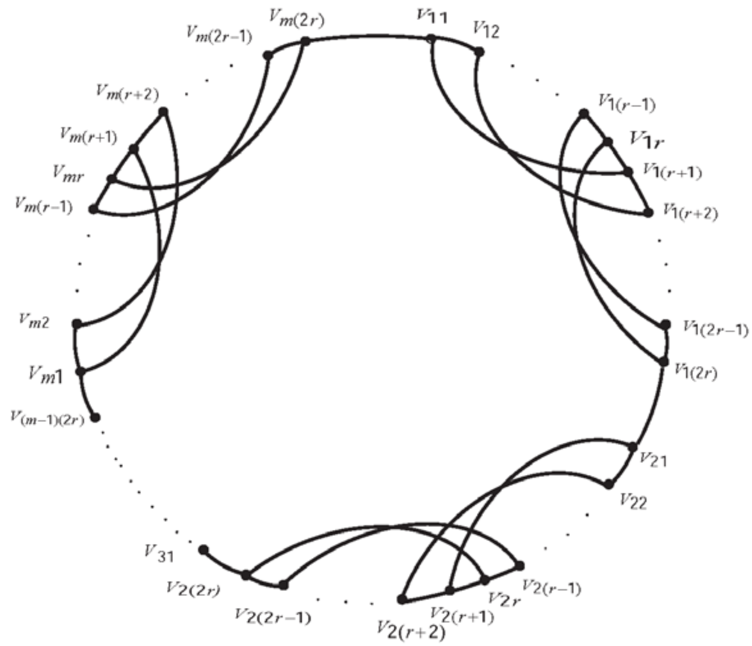

Figure 2

Theorem 5. $\gamma(G_0) = \begin{cases} m\left\lceil \frac{n}{4} \right\rceil & \textit{if } r \equiv 2 \pmod 4 \\ m\left(\left\lceil \frac{n}{4} \right\rceil + 1\right) & \textit{if } r \equiv 3 \pmod 4. \end{cases}$

Proof. We suppose that $r \equiv 2 \pmod 4$. We consider $S_i = \{v_{i2}, v_{i6}, ..., v_{i(r-4)}, v_{ir}, v_{i(r+4)}, ..., v_{i(2r-2)}\}$. The set $S_0 = \bigcup_{i=1}^{m} S_i$ is a dominating set for G_0, so $\gamma(G_0) \leq |S_0| = m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) = m\left\lceil \frac{n}{4} \right\rceil$. If S is a dominating set of G and $|S| < m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right)$, then there is $i \in \{1, ..., m\}$, such that $|S \cap V(G_i')| \leq 2\left\lfloor \frac{r}{4} \right\rfloor$. This contradicts Lemma 2, so $\gamma(G_0) = m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) = m\left\lceil \frac{n}{4} \right\rceil$. For case $r \equiv 3 \pmod 4$, a same argument in case $r \equiv 2 \pmod 4$, shows $\gamma(G_0) = m\left(\left\lceil \frac{n}{4} \right\rceil + 1\right)$.

Theorem 6. *If* $r \equiv 1 \pmod 4$, *then* $\gamma(G_0) = m\left\lceil \frac{n}{4} \right\rceil - \left\lceil \frac{m}{3} \right\rceil$.

Proof. Suppose that $r = 4k+1$ and S_i is a dominating set for G_i. If $|\{v_{i1}, v_{in}\} \cap S| = 2$, then $|S_i| > 2k+1$. Because if $|S| = 2k+1$, and $\{v_{i1}, v_{in}\} \subset S_i$, then for each vertex $v \in S_i \setminus \{v_{i1}, v_{in}\}$, $|N_p(v)| = 4$ and $|\{v_{i3}, v_{i4}, \ldots, v_{i(r-1)}\}| = |\{v_{i(r+2)}, v_{i(r+3)}, \ldots, v_{i(2r-2)}\}|$. This is impossible, so $|S_i| > 2k+1$. We consider

