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CONSTRUCTION OF PLANAR CMC 4-NOIDS OF GENUS $g = 0$

J. DORFMEISTER[*, †] and M. SCHUSTER[*, ‡]

[*]Universität Augsburg, Universitätsstrasse 14
D-86159 Augsburg, Germany

[†]TU Muenchen, Boltzmannstr.3
D-85747 Garching, Germany

Abstract

In this paper, we use the loop group method [10] to construct planar 4-noids of genus $g = 0$ with embedded ends. This is the first such construction using the loop group method.

1. Introduction

In this note, we want to provide a first step in the direction of constructing k-noids of genus $g = 0$ with embedded ends using a generalized Weierstraß representation, the so called DPW-method. Up to now, we are only able to construct planar 4-noids, but it is our hope, that some of the methods used and developed here, may be useful for an arbitrary number of ends or arbitrary 4-noids.

In some sense, this paper is a continuation of [11], which mainly deals with the construction of trinoids but also includes a number of more

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general results concerning the construction of k-noids. The paper [11] includes some ideas and methods which may help in the investigation of trinoids and more general of k-noids within the framework of the so-called DPW-method. Unfortunately, despite the efforts undertaken so far, general results on the construction or classification results, like those obtained in [15] by completely different methods, are not yet in the realm of consideration using the DPW-method. Nevertheless, it is already possible to construct quite a number of examples, and, by providing some method to construct 4-noids, we hope to make at least a small step towards a more comprehensive understanding of k-noids.

In [10], a method to construct all constant mean curvature surfaces from simply connected domains, except the sphere, is presented, using so called holomorphic (or also meromorphic) potentials as input data, which depend on a spectral parameter $\lambda \in \mathbb{S}^1$. This way a whole $\mathbb{S}^1$-family of a CMC-surface is built, the so-called associated family. Construction of a CMC-surface defined on a non-simply connected domain then is reduced to the construction of a CMC-surface defined on a simply-connected domain, which is invariant under a prescribed fundamental group. Actually, such an invariance can only be obtained for certain values of λ. (We will always normalize things so that $\lambda = 1$ will work.) First steps in understanding the interplay of holomorphic potentials and symmetries of the surfaces resulting from the invariance of the potential are described in [8] and [9]. If one wants to construct a CMC-immersion from a Riemann surface $\mathcal{M}$ to $\mathbb{R}^3$, then one can start from some potential η defined on the universal cover $\hat{\mathcal{M}}$ of $\mathcal{M}$, which is invariant under $\pi_1(\mathcal{M}) : \gamma^*\eta = \eta$ for all $\gamma \in \pi_1(\mathcal{M})$. Then the procedure outlined in [10] requires to solve the ordinary differential equation $\mathrm{d}C = C\eta$, to perform an Iwasawa splitting, $C = FV_+$, and to apply the Sym-Bobenko-formula to obtain the CMC-immersions φ_λ associated with η. Clearly, since η is invariant under $\pi_1(\mathcal{M})$, one obtains $C(\gamma \cdot z, \lambda) = \varrho(\gamma, \lambda)C(z, \lambda)$ for all $z \in \widetilde{\mathcal{M}}$, $\lambda \in \mathbb{S}^1$, $\gamma \in \pi_1(\mathcal{M})$. For $\varphi_{\lambda=1}$ to be invariant under $\pi_1(\mathcal{M})$, it is necessary and sufficient that

(a) $\varrho(\gamma, \lambda)$ is unitary for all $\gamma \in \pi_1(\mathcal{M})$,

(b) $\varrho(\gamma, \lambda = 1) = \pm I$ for all $\gamma \in \pi_1(\mathcal{M})$, and

(c) $\partial_\lambda \varrho(\gamma, \lambda)|_{\lambda=1} = 0$ for all $\gamma \in \pi_1(\mathcal{M})$.

