Far East J. Theo. Stat. **20**(2) (2006), 241-256

JENSEN'S INEQUALITY FOR g-EXPECTATION

FAN SHENG-JUN

College of Sciences
CUMT, Xuzhou, Jiangsu 221008
P. R. China
e-mail: f_s_j@126.com

Abstract

In 2003, Jiang gave a sufficient condition under which Jensen's inequality of bivariate function for g-expectation holds under the situation that g does not depend on y, and in 2004, he gave a sufficient and necessary condition under which Jensen's inequality for g-expectation holds in general under the situation that g does not depend on y and is continuous in t. In this paper, after investigating some relationship between the generator g and the conditional g-expectation system, under the most elementary conditions with respect to g-expectation, a sufficient and necessary condition under which Jensen's inequality of bivariate function for g-expectation holds is obtained. At the same time, it is proved that under the condition g is continuous in t, if Jensen's inequality for g-expectation holds in general, then g must not depend on y.

1. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the σ-algebra generated by $(B_t)_{t\geq 0}$. We always assume that $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete.

2000 Mathematics Subject Classification: 60H10.

Keywords and phrases: backward stochastic differential equation, g-expectation, Jensen's inequality, comparison theorem, generator.

This work is supported by Science and Technology Foundation of CUMT (No. 200409) and National Natural Science Foundation of China (No. 10671205).

Received February 5, 2006

Let $T > 0$ be a given real number. In this paper, we always work in the space $(\Omega, \mathcal{F}_T, P)$, and only consider processes indexed by $t \in [0, T]$. For any positive integer d, let $|z|$ denote Euclidean norm of $z \in \mathbf{R}^d$. 1_A denotes the indicator of event A. $\mathbf{R}^+$ denotes the set of non negative real numbers.

For notational simplicity, we use $L_T^2 = L^2(\Omega, \mathcal{F}_T, P)$, $L_t^2 = L^2(\Omega, \mathcal{F}_T, P)$ when $t \in [0, T]$, and define the following usual space of processes:

$$\mathcal{H}_2 = \left\{\phi : \phi \text{ progressively measurable; } \mathbf{E}\left[\int_0^T |\phi(t)|^2 \mathrm{d}t\right] < +\infty\right\}.$$

Let us consider a function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ such that the process $(g(\omega, t, y, z))_{t\in[0,T]}$ is progressively measurable for each (y, z) in $\mathbf{R} \times \mathbf{R}^d$, and furthermore, g satisfies some of the following assumptions:

(A1) There exists a constant $\mu \geq 0$, such that, P-a.s., we have

$$\forall (t, y_i, z_i) \in [0, T] \times \mathbf{R}^{1+d} (i = 1, 2),$$

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|).$$

(A2) The process $(g(\omega, t, 0, 0))_{t\in[0,T]}$ belongs to $\mathcal{H}_2$.

(A3) P-a.s., we have $\forall (t, y) \in [0, T] \times \mathbf{R}$, $g(t, y, 0) = 0$.

(A4) P-a.s., we have $\forall (y, z) \in \mathbf{R}^{1+d}$, $t \to g(t, y, z)$ is right-continuous in $t \in [0, T)$ and left-continuous in T.

Remark 1.1. The assumption (A3) implies the assumption (A2).

It is now well known that under the assumptions (A1) and (A2), for any random variable ξ in L_T^2, the following backward stochastic differential equation (BSDE for short in the remaining of this paper):

$$y_u = \xi + \int_u^T g(s, y_s, z_s)\mathrm{d}s - \int_u^T z_s \cdot \mathrm{d}B_s, \; u \in [0, T], \tag{1}$$

has a unique adapted and square-integrable solution, say

$(y^{\xi}(t), z^{\xi}(t))_{t \in [0, T]}$, such that (y, z) is in the space $\mathcal{H}_2 \times \mathcal{H}_2$ (see [8] for details). The function g is called the *generator* of BSDE (1). In [10], under the situation that (A3) holds, $y^{\xi}(0)$ is called *g-expectation* of ξ, denoted by $\varepsilon_g[\xi]$, and $y^{\xi}(t)$ is called *conditional g-expectation* of ξ with respect to $\mathcal{F}_t$, denoted by $\varepsilon_g[\xi \mid \mathcal{F}_t]$.

For the convenience of readers, we list some basic properties of conditional g-expectation and BSDEs, which will be used in the following of this paper.