$$S_i' = \{v_{i3}, v_{i7}, \ldots, v_{i(r-2)}, v_{i(r+1)}, v_{i(r+5)}, \ldots, v_{i(2r-4)}, v_{i(2r)}\},$$

$$S_i'' = \{v_{i4}, v_{i8}, \ldots, v_{i(r-5)}, v_{i(r-1)}, v_{i(r+2)}, v_{i(r+6)}, \ldots, v_{i(2r-3)}\},$$

$$S_i''' = \{v_{i1}, v_{i5}, \ldots, v_{i(r-4)}, v_{ir}, v_{i(r+3)}, v_{i(r+7)}, \ldots, v_{i(2r-2)}\}$$

and

$$S_i = S_i' \cup S_{i+1}'' \cup S_{i+2}'''.$$

Now if $m \equiv 0 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-2}$ is a dominating set for G_0. If $m \equiv 1 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-3} \cup S_m'$ is a dominating set for G_0 and if $m \equiv 2 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-4} \cup S_{m-1}' \cup S_m'$ is a dominating set for G_0. So $\gamma(G_0) \le |S_0| = m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) - \left\lfloor \frac{m}{3} \right\rfloor = m\left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{m}{3} \right\rfloor$, by Lemma 4, we have $\gamma(G_0) = m\left\lceil \frac{n}{4} \right\rceil - \left\lceil \frac{m}{3} \right\rceil$.

Theorem 7. *If* $r \equiv 0 \pmod 4$, *then*

$$\gamma(G_0) = \begin{cases} m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) - 2\left\lfloor \frac{m}{3} \right\rfloor - 1 & \text{if } m \equiv 2 \pmod 3 \\ m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) - 2\left\lfloor \frac{m}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. First we suppose

$$S_i' = \{v_{i3}, v_{i6}, \ldots, v_{i(r-1)}, v_{i(r+1)}, v_{i(r+5)}, v_{i(r+9)}, \ldots, v_{i(2r-3)}\},$$

$$S_i'' = \{v_{i1}, v_{i2}, v_{i6}, v_{i10}, \ldots, v_{ir-2}, v_{i(r+4)}, v_{i(r+8)}, \ldots, v_{i(2r-4)}, v_{i(2r)}\},$$

$S_i''' = \{v_{i4}, v_{i8}, ..., v_{ir}, v_{i(r+2)}, v_{i(r+6)}, ..., v_{i(2r-2)}\}.$

We also suppose $S_i = S_i' \cup S_{i+1}'' \cup S_{i+2}'''$. If $m \equiv 0 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-2}$ is a dominating set for G_0. If $m \equiv 1 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-2} \cup S_m'$ is a dominating set for G_0. So if $m \equiv 0$ or $1 \pmod 3$, then $\gamma(G_0) \leq |S_0| = m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) - 2\left\lfloor \frac{m}{3} \right\rfloor$. Now if $m \equiv 2 \pmod 3$, then the set $S_0 = S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{m-4} \cup S_{m-1}'' \cup S_m'''$ is a dominating set for G_0. So $\gamma(G_0) \leq |S_0| = m\left(2\left\lfloor \frac{r}{4} \right\rfloor + 1\right) - 2\left\lfloor \frac{m}{3} \right\rfloor - 1$, but by Lemma 3, $\gamma(G_0) = |S_{G_0}|$.

3. Connected, Independent and Total Domination Number

In this section we study $\gamma_c(G_0)$, $i(G_0)$ and $\gamma_t(G_0)$.

Lemma 8. $\gamma_c(G) = r - 1$.

Proof. Obviously $S_0 = \{v_2, v_3, ..., v_r\}$ is a connected dominating set for G, so $\gamma_c(G) \leq r - 1$. Now we suppose S is an arbitrary connected dominating set for G. If $\langle S \rangle$ is a path of length l where at most $r - 2$, then for the first and last vertices of this path, we have $|N_p[x]| = |N_p[y]| = 3$ and for other vertices of this path $|N_p[z]| = 2$, so $\cup_{x \in S} N[x] \leq 2 \times 3 + (r-4) \times 2 = 2r - 2 = n - 2$, so S cannot dominate all vertices.