The simplest non-simply connected domain is $\mathbb{C}\setminus\{0\}$, which gives CMC-ends and in particular CMC-cylinders. In [18], many examples of CMC-cylinders are presented. The "next easier" topological space perhaps are trinoids and more generally k-noids of genus 0, that is immersions from $\mathcal{M} = \mathbb{S}^2\setminus\{z_1, ..., z_k\} \cong \mathbb{C}\setminus\{z_1, ..., z_{k-1}\}$ to $\mathbb{R}^3$. In this case, the fundamental group is the free group of $k - 1$ generators.

As mentioned above, for a CMC-immersion $\varphi : \hat{\mathcal{M}} \to \mathbb{R}^3$ derived from an invariant potential to descend to $\mathcal{M}$ for $\lambda = 1$, one needs certainly that all monodromy matrices are unitary. As outlined in [12, 11], it suffices to find some η such that all the monodromy matrices $\varrho(\gamma, \lambda)$ are simultaneously r-unitarizable (for an explanation of notation see Section 2). It turns out that there is a fairly simple criterion for two matrices to be simultaneously unitarizable (see, e.g., [12] and the references therein). For three matrices things are much more complicated (again see [12]). However, for k matrices $\varrho_1, ..., \varrho_k$ the situation is easy again: it suffices to show that three consecutive matrices ϱ_j, ϱ_{j+1}, ϱ_{j+2}, $j = 0, ..., k - 2$ are simultaneously unitarizable.

From this point of view, it is most important to discuss CMC-immersions from Riemann surfaces $\mathcal{M}$, for which $\pi_1(\mathcal{M})$ is generated by three elements. Perhaps the simplest case is $\mathcal{M} = \mathbb{S}^2\setminus\{0, 1, \infty, a\}$, where $a \notin \{0, 1, \infty\}$. This is the situation to which this paper is devoted. Actually, we present a large family of CMC 4-noids. We do not attempt, however, to construct all CMC 4-noids in this paper.

At this point we would like to point out that by completely different methods, [16], planar CMC-k-noids have been investigated. However, the beautiful classification of [16] refers to a class of CMC-immersions, which is much more restricted from the class of CMC-immersions considered in this paper.

In the following, we will work with off-diagonal potentials $\eta = \begin{pmatrix} 0 & \nu \\ \tau & 0 \end{pmatrix}$ (see Section 3 for details). We find it convenient (and in fact it is one of the main concepts of this paper) to translate the first order matrix equation $\mathrm{d}C = C\eta$ to a second order differential equation $y'' + \frac{\nu}{\nu'} y + \nu\tau y = 0$. In the case of trinoids, this leads to a hypergeometric equation. Since there are many results for the hypergeometric equation, it is no surprise that the case of trinoids is fairly well understood. In the case of 4-noids, however, this leads to a Heun equation, and unfortunately, there are not many results for these types of equations. Therefore, we seek to simplify the setting in such a way, that results for the hypergeometric equations can be used to obtain sufficient criteria for the simultaneous unitarizability of the monodromy matrices in the case of 4-noids.

The paper is organized as follows: In Section 2, we give a short outline of the DPW method and the theory involved with it, Sections 3 and 4 mainly repeat the results of [11] that we will use and at the same time provides the notations and definitions we will work with hereafter. Section 5 introduces the holomorphic potentials we will use for construction of planar 4-noids and the transformations of the associated Heun equation that we consider in order to derive conditions on the simultaneous unitarizability of the monodromy matrices, Section 6 is devoted to the spherical triangle inequalities and their relation to the simultaneous unitarizability of monodromy matrices for trinoids, and in Section 7 we carryover these results to the limit case of 4-noids. And in Section 8 at last, we are able to state our main theorem about the construction of planar 4-noids.