Proposition 1.1 (See [4, 10])**.** *Let the generator g satisfy* (A1) *and* (A3). *Then for each $t \in [0, T]$ and for any $(\xi, \eta) \in L_T^2 \times L_T^2$, we have*

(a) (*Monotonicity*) *If $\xi \geq \eta$, P-a.s., then $\varepsilon_g[\xi \mid \mathcal{F}_t] \geq \varepsilon_g[\eta \mid \mathcal{F}_t]$, P-a.s.*

(b) *For each constant $c \in \mathbf{R}$, we have $\varepsilon_g[c \mid \mathcal{F}_t] = c$, P-a.s.*

(c) *For each $A \in \mathcal{F}_t$, we have $\varepsilon_g[\xi 1_A + \eta 1_{A^C} \mid \mathcal{F}_t] = \varepsilon_g[\xi \mid \mathcal{F}_t] 1_A + \varepsilon_g[\eta \mid \mathcal{F}_t] 1_{A^C}$, P-a.s.*

(d) *There exists a universal constant K, such that $\mathbf{E} \mid \varepsilon_g[\xi \mid \mathcal{F}_t] - \varepsilon_g[\eta \mid \mathcal{F}_t] |^2 \leq K\mathbf{E} \mid \xi - \eta \mid^2$.*

The following Proposition 1.2 can be regarded as the greatest achievements of theory of BSDEs, readers can see the proof in [5, 9].

Proposition 1.2 (Comparison theorem)**.** *Let both g and g' satisfy* (A1) *and* (A2), *let $(\xi, \xi') \in L_T^2 \times L_T^2$. Moreover, let $(y_u, z_u)_{u \in [0, T]}$ and $(y'_u, z'_u)_{u \in [0, T]}$, respectively, be the unique solutions of BSDE* (1) *and the following BSDE*

$$y_u = \xi' + \int_u^T g'(s, y_s, z_s) \mathrm{d}s - \int_u^T z_s \cdot \mathrm{d}B_s, \; u \in [0, T].$$

If P-a.s., $\xi \geq \xi'$ and P-a.s., for each $s \in [0, T]$, $g(s, y'_s, z'_s) \geq g'(s, y'_s, z'_s)$, then for each $t \in [0, T]$, we have

$$P\text{-a.s.}, \; y_t \geq y'_t.$$

The following Proposition 1.3 is often called the *Representation theorem* of generator for BSDEs, readers can see the proof in [1].

Proposition 1.3 (Representation theorem)**.** *Let the assumptions* (A1), (A3) *and* (A4) *hold for the generator* g. *Then for each* $(t, y, z) \in [0, T) \times \mathbf{R}^{1+d}$, *we have*

$$L^2 - \lim_{n\to\infty} n\{\varepsilon_g[y + z \cdot (B_{t+1/n} - B_t) | \mathcal{F}_t] - y\} = g(t, y, z).$$

Since the notion of g-expectation was introduced, many properties of g-expectation have been studied in [1, 4, 10]. Some properties of classical mathematical expectation are preserved (monotonicity for instance), and some important results on Jensen's inequality for g-expectation were obtained in [1-3, 6-7]. The following Propositions 1.4 and 1.5 come from [7] and [6], respectively.

Proposition 1.4 (See [7])**.** *Let* g *satisfy* (A1), (A3) *and* (A4). *If* g *does not depend on* y, *then the following two conditions are equivalent*:

(1) g *is a super-homogeneous generator in* z, *i.e.*, P-*a.s.*, *we have*

$$\forall (t, z, \lambda) \in [0, T] \times \mathbf{R}^{d+1}, \quad g(t, \lambda z) \geq \lambda g(t, z).$$

(2) *Jensen's inequality for* g-*expectation on convex function holds in general*, *i.e.*, *for each convex function* $\varphi(x) : \mathbf{R} \to \mathbf{R}$ *and each* $\xi \in L_T^2$, *if* $\varphi(\xi) \in L_T^2$, *then for each* $t \in [0, T]$, *we have*, P-*a.s.*,

$$\varepsilon_g[\varphi(\xi) | \mathcal{F}_t] \geq \varphi[\varepsilon_g[\xi | \mathcal{F}_t]].$$

Remark 1.2. Similarly, by replacing "$\geq$" with "$\leq$" in above two inequality, we can give the definitions that sub-homogeneous generator in z and Jensen's inequality for g-expectation on concave function holds in general. Similar to Proposition 1.4, we can prove that under the same assumptions as Proposition 1.4, the above two conditions are also equivalent.

Proposition 1.5 (See [6])**.** *Suppose* g *satisfies* (A1), (A3) *and* g *does not depend on* y, *moreover let* g *be a sub-linear generator in* z, *i.e.*, g *also satisfies the following two conditions*:

(1) *P-a.s.*, $\forall(t, z, \lambda) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^+$, $g(t, \lambda z) = \lambda g(t, z)$; (*positive-homogeneity*).

(2) *P-a.s.*, $\forall(t, z_1, z_2) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^d$, $g(t, z_1 + z_2) \leq g(t, z_1) + g(t, z_2)$, (*sub-additive*).