Lemma 9. $i(G) = \gamma(G)$.

Proof. Since the set S_0 introduced in Lemma 1, is independent dominating set for G, so $i(G) \leq \gamma(G)$, and therefore $i(G) = \gamma(G)$.

Lemma 10. $$\gamma_t(G) = \begin{cases} 2\left\lfloor \frac{r}{3} \right\rfloor & \text{if } r \equiv 0 \pmod 3 \\ 2\left\lfloor \frac{r}{3} \right\rfloor + 1 & \text{if } r \equiv 1 \pmod 3 \text{ and } r \text{ is even} \\ 2\left\lfloor \frac{r}{3} \right\rfloor + 2 & \text{otherwise.} \end{cases}$$

Proof. First we assume $r \equiv 0 \pmod 3$, so $r = 3l$. It is easy to verify that the set $S_0 = \{v_2, v_{r+2}, v_5, v_{r+5}, ..., v_{r-1}, v_{2r-1}\}$ is a total dominating set for G. This implies that $\gamma_t(G) \le |S_0| = 2l$. Now we suppose that S is an arbitrary total dominating set for G. For each vertex $v_x \in S$, $|N_p[x]| \le 3$, so $\left\lceil \frac{n}{3} \right\rceil \le \gamma_t(G)$, this implies that $\gamma_t(G) \ge \left\lceil \frac{2 \times 3l}{3} \right\rceil = 2l$, therefore $\gamma_t(G) = 2l = 2\left\lfloor \frac{r}{3} \right\rfloor$.

If $r \equiv 2 \pmod 3$, then $r = 3l + 2$ and the set $S_1 = \{v_2, v_{r+2}, v_5, v_{r+5}, ..., v_{r-3}, v_{2r-3}, v_r, v_{2r}\}$ is a total dominating set for G, so $\gamma_t(G) \le |S_0| = 2l + 2$. In this case, we have $\gamma_t(G) \ge \left\lceil \frac{2(3l+2)}{3} \right\rceil = 2l + 2$. So $\gamma_t(G) = 2l + 2$.

Now we suppose $r = 3l + 1$ and S is an arbitrary total dominating set for G, obviously $|S| \ge 2l + 1$. If r is even, then the set

$$S_2 = \{v_4, v_5, v_{10}, v_{11}, ..., v_{r-12}, v_{r-11}, v_{r-6}, v_{r-5}, v_{r-4}, v_{r+1}, v_{r+2}, v_{r+7}, v_{r+8}, ..., v_{2r-2}, v_{2r-1}\},$$

therefore $\gamma_t(G) = 2l + 1 = 2\left\lfloor \frac{r}{3} \right\rfloor + 1$.

Now we suppose r is odd and S is a total dominating set for G, such that $|S| = 2l + 1$. If $\{v_1, v_{2r}\} \cap S \ne \varnothing$, for example $v_1 \in S$, then $\{v_2, v_{r+1}\} \cap S \ne \varnothing$, (for example $v_2 \in S$). Since $|\{v_{r+3}, v_{r+4}, ..., v_{2r}\}| = |\{v_4, v_5, ..., v_r\}| + 1$, so there is a vertex $v_i \in S \setminus \{v_1\}$ such that $|N_p[v_i]| < 3$, and this is contradiction, because for each vertex $v_i \in S \setminus \{v_1\}$, $|N_p[v_i]| = 3$.

So $\{v_1, v_{2r}\} \cap S = \varnothing$ and there are vertices v_x, v_y, v_z such that $|x - y| = 1$, $|z - y| = 1$ and $x < y < z$.

Now there are four cases:

Case 1. $x = r - 1$, $y = r$ and $z = r + 1$.

In this case $|\{v_1, v_2, ..., v_r\} \setminus A| = |\{v_{r+1}, v_{r+2}, ..., v_{2r}\} \setminus A| = r - 4$, where $A = N[x] \cup N[y] \cup N[z]$. But r is odd, so the vertices v_{r-3}, v_{r-4}, v_{r-5}, v_{2r-4}, v_{2r-3} and v_{2r-2} must be dominated by two adjacent vertices and it is a contradiction.