2. Basic Definitions and Results

2.1. Associated families and orthogonal frames

Let us start with a quick review of the characterization of CMC-surfaces, their appearance in associated families and the means of describing and constructing them. We first restrict to CMC-immersions defined on a simply connected domain $\widetilde{\mathcal{M}}$. We consider a surface $\Psi : \widetilde{\mathcal{M}}$

$\to \mathbb{R}^3$, where $\widetilde{\mathcal{M}}$ is a simply connected Riemann surface. We will always assume Ψ is conformal. Then we can rewrite the induced metric as $\mathrm{d}s^2 = e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$, $u : \mathbb{D} \to \mathbb{R}$. Moreover, if $N = \dfrac{\Psi_x \times \Psi_y}{|\Psi_x \times \Psi_y|}$ denotes the Gauss map, then we have

$$\langle \Psi_x, \Psi_x \rangle = \langle \Psi_y, \Psi_y \rangle = e^u, \quad \langle \Psi_x, \Psi_y \rangle = 0.$$

Therefore, the frame $U = (e^{-\frac{u}{2}}\Psi_x, e^{-\frac{u}{2}}\Psi_y, N)$ is an orthogonal matrix of determinant 1. By possibly rotating the surface, we can assume $U(0, 0) = I$.

It is well known (compare for example [6, Appendix]), that Ψ is a CMC-surface if and only if U is a solution to the Lax pair equations

$$U^{-1}U_z = A \quad \text{and} \quad U^{-1}U_{\bar{z}} = B, \tag{2.1.1}$$

where $Q = \langle \Psi_{zz}, N \rangle$ is the torsion invariant (i.e., the coefficient of the Hopf differential $\langle \Psi_{zz}, N \rangle \mathrm{d}z^2$),

$$A = \begin{pmatrix} 0 & \frac{i}{2}u_z & -\left(Q + \frac{1}{2}e^u H\right)e^{-u/2} \\ -\frac{i}{2}u_z & 0 & -i\left(Q - \frac{1}{2}e^u H\right)e^{-u/2} \\ \left(Q + \frac{1}{2}e^u H\right)e^{-u/2} & i\left(Q - \frac{1}{2}e^u H\right)e^{-u/2} & 0 \end{pmatrix}$$

and $B = \overline{A}$. The integrability conditions for the differential equations (2.1.1) are

$$u_{zz} + \frac{1}{2}e^u H^2 - 2e^{-u}|Q|^2 = 0, \tag{2.1.2}$$

$$Q_{\bar{z}} = \frac{1}{2}e^u H_z = 0. \tag{2.1.3}$$

Clearly, Ψ is a CMC-surface iff Q is holomorphic. Then (2.1.3) is void and from (2.1.2) we see that we can replace Q by $\lambda^{-2}Q$, where $\lambda \in \mathbb{S}^1$ and still obtain a solution U to (2.1.1). We then have a CMC-immersion with metric e^u, mean curvature H and torsion invariant $\lambda^{-2}Q$. Hence, CMC-surfaces come in associated families. The parameter λ is called *spectral parameter*. This way, one defines an extended orthogonal frame, that is an orthogonal frame (depending on λ) for the associated family of CMC-surfaces. See below for an alternate method constructing an extended unitary frame.

2.2. Unitary frames

Using the isomorphism between $\mathbb{R}^3$ and $\mathfrak{su}_2$, given by the spinor map

$$J : \mathbb{R}^3 \to \mathfrak{su}_2, \quad (x, y, z) \mapsto J(x, y, z) = \frac{1}{2}\begin{pmatrix} -iz & -ix-y \\ -ix+y & iz \end{pmatrix} \quad (2.2.1)$$

we lift the moving frame $U \in \mathrm{SO}_3$ of a CMC-surface to a moving frame $F \in \mathrm{SU}_2$ (also compare [6, Appendix]). The automorphisms of $\mathbb{R}^3$ respecting the cross product are the elements of SO_3, so we regard the moving frame U as such an automorphism. Knowing the automorphisms of $\mathbb{R}^3$, it is easy to determine the group of automorphisms of $\mathfrak{su}_2$, this is given by $J \circ \mathrm{SO}_3 \circ J^{-1}$. On the other hand, we know that the group SU_2 acts via conjugation on $\mathfrak{su}_2$ and that SU_2 is a 2-to-1 cover of $\mathrm{Aut}\,\mathfrak{su}_2$. Consequently, also SU_2 and SO_3 are in a 2-to-1 correspondence. Hence, every $U \in \mathrm{SO}_3$ determines an associated $F \in \mathrm{SU}_2$ uniquely up to sign.