Then Jensen's inequality of bivariate function for g-expectation holds, i.e., for any non negative variables $(\xi, \eta) \in L_T^2 \times L_T^2$, *and any semi-negative definite bivariate function* $f(x, y) : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$, *if* $f(\xi, \eta) \in L_T^2$, *then for each* $t \in [0, T]$, *we have, P-a.s.*,

$$\varepsilon_g[f(\xi, \eta) | \mathcal{F}_t] \leq f(\varepsilon_g[\xi | \mathcal{F}_t], \varepsilon_g[\eta | \mathcal{F}_t]).$$

Remark 1.3. A bivariate function $f(x, y) : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ is semi-negative definite means that it satisfies the following two conditions:

(1) $f(x, y) \in C^2(\mathbf{R}^+ \times \mathbf{R}^+)$ and for each $(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+$, $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \geq 0$;

(2) for each $(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+$, the Hessian-matrix $A(x, y) = \begin{pmatrix} \partial^2 f / \partial x^2 & \partial^2 f / \partial x \partial y \\ \partial^2 f / \partial y \partial x & \partial^2 f / \partial y^2 \end{pmatrix}$ is semi-negative definite.

2. Main Results

In this section, we always assume that the generator g satisfies (A1) and (A3). Similar to the definitions in [7] and [6], we put forward the following definition.

Definition 2.1. Let the generator g satisfy (A1) and (A3). Then we say g is a *super-homogeneous* (resp., *sub-homogeneous*, *homogeneous*) *generator* in (y, z) if g also satisfies the following condition: P-a.s., we have

$$\forall(t, y, z) \in [0, T] \times \mathbf{R}^{1+d}, \forall\lambda \in \mathbf{R}, g(t, \lambda y, \lambda z) \geq \lambda g(t, y, z) \text{(resp., } \leq, =).$$

We say g is a *positive-homogeneous generator* in (y, z) if g also satisfies the following condition: P-a.s., we have

$$\forall(t, y, z) \in [0, T] \times \mathbf{R}^{1+d}, \forall\lambda \in \mathbf{R}^+, g(t, \lambda y, \lambda z) = \lambda g(t, y, z).$$

We say g is a *super-additive* (resp., *sub-additive*, *additive*) *generator* in (y, z) if g also satisfies the following condition: P-a.s., $\forall(t, y_i, z_i) \in [0, T] \times \mathbf{R}^{1+d} (i = 1, 2)$,

$$g(t, y_1 + y_2, z_1 + z_2) \geq g(t, y_1, z_1) + g(t, y_2, z_2) \text{ (resp., } \leq, =).$$

Remark 2.1. If g is a super-additive generator in (y, z), then P-a.s., for $(t, y_1, y_2, z) \in [0, T] \times \mathbf{R}^{1+1+d}$, we have

$$g(t, y_1, z) - g(t, y_2, z) \geq g(t, y_1 - y_2, 0) = 0.$$

Thus g must not depend on y. Hence g is a super-additive (resp., sub-additive) generator in (y, z) if and only if g does not depend on y and is a super-additive (resp., sub-additive) generator in z.

Definition 2.2. Let the generator g satisfy (A1) and (A3). Then we say the conditional g-expectation system is *super-homogeneous* (resp., *sub-homogeneous*, *homogeneous*) if $\varepsilon_g[\cdot | \mathcal{F}_t]$ satisfies the following condition:

$$\forall(\xi, t) \in L_T^2 \times [0, T], \forall\lambda \in \mathbf{R}, P\text{-a.s., } \varepsilon_g[\lambda\xi | \mathcal{F}_t] \geq \lambda\varepsilon_g[\xi | \mathcal{F}_t] \text{ (resp., } \leq, =).$$

We say the conditional g-expectation system is *positive-homogeneous* if $\varepsilon_g[\cdot | \mathcal{F}_t]$ satisfies the following condition:

$$\forall(\xi, t) \in L_T^2 \times [0, T], \forall\lambda \in \mathbf{R}^+, P\text{-a.s., } \varepsilon_g[\lambda\xi | \mathcal{F}_t] = \lambda\varepsilon_g[\xi | \mathcal{F}_t]. \quad (2)$$

We say the conditional g-expectation system is *super-additive* (resp., *sub-additive*, *additive*) if $\varepsilon_g[\cdot | \mathcal{F}_t]$ satisfies the following condition:

$$\forall(\xi_i, t) \in L_T^2 \times [0, T] (i = 1, 2), P\text{-a.s.,}$$

$$\varepsilon_g[\xi_1 + \xi_2 | \mathcal{F}_t] \geq \varepsilon_g[\xi_1 | \mathcal{F}_t] + \varepsilon_g[\xi_2 | \mathcal{F}_t] \text{(resp., } \leq, =).$$

Remark 2.2. It is easy to prove that (2) is equivalent to the following (3):

$$\forall(\xi, t) \in L_T^2 \times [0, T], \forall\lambda \in \mathbf{R}^+, P\text{-a.s., } \varepsilon_g[\lambda\xi | \mathcal{F}_t] \geq \lambda\varepsilon_g[\xi | \mathcal{F}_t] \text{ (resp.,} \leq), \quad (3)$$

considering that

$$\varepsilon_g[\lambda\xi \mid \mathcal{F}_t] \geq \lambda\varepsilon_g\left[\frac{1}{\lambda}\lambda\xi \mid \mathcal{F}_t\right] \geq \lambda\frac{1}{\lambda}\varepsilon_g[\lambda\xi \mid \mathcal{F}_t] = \varepsilon_g[\lambda\xi \mid \mathcal{F}_t] \text{ (resp., } \leq).$$

The following Theorems 2.1-2.2 investigate some relationship between the generator g of BSDE and the conditional g-expectation system.