Case 2. $x = r$, $y = r + 1$ and $z = r + 2$, the proof is similar to the proof of Case 1.

Case 3. $\{v_x, v_y, v_z\} \subset \{v_1, v_2, ..., v_r\}$, we consider $B = \{v_1, v_2, ..., v_{x-2}\}$.

If $|B| \equiv 0 \pmod 6$, then the vertices v_{r-1}, v_r, v_{r+1}, v_{2r}, v_{2r-1} and v_{2r-2} must be dominated by two adjacent vertices and this is impossible.

If $|B| \equiv 1 \pmod 6$, then the vertices v_r, v_{r+1}, v_{r+2}, v_1, v_{2r-1} and v_{2r} must be dominated by two adjacent vertices and this is impossible.

If $|B| \equiv 2 \pmod 6$, then the vertices v_{r+1}, v_{r+2}, v_{r+3}, v_1, v_2 and v_{2r} must be dominated by two adjacent vertices and this is impossible.

If $|B| \equiv 3 \pmod 6$, then the vertices v_{r-2}, v_{r-1}, v_r, v_1, v_{2r} and v_{2r-1} must be dominated by two adjacent vertices and this is impossible.

If $|B| \equiv 4 \pmod 6$, then the vertices v_{r-1}, v_r, v_{r+1}, v_1, v_2 and v_{2r} must be dominated by two adjacent vertices and this is impossible.

If $|B| \equiv 5 \pmod 6$, then the vertices v_{r-2}, v_{r-1}, v_r, v_1, v_2 and v_3 must be dominated by two adjacent vertices and this is impossible.

Case 4. $\{v_x, v_y, v_z\} \subset \{v_{r+1}, v_{r+2}, ..., v_{2r}\}$, a same argument described in Case 3 settles this case.

So $|S| > 2l + 1$, but the set $S_3 = \{v_2, v_{r+2}, v_5, v_{r+5}, ..., v_{r-2}, v_{2r-2}, v_{r-1}, v_{2r-1}\}$ is a total dominating set for G. This implies $\gamma_t(G) \leq 2l + 2 = 2\left\lfloor \frac{r}{3} \right\rfloor + 2$, so $\gamma_t(G) = 2l + 2 = 2\left\lfloor \frac{r}{3} \right\rfloor + 2$.

Lemma 11. $\gamma_c(G') = \gamma_c(G)$.

Proof. Obviously $\gamma_c(G') > r - 2$, but the set S_0 in Lemma 1 is a connected dominating set for G', so $\gamma_c(G') \le r-1$, therefore $\gamma_c(G') = r-1$.

Lemma 12. $\gamma_t(G') = \begin{cases} \gamma_t(G) - 2 & \textit{if } r \equiv 1 \pmod 3 \textit{ and } r \textit{ is odd} \\ \gamma_t(G) & \textit{otherwise.} \end{cases}$

Proof. If $r \equiv 0 \pmod 3$, then $r = 3l$. Since the set S_0 introduced in Lemma 10 is a total dominating set for G', so $\gamma_t(G') \le 2l$. On the other hand, $\gamma_t(G') \ge \left\lceil \frac{n(G')}{3} \right\rceil = \left\lceil \frac{6l-2}{3} \right\rceil = 2l$. Therefore $\gamma_t(G') = 2l$.

If $r \equiv 2 \pmod 3$, then $r = 3l + 2$. In this case we suppose that S' is an arbitrary total dominating set for G'. It is simple to see $|S'| > 2l$.

If $|S'| = 2l + 1$, then there are three cases:

Case 1. v_r and v_{r+1} belong to S'. But $|N[v_r] \cup N[v_{r+1}]| = 4$, so $6l-2$ other vertices dominate by $2l-1$ vertices of S', but this is impossible, (because at most $6l-3$ vertices are dominated by $2l-1$ vertices).