2.3. Extended unitary frames

In order to obtain an extended unitary frame (i.e., a frame for the associated family of CMC-surfaces) from a given unitary frame, one proceeds as follows: Consider the Maurer-Cartan-Form $\alpha := F^{-1}\mathrm{d}F \in \mathfrak{su}_2$. First, split α into a $(1, 0)$ and a $(0, 1)$ part: $\alpha = \alpha'\mathrm{d}z + \alpha''\mathrm{d}\bar{z}$. Next, we decompose α' and α'' as $\alpha' = \alpha'_k + \alpha'_p$ and $\alpha'' = \alpha''_k + \alpha''_p$ into a diagonal part (denoted by the index k) and an off-diagonal part (indicated by p). Writing $\alpha'_k + \alpha''_k = \alpha_0$, we have

$$\alpha = \alpha'_p\mathrm{d}z + \alpha_0 + \alpha''_p\mathrm{d}\bar{z}.$$

If one introduces a spectral parameter $\lambda \in \mathbb{S}^1$ in the following way:

$$\alpha_\lambda = \lambda^{-1}\alpha'_p dz + k_0 + \lambda\alpha''_p d\bar{z},$$

then one obtains the following result ([10] and [21]): The extended moving frame F obtained by integrating α is the frame for an associated family of CMC-surfaces if and only if α_λ is integrable, i.e., iff it satisfies the integrability condition

$$d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0.$$

2.4. Loop groups

For each real constant r, $0 < r \leq 1$, let $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$ denote the group of maps $g(\lambda)$ from C_r, the circle of radius r, to $(\mathrm{SL}_2\mathbb{C})_\sigma$, which satisfy the twisting condition

$$g(-\lambda) = \sigma(g(\lambda)), \tag{2.4.1}$$

where $\sigma : \mathrm{SL}_2\mathbb{C} \to \mathrm{SL}_2\mathbb{C}$ is defined by conjugation with the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and a topological condition discussed below.

The Lie algebras of these groups, which we denote by $\Lambda_r(\mathfrak{sl}_2\mathbb{C})_\sigma$, consist of maps $x : C_r \to \mathfrak{sl}_2\mathbb{C}$, which satisfy a similar condition, namely

$$x(-\lambda) = \sigma_3 x(\lambda)\sigma_3. \tag{2.4.2}$$

In order to make these loop groups complex Banach Lie groups, we require that each matrix coefficient, considered as a function on C_r, is contained in the Wiener Algebra

$$\mathcal{A}_{(r)} = \left\{ q(\lambda) : \lambda \in C_r,\ q = \sum_{n\in\mathbb{Z}} q_n\lambda^n,\ \sum_{n\in\mathbb{Z}} |q_n r^n| < \infty \right\}. \tag{2.4.3}$$

For $r = 1$, we will always omit the subscript "r".