Theorem 2.1. *Let the generator g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent*:

(1) *g is a super-homogeneous* (*resp*., *sub-homogeneous*, *homogeneous*, *positive-homogeneous*) *generator in* (y, z).

(2) *the conditional g-expectation system is super-homogeneous* (*resp*., *sub-homogeneous*, *homogeneous*, *positive-homogeneous*).

Theorem 2.2. *Let the generator g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent*:

(1) *g does not depend on y and is a super-additive* (*resp*., *sub-additive*, *additive*) *generator in z*.

(2) *the conditional g-expectation system is super-additive* (*resp*., *sub-additive*, *additive*).

Remark 2.3. (1) $\Rightarrow$ (2) in Theorems 2.1-2.2 does not need the condition (A4). (1) $\Rightarrow$ (2) in Theorem 2.2 has been proved in [6].

The following Theorems 2.3-2.4 are the main results of this paper.

Theorem 2.3. *Let the generator g satisfy* (A1), (A3) *and* (A4). *If Jensen's inequality for g-expectation on convex* (*resp*., *concave*) *function holds in general*, *then g must not depend on y*.

Theorem 2.4. *Let the generator g satisfy* (A1) *and* (A3). *Then the following conditions are equivalent*:

(1) *Jensen's inequality of bivariate function for g-expectation holds*.

(2) *For any non negative variables* $(\xi_1, \xi_2) \in L_T^2 \times L_T^2$ *and any* $(t, \lambda_i) \in [0, T] \times \mathbf{R}^+ (i = 1, 2, 3)$,

$$\varepsilon_g[\lambda_1\xi_1 + \lambda_2\xi_2 - \lambda_3 \mid \mathcal{F}_t] \leq \lambda_1\varepsilon_g[\xi_1 \mid \mathcal{F}_t] + \lambda_2\varepsilon_g[\xi_2 \mid \mathcal{F}_t] - \lambda_3, \ P\text{-}a.s.$$

(3) *For any non negative variables* $(\xi, \eta) \in L_T^2 \times L_T^2$ *and any* $(t, \lambda) \in [0, T] \times \mathbf{R}^+$,

$$\begin{cases} \text{(i)} & \varepsilon_g[\xi - \lambda \,|\, \mathcal{F}_t] \le \varepsilon_g[\xi \,|\, \mathcal{F}_t] - \lambda,\ P\text{-}a.s., \\ \text{(ii)} & \varepsilon_g[\lambda\xi \,|\, \mathcal{F}_t] = \lambda\varepsilon_g[\xi \,|\, \mathcal{F}_t],\ P\text{-}a.s., \\ \text{(iii)} & \varepsilon_g[\xi + \eta \,|\, \mathcal{F}_t] \le \varepsilon_g[\xi \,|\, \mathcal{F}_t] + \varepsilon_g[\eta \,|\, F_t],\ P\text{-}a.s. \end{cases} \tag{4}$$

3. The Proof of Main Results

Lemma 3.1. *Let the generator g satisfy* (A1), (A3) *and* (A4). *Then for each* $\lambda \in \mathbf{R}$, *the following two conditions are equivalent*:

(1) P-*a.s.*, $\forall (t, y, z) \in [0, T] \times \mathbf{R}^{1+d}$, $g(t, \lambda y, \lambda z) \ge \lambda g(t, y, z)$ (*resp.*, $\le, =$).

(2) $\forall (\xi, t) \in L_T^2 \times [0, T]$, P-*a.s.*, $\varepsilon_g[\lambda\xi \,|\, \mathcal{F}_t] \ge \lambda\varepsilon_g[\xi \,|\, \mathcal{F}_t]$ (*resp.*, $\le, =$).

Proof. Only need to prove the case "$\ge$", similarly we can prove the remaining.

It is obvious when $\lambda = 0$ considering (A3) and (b) of Proposition 1.1. In the following we prove the case $\lambda \ne 0$. The method of proof comes from [1] and [7].