Case 2. $|\{v_r, v_{r+1}\} \cap S'| = 1$, without loss of generality we suppose that $v_r \in S'$ so $v_{r-1} \in S'$ and for each vertex $v_i \in S' \setminus \{v_2\}$, $|N_p(v_i)| = 3$. This implies $\{v_2, v_{r+2}\} \cap S' \ne \varnothing$, so $\{v_3, v_{r+3}\} \subset S'$ and this is impossible, because $|\{v_{r+5}, v_{r+6}, \ldots, v_{2r-2}\}| = |\{v_5, v_6, \ldots, v_{r-3}\}| + 1$.

Case 3. $\{v_r, v_{r+1}\} \cap S' = \varnothing$, so $\{v_{r-1}, v_{r+2}\} \subset S'$ and also we have $\{v_{r-2}, v_{2r-1}\} \cap S' \ne \varnothing$ and $\{v_2, v_{r+3}\} \cap S' \ne \varnothing$. For example $\{v_2, v_{r-2}\} \subset S'$, this is impossible, since $|\{v_{r+4}, v_{r+5}, \ldots, v_{2r-3}\}| = |\{v_4, v_5, \ldots, v_{r-4}\}| + 1$.

So $|S'| \ge 2l + 2$, but the set $S_0' = \{v_3, v_{r+3}, v_6, v_{r+6}, \ldots, v_{r-2}, v_{2r-2}, v_r, v_{r+1}\}$ is a total dominating set for G', so $\gamma_t(G') \le |S_0'| = 2l + 2$. Combining the two inequalities, we obtain $\gamma_t(G') = 2l + 2$.

Now we suppose $r \equiv 1 \pmod 3$, so $r = 3l + 1$. If r is odd, then the set $S_0 = \{v_5, v_6, v_{11}, v_{12}, \ldots, v_{r-2}, v_{r-1}, v_{r+2}, v_{r+3}, v_{r+8}, v_{r+9}, \ldots, v_{2r-5}, v_{2r-4}\}$ is a total dominating set for G', so $|S_t| \leq |S_0| = 2l$. But $|S_t| \geq \left\lceil \frac{n(G')}{3} \right\rceil$ $= 2l$, therefore $\gamma_t(G') = \gamma_t(G) - 2$. If r is even, then the set S_2 introduced in Lemma 10 is a total dominating set for G', so $\gamma(G') \leq 2l + 1$. If $\gamma(G') = 2l$ and S' is a total dominating set for G' such that $|S'| = 2l$, then for each vertex $v_i \in S'$, $|N_p[v_i]| = 3$. So $\{v_r, v_{r+1}, v_2, v_{2r-1}\} \cap S' = \varnothing$, this implies that $\{v_{r-1}, v_{r-2}, v_{r+2}, v_{r+3}\} \subset S'$. So $\{v_3, v_4, v_{r+4}\} \cap S' = \varnothing$ and $\{v_5, v_6\} \subset S'$. Since r is even we can assume $r = 6l' + 4$. Therefore the vertices v_{r-4}, v_{r-5}, v_{r-6}, v_{2r-3}, v_{2r-4} and v_{2r-5} must be dominated by two adjacent vertices of S', and this is impossible. So $\gamma_t(G') = 2l + 1 = \gamma(G)$.

Theorem 13. $\gamma_c(G_0) = m(r-1)$.

Proof. It is an immediate consequence by Lemmas 8 and 11.

Theorem 14. $i(G_0) = \gamma(G_0)$.

Proof. Since the set S_0 in Theorems 5, 6 and 7 is an independent dominating set for G_0, so $i(G_0) = \gamma(G_0)$.

Theorem 15. *If* $r \equiv 0 \pmod 3$, *then* $\gamma_t(G_0) = 2m\left\lfloor \frac{r}{3} \right\rfloor$.