Furthermore, we will use the following subgroups of $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$: Let B be a subgroup of $\mathrm{SL}_2\mathbb{C}$ and $\Lambda^+_{r,B}(\mathrm{SL}_2\mathbb{C})_\sigma$ be the group of maps in

$\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$, which can be extended to holomorphic maps on

$$I^{(r)} = \{\lambda \in \mathbb{C} : |\lambda| < r\}, \tag{2.4.4}$$

the interior of the circle C_r, and take values in B at $\lambda = 0$. Analogously, let $\Lambda^-_{r,B}(\mathrm{SL}_2\mathbb{C})_\sigma$ be the group of maps in $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$, which can be extended to the exterior

$$E^{(r)} = \{\lambda \in \mathbb{C}P_1 : |\lambda| > r\} \tag{2.4.5}$$

of C_r and take values in B at $\lambda = \infty$. If $B = \{I\}$, then we write the subscript $*$ instead of B, and if $B = \mathrm{SL}_2\mathbb{C}$, then we omit the subscript B entirely.

Also, by an abuse of notation, we will denote by $\Lambda_r(\mathrm{SU}_2)_\sigma$ the subgroup of maps in $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$, which can be extended holomorphically to the open annulus

$$A^{(r)} = \left\{\lambda \in \mathbb{C} : r < |\lambda| < \frac{1}{r}\right\} \tag{2.4.6}$$

and take values in SU_2 on the unit circle.

Corresponding to these subgroups, we analogously define Lie subalgebras of $\Lambda_r(\mathfrak{sl}_2\mathbb{C})_\sigma$.

We will need the following results from [19] and [10]:

• For each solvable subgroup B of $\mathrm{SL}_2\mathbb{C}$, which satisfies $\mathrm{SU}_2 \cdot B = \mathrm{SL}_2\mathbb{C}$ and $\mathrm{SU}_2 \cap B = \{I\}$, multiplication

$$\Lambda_r(\mathrm{SU}_2)_\sigma \times \Lambda^+_{r,B}(\mathrm{SL}_2\mathbb{C})_\sigma \to \Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma \tag{2.4.7}$$

is a diffeomorphism onto. The associated splitting

$$C = FV_+ \tag{2.4.8}$$

of an element $C \in \Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$ such that $F \in \Lambda_r(\mathrm{SU}_2)_\sigma$ and $V_+ \in \Lambda^+_{r,B}(\mathrm{SL}_2\mathbb{C})_\sigma$ is called *r-Iwasawa splitting*. (In the case $r = 1$, we will just speak of Iwasawa splitting.)

• Multiplication

$$\Lambda^-_{r,*}(\mathrm{SL}_2\mathbb{C})_\sigma \times \Lambda^+_r(\mathrm{SL}_2\mathbb{C})_\sigma \to \Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma \tag{2.4.9}$$

is a diffeomorphism onto the open and dense subset $\Lambda^-_{r_+}(\mathrm{SL}_2\mathbb{C})_\sigma \cdot \Lambda^+_r(\mathrm{SL}_2\mathbb{C})_\sigma$ of $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$, called the "big cell". The associated splitting

$$g = g_- g_+ \tag{2.4.10}$$

of an element g of the big cell, where $g_- \in \Lambda^-_{r,*}(\mathrm{SL}_2\mathbb{C})_\sigma$ and $g_+ \in \Lambda^+_r(\mathrm{SL}_2\mathbb{C})_\sigma$ will be called *Birkhoff factorization*.

2.5. Weierstraß representation

The so-called generalized Weierstraß representation presents a way to construct a surface with constant mean curvature $H \neq 0$ (or rather its associated family) on a simply-connected domain $\hat{\mathcal{M}}$ (excluding the sphere!) from a prescribed holomorphic potential. This is done in the following 4 steps:

Step 1. Choose any holomorphic 2×2-matrix differential form $\eta = A(z, \lambda)\mathrm{d}z \in \Lambda(\mathfrak{sl}_2\mathbb{C})_\sigma \times \Omega(\hat{\mathcal{M}})$, of which the diagonal elements are even functions of $\lambda \in \mathbb{C}^*$, and the off-diagonal elements are odd functions of $\lambda \in \mathbb{C}^*$, and the powers of λ are ≥ -1. Assume $\det A_{-1} \neq 0$, where A_{-1} is the coefficient matrix of the λ^{-1}-term of η.

Step 2. Then