(1) $\Rightarrow$ (2) For the given $\lambda \in \mathbf{R}$ and any $\xi \in L_T^2$, we can suppose that $(y_u, z_u)_{u \in [0, T]}$ and $(y'_u, z'_u)_{u \in [t, T]}$, respectively, be the unique solutions of BSDE (1) and the following BSDE

$$y_u = \lambda\xi + \int_u^T g(s, y_s, z_s)\mathrm{d}s - \int_u^T z_s \cdot \mathrm{d}B_s,\ u \in [0, T]. \tag{5}$$

Then we have

$$\begin{aligned} \lambda y_u &= \lambda\xi + \int_u^T \lambda g(s, y_s, z_s)\mathrm{d}s - \int_u^T \lambda z_s \cdot \mathrm{d}B_s \\ &= \lambda\xi + \int_u^T \tilde{g}(s, \lambda y_s, \lambda z_s)\mathrm{d}s - \int_u^T \lambda z_s \cdot \mathrm{d}B_s,\ u \in [0, T], \end{aligned} \tag{6}$$

where

$$\tilde{g}(s, y, z) := \lambda g(s, y/\lambda, z/\lambda),\ \forall (s, y, z) \in [0, T] \times \mathbf{R}^{1+d}.$$

We can prove that $\widetilde{g}$ satisfies the assumptions (A1) and (A2) when $s \in [0, T]$. Thus from the existence and uniqueness of solution of the following BSDE (7), by (6), we can conclude that $(\lambda y_u, \lambda z_u)_{u \in [0, T]}$ is just the square-integrable adapted solution $(\widetilde{y}_u, \widetilde{z}_u)_{u \in [0, T]}$ of BSDE (7):

$$y_u = \lambda\xi + \int_u^T \widetilde{g}(s, y_s, z_s)\mathrm{d}s - \int_u^T z_s \cdot \mathrm{d}B_s, \; u \in [0, T]. \tag{7}$$

By the condition (1) and the definition of $\widetilde{g}$, we can get that, P-a.s.,

$$g(s, \widetilde{y}_s, \widetilde{z}_s) = g(s, \lambda y_s, \lambda z_s) \geq \lambda g(s, y_s, z_s) = \widetilde{g}(s, \lambda y_s, \lambda z_s)$$

$$= \widetilde{g}(s, \widetilde{y}_s, \widetilde{z}_s), \; \forall s \in [0, T].$$

Thus by Comparison theorem, comparing BSDE (5) with BSDE (7), we can conclude that, for each $t \in [0, T]$,

$$P\text{-a.s.}, \; \varepsilon_g[\lambda\xi \mid \mathcal{F}_t] = y'_t \geq \widetilde{y}_t = \lambda y_t = \lambda\varepsilon_g[\xi \mid \mathcal{F}_t].$$

(2) $\Rightarrow$ (1) For the given $\lambda \in \mathbf{R}$ and each $(t, y, z) \in [0, T) \times R^{1+d}$, let us choose a large enough n such that $t + 1/n \leq T$, and choose $\xi = y + z \cdot (B_{t+1/n} - B_t)$, which obviously $\xi \in L_T^2$. By the condition (2), we have, P-a.s.,

$$\varepsilon_g[\lambda\xi \mid \mathcal{F}_t] \geq \lambda\varepsilon_g[\xi \mid \mathcal{F}_t].$$

So we have, P-a.s.,

$$\varepsilon_g\{\lambda[y + z \cdot (B_{t+1/n} - B_t)] \mid \mathcal{F}_t\} - \lambda y \geq \lambda\{\varepsilon_g[y + z \cdot (B_{t+1/n} - B_t) \mid \mathcal{F}_t] - y\}. \tag{8}$$

Due to Proposition 1.3, we know there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} n_k\{\varepsilon_g[\lambda y + \lambda z \cdot (B_{t+1/n_k} - B_t) \mid \mathcal{F}_t] - \lambda y\} = g(t, \lambda y, \lambda z), \; P\text{-a.s.},$$

$$\lim_{k \to \infty} n_k\{\varepsilon_g[y + z \cdot (B_{t+1/n_k} - B_t) \mid \mathcal{F}_t] - y\} = g(t, y, z), \; P\text{-a.s.}$$

Coming back to (8), we have, P-a.s.,

$$g(t, \lambda y, \lambda z) \geq \lambda g(t, y, z).$$

By (A4), we know that for each (y, z), the process $t \to g(t, y, z)$ is left-continuous in T. Hence we have, P-a.s.,

$$\begin{aligned} g(T, \lambda y, \lambda z) &= \lim_{\varepsilon \to 0} g(T - \varepsilon, \lambda y, \lambda z) \\ &\geq \lim_{\varepsilon \to 0} \lambda g(T - \varepsilon, y, z) = \lambda g(T, y, z). \end{aligned}$$

Thus by (A1) and (A4), we know that, P-a.s.,

$$\forall (t, y, z) \in [0, T] \times \mathbf{R}^{1+d}, \quad g(t, \lambda y, \lambda z) \geq \lambda g(t, y, z).$$

The proof is complete.

Remark 3.1. (1) $\Rightarrow$ (2) in Lemma 3.1 does not need the condition (A4).

Proof of Theorem 2.1. By Lemma 3.1, we immediately obtain Theorem 2.1.

Proof of Theorem 2.2. Only need to prove the case "sub-additive" of (2) $\Rightarrow$ (1). The method of proof comes from [1] and [7].