Proof. The set $S_0 = \bigcup_{i=1}^{m} S_i$ with $S_i = \{v_{i2}, v_{i(r+2)}, v_{i5}, v_{i(r+5)}, \ldots, v_{i(r-1)}, v_{i(2r-1)}\}$ is a total dominating set for G_0, so $\gamma_t(G_0) \leq |S_0| = 2m\left\lfloor \frac{r}{3} \right\rfloor$. On the other hand by Lemma 12, we have $\gamma_t(G_i') = 2\left\lfloor \frac{r}{3} \right\rfloor$ for each $1 \leq i \leq m$. Therefore $\gamma_t(G_0) = 2m\left\lfloor \frac{r}{3} \right\rfloor$.

Theorem 16. *If* $r \equiv 2 \pmod 3$, *then* $\gamma_t(G_0) = 2m\left\lceil \frac{r}{3} \right\rceil$.

Proof. A same argument described in Theorem 15 can be used in this theorem.

Theorem 17. *If* $r \equiv 1 \pmod 3$, *then*

$$\gamma_t(G_0) = \begin{cases} m\left(2\left\lfloor \frac{r}{3} \right\rfloor + 1\right) & r \text{ is even} \\ 2m\left\lceil \frac{r}{3} \right\rceil - 2\left\lfloor \frac{m}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. First we suppose r is even. The set $S_0 = \bigcup_{i=1}^{m} S_i$ with

$$S_i = \{v_{i4}, v_{i5}, v_{i10}, v_{i11}, \ldots, v_{i(r-12)}, v_{i(r-11)}, v_{i(r-6)}, v_{i(r-5)}, v_{i(r-4)}, v_{i(r+1)}, v_{i(r+2)}, v_{i(r+7)}, v_{i(r+8)}, \ldots, v_{i(2r-9)}, v_{i(2r-8)}, v_{i(2r-2)}, v_{i(2r-1)}\}$$

is a total dominating set for G_0, so $\gamma_t(G_0) \le |S_0| = m\left(2\left\lfloor \frac{r}{3} \right\rfloor + 1\right)$. If $\gamma_t(G_0) < m\left(2\left\lfloor \frac{r}{3} \right\rfloor + 1\right)$, then there is $i \in \{1, 2, \ldots, m\}$ such that $\gamma_t(G_i') < 2\left\lfloor \frac{r}{3} \right\rfloor + 1$ and this contradicts Lemma 12.

Next, we suppose r is odd. We consider

$$S_i' = \{v_{i1}, v_{i2}, v_{i9}, v_{i10}, v_{i15}, v_{i16}, \ldots, v_{i(r-4)}, v_{i(r-3)}, v_{i(r+4)}, v_{i(r+5)}, v_{i(r+6)}, v_{i(r+7)}, v_{i(r+12)}, v_{i(r+13)}, v_{i(r+17)}, v_{i(r+18)}, \ldots, v_{i(2r-1)}, v_{i(2r)}\}$$

and

$$S_i'' = \{v_{i5}, v_{i6}, v_{i11}, v_{i12}, \ldots, v_{i(r-2)}, v_{i(r-1)}, v_{i(r+2)}, v_{i(r+3)}, v_{i(r+8)}, v_{i(r+9)}, \ldots, v_{i(2r-5)}, v_{i(2r-4)}\}.$$

If m is even, then the set $S_0 = S_1' \cup S_2'' \cup S_3' \cup S_4'' \cup \cdots \cup S_{m-1}' \cup S_m''$ is a total dominating set for G_0. If m is odd number, then the set $S_0 = S_1' \cup S_2'' \cup S_3' \cup S_4'' \cup \cdots \cup S_{m-2}' \cup S_{m-1}'' \cup S_m'$ is a total dominating set for G_0. So $\gamma_t(G_0) \le |S_0'| = 2m\left\lceil \frac{r}{3} \right\rceil - 2\left\lfloor \frac{m}{2} \right\rfloor$. If $\gamma_t(G_0) < 2m\left\lceil \frac{r}{3} \right\rceil - 2\left\lfloor \frac{m}{2} \right\rfloor$, then there is $i \in \{1, 2, \ldots, m\}$ such that $\gamma_t(G_i') < 2\left\lceil \frac{r}{3} \right\rceil$ and this contradicts Lemma 12.

Problem. What are the domination numbers of the Hamiltonian 4-regular graphs?

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