(2) $\Rightarrow$ (1) For the given $(t, y_i, z_i) \in [0, T) \times \mathbf{R}^{1+d}$ $(i = 1, 2)$, let us choose a large enough n such that $t + 1/n \leq T$, and choose $\xi_i = y_i + z_i \cdot (B_{t+1/n} - B_t)$ $(i = 1, 2)$, which obviously $\xi_i \in L_T^2$. Then by the condition (2), we can get that, P-a.s.,

$$\varepsilon_g[\xi_1 + \xi_2 \mid \mathcal{F}_t] \leq \varepsilon_g[\xi_1 \mid \mathcal{F}_t] + \varepsilon_g[\xi_2 \mid \mathcal{F}_t],$$

so we have, P-a.s.,

$$\begin{aligned} &\varepsilon_g\left\{\sum_{i=1}^{2} [y_i + z_i \cdot (B_{t+1/n} - B_t)] \mid \mathcal{F}_t\right\} - \sum_{i=1}^{2} y_i \\ &\leq \sum_{i=1}^{2} \{\varepsilon_g[y_i + z_i \cdot (B_{t+1/n} - B_t) \mid \mathcal{F}_t] - y_i\}. \end{aligned}$$

Thus similar to the proof of Lemma 3.1, making use of Proposition 1.3, (A1) and (A4), we can deduce that P-a.s., for $(t, y_i, z_i) \in [0, T] \times \mathbf{R}^{1+d} (i = 1, 2)$, we have

$$g(t, y_1 + y_2, z_1 + z_2) \leq g(t, y_1, z_1) + g(t, y_2, z_2).$$

Hence it follows that (1) holds by Remark 2.1.

Proof of Theorem 2.3. Only need to prove the case "convex function", similarly we can prove the remaining part of this theorem.

In fact, since Jensen's inequality for g-expectation on convex function holds in general, we know that for each $\xi \in L_T^2$ and convex function $\varphi(x)$, if $\varphi(\xi) \in L_T^2$, then for each $t \in [0, T]$, we have, P-a.s.,

$$\varepsilon_g[\varphi(\xi) | \mathcal{F}_t] \geq \varphi(\varepsilon_g[\xi | \mathcal{F}_t]).$$

Thus for any given $(t, y, z) \in [0, T) \times \mathbf{R}^{1+d}$, let us pick a large enough n such that $t + 1/n \leq T$, and choose $\xi = y + z \cdot (B_{t+1/n} - B_t)$, and for any given $a \in \mathbf{R}$, let $\varphi(x) = x + a$, which obviously $\xi \in L_T^2$, $\varphi(x)$ is a convex function and $\varphi(\xi) \in L_T^2$. Then we can get that, P-a.s.,

$$\varepsilon_g[\xi + a | \mathcal{F}_t] = \varepsilon_g[\varphi(\xi) | \mathcal{F}_t] \geq \varphi(\varepsilon_g[\xi | \mathcal{F}_t]) = \varepsilon_g[\xi | \mathcal{F}_t] + a.$$

So we have, P-a.s.,

$$\begin{aligned} &\varepsilon_g[y + z \cdot (B_{t+1/n} - B_t) + a | \mathcal{F}_t] - (a + y) \\ &\geq \varepsilon_g[y + z \cdot (B_{t+1/n} - B_t) | \mathcal{F}_t] - y. \end{aligned}$$

Thus similar to the proof of Lemma 3.1, making use of Proposition 1.3, (A1) and (A4), we can deduce that P-a.s.,

$$\forall (t, y, z, a) \in [0, T] \times \mathbf{R}^{1+d+1}, \quad g(t, a + y, z) \geq g(t, y, z).$$

Given (t, z), let both y and a change arbitrarily. Then we can conclude that the generator g must not depend on y.

Remark 3.2. By the proof of this theorem, we know that when (A1), (A3) and (A4) hold for the generator g, if for each $t \in [0, T]$ and $(\xi, a) \in L_T^2 \times \mathbf{R}$, $\varepsilon_g[\xi + a \mid \mathcal{F}_t] \geq \varepsilon_g[\xi \mid \mathcal{F}_t] + a$ $(\leq)$, P-a.s., then g must not depend on y.

Combining Proposition 1.4 and Theorem 2.3, we can obtain the following corollary:

Corollary 3.1. *Let g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent*:

(1) *g does not depend on y and is a super-homogeneous* (*sub-homogeneous*) *generator in z.*

(2) *Jensen's inequality for g-expectation on convex* (*concave*) *function holds in general.*

Proof of Theorem 2.4. We can easily prove that (1) $\Rightarrow$ (2) by choosing the semi-negative bivariate functions $f(x, y) = \lambda_1 x + \lambda_2 y - \lambda_3$. (2) $\Rightarrow$ (3) is clear by Remark 2.2. In the following, we prove that (3) $\Rightarrow$ (1). The approach of the following proof partly derives from [6]. Firstly, we prove the following Proposition 3.1:

Proposition 3.1. *Let g satisfy* (A1) *and* (A3). *If the condition* (3) *in Theorem* 2.4 *holds, then for each $t \in [0, T]$ and $\eta \in L_t^2$ and for each non negative variable $\xi \in L_T^2$,*

$$\varepsilon_g[\xi + \eta \mid \mathcal{F}_t] \leq \varepsilon_g[\xi \mid \mathcal{F}_t] + \eta$$

and for each $t \in [0, T]$ and any non negative variables $\xi \in L_T^2$ and $\eta \in L_t^2$,

$$\varepsilon_g[\eta\xi \mid \mathcal{F}_t] = \eta\varepsilon_g[\xi \mid \mathcal{F}_t].$$

Proof. Given $t \in [0, T]$, let $\{A_i\}_{i=1}^m$ be an $\mathcal{F}_t$-measurable partition of Ω (i.e., A_i are disjoint, $\mathcal{F}_t$-measurable and $\bigcup A_i = \Omega$) and let $\lambda_i \in \mathbf{R}$ $(i = 1, 2, \ldots, m)$. From (c) of Proposition 1.1, (i), (iii) and (b) of Proposition 1.1, we can deduce that for each non negative variable $\xi \in L_T^2$,

$$\varepsilon_g\left[\xi + \sum_{i=1}^{m} \lambda_i 1_{A_i} \mid \mathcal{F}_t\right] = \varepsilon_g\left[\sum_{i=1}^{m} 1_{A_i}(\xi + \lambda_i) \mid \mathcal{F}_t\right]$$

$$= \sum_{i=1}^{m} 1_{A_i} \varepsilon_g[\xi + \lambda_i \mid \mathcal{F}_t]$$

$$\leq \sum_{i=1}^{m} 1_{A_i}[\varepsilon_g[\xi \mid \mathcal{F}_t] + \lambda_i]$$

$$= \varepsilon_g[\xi \mid \mathcal{F}_t] + \sum_{i=1}^{m} \lambda_i 1_{A_i}.$$

Moreover, if $\lambda_i \geq 0$ $(i = 1, 2, \ldots, m)$, then by (c) of Proposition 1.1 and (ii) we also have

$$\varepsilon_g\left[\sum_{i=1}^{m} \lambda_i 1_{A_i} \xi \mid \mathcal{F}_t\right] = \sum_{i=1}^{m} 1_{A_i} \varepsilon_g[\lambda_i \xi \mid \mathcal{F}_t] = \sum_{i=1}^{m} 1_{A_i} \lambda_i \varepsilon_g[\xi \mid \mathcal{F}_t].$$

In other words, for any non negative variable $\xi \in L_T^2$ and any simple function $\eta \in L_t^2$,

$$\varepsilon_g[\xi + \eta \mid \mathcal{F}_t] \leq \varepsilon_g[\xi \mid \mathcal{F}_t] + \eta$$

and for any non negative variable $\xi \in L_T^2$ and any non negative simple function $\eta \in L_t^2$,

$$\varepsilon_g[\eta\xi \mid \mathcal{F}_t] = \eta\varepsilon_g[\xi \mid \mathcal{F}_t].$$

Thus from the continuity of $\varepsilon_g[\cdot \mid \mathcal{F}_t]$ in L^2 given by (d) of Proposition 1.1, it follows that Proposition 3.1 is true.

Now, let us come back the proof of Theorem 2.4. Given a semi-negative bivariate function $f(x, y) \in C^2(\mathbf{R}^+ \times \mathbf{R}^+)$, let non negative variables $(\xi, \eta) \in L_T^2 \times L_T^2$ and $f(\xi, \eta) \in L_T^2$. For each $t \in [0, T]$, let $(x_0, y_0) = (\varepsilon_g[\xi \mid \mathcal{F}_t], \varepsilon_g[\eta \mid \mathcal{F}_t])$, by (a) and (b) of Proposition 1.1, we know that $(x_0, y_0) \in \mathbf{R}^+ \times \mathbf{R}^+$. For the given $t \in [0, T]$ and any $n \in \mathbf{N}$, we define

$$\Omega_{t,n} = \left\{ \left| \frac{\partial f}{\partial x}(x_0, y_0) \right| + \left| \frac{\partial f}{\partial y}(x_0, y_0) \right| + | f(x_0, y_0) | \le n \right\}.$$

From [6], we know that, P-a.s.,

$$\varepsilon_g[1_{\Omega_{t,n}} f(\xi, \eta) | \mathcal{F}_t] \le \varepsilon_g\Big[1_{\Omega_{t,n}} f(x_0, y_0) + 1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0)(\xi - x_0) + 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0)(\eta - y_0) | \mathcal{F}_t\Big]. \tag{9}$$

By the definition of $\Omega_{t,n}$, since $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$, $f(x_0, y_0)$ are all $\mathcal{F}_t$ -measurable, we can conclude that

$$1_{\Omega_{t,n}} f(x_0, y_0) - 1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0) x_0 - 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0) y_0 \in L_t^2.$$

Considering that ξ, η, $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ are all non negative, it follows that by Proposition 3.1 and (iii), P-a.s.,

$$\begin{aligned}
&\varepsilon_g\Big[1_{\Omega_{t,n}} f(x_0, y_0) + 1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0)(\xi - x_0) + 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0)(\eta - y_0) | \mathcal{F}_t\Big] \\
&\le 1_{\Omega_{t,n}} f(x_0, y_0) - 1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0) x_0 - 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0) y_0 \\
&\quad + \varepsilon_g\Big[1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0)\xi + 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0)\eta | \mathcal{F}_t\Big] \\
&\le 1_{\Omega_{t,n}} f(x_0, y_0) - 1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0) x_0 - 1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0) y_0 \\
&\quad + \varepsilon_g\Big[1_{\Omega_{t,n}} \frac{\partial f}{\partial x}(x_0, y_0)\xi | \mathcal{F}_t\Big] + \varepsilon_g\Big[1_{\Omega_{t,n}} \frac{\partial f}{\partial y}(x_0, y_0)\eta | \mathcal{F}_t\Big] \\
&= 1_{\Omega_{t,n}} f(x_0, y_0).
\end{aligned} \tag{10}$$

Combining (9) with (10), we can conclude that, P-a.s.,

$$\varepsilon_g[1_{\Omega_{t,n}} f(\xi, \eta) | \mathcal{F}_t] \leq 1_{\Omega_{t,n}} f(x_0, y_0).$$

Then according to [6] again, we can know that, P-a.s.,

$$\varepsilon_g[f(\xi, \eta) | \mathcal{F}_t] \leq f(\varepsilon_g[\xi | \mathcal{F}_t], \varepsilon_g[\eta | \mathcal{F}_t]).$$

Because that $t \in [0, T]$ is any given, we know that Jensen's inequality of bivariate function for g-expectation holds.

Remark 3.3. Proposition 1.5 can be regarded as a corollary of Theorem 2.4. In fact, by Theorems 2.1-2.2, we can prove that when (A1), (A3) and (A4) hold for the generator g, g does not depend on y and is sub-linear generator in z, which are the conditions of Proposition 1.5, if and only if (4) holds for any variables $(\xi, \eta) \in L_T^2 \times L_T^2$ and any $(t, \lambda) \in [0, T] \times \mathbf{R}^+$.

Corollary 3.2. *Let the generator g satisfy* (A1) *and* (A3), *and g do not depend on y. Then the following conditions are equivalent*:

(1) *Jensen's inequality of bivariate function for g-expectation holds*.

(2) *For any non negative variables $(\xi, \eta) \in L_T^2 \times L_T^2$ and any $(t, \lambda) \in [0, T] \times \mathbf{R}^+$*,

$$\begin{cases} \varepsilon_g[\lambda\xi | \mathcal{F}_t] = \lambda\varepsilon_g[\xi | \mathcal{F}_t], \ P\text{-}a.s., \\ \varepsilon_g[\xi + \eta | \mathcal{F}_t] \leq \varepsilon_g[\xi | \mathcal{F}_t] + \varepsilon_g[\eta | \mathcal{F}_t], \ P\text{-}a.s. \end{cases}$$

Remark 3.4. Proposition 1.5 can also be regarded as a corollary of Corollary 3.2.

References

[1] P. Briand et al., A converse comparison theorem for BSDEs and related properties of g-expectation, Electron. Comm. Probab. 5 (2000), 101-117.

[2] Z. Chen, R. Kulperger and L. Jiang, Jensen's inequality for g-expectation: part 1, C. R. Acad. Sci. Paris. Sér. I337 (2003), 725-730.

[3] Z. Chen, R. Kulperger and L. Jiang, Jensen's inequality for g-expectation: part 2, C. R. Acad. Sci. Paris. Sér. I337 (2003), 797-800.

[4] F. Coquet et al., Filtration consistent nonlinear expectations and related g-expectation, Probab. Theory Related Fields 123 (2002), 1-27.

[5] N. El Karoui, S. G. Peng and M. C. Quenez, Backward stochastic differential equation in finance, Math. Finance 7(1) (1997), 1-71.

[6] L. Jiang, On Jensen's inequality of bivariate function for g-expectation, J. Shandong Univ. 38(5) (2003), 13-22 (in Chinese).

[7] L. Jiang and Z. Chen, On Jensen's inequality for g-expectation, Chinese Ann. Math. Ser. B 25(3) (2004), 401-412.

[8] E. Pardoux and S. G. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990), 55-61.

[9] S. G. Peng, A general dynamic programming principle and Hamilton-Bellman equation, Stochastics 38(2) (1992), 119-134.

[10] S. G. Peng, BSDE and related g-expectations, Backward Stochastic Differential Equations, N. El Karoui and L. Mazliak, eds., Addison Welsey Longman, Harlow, 1997, pp. 141-159.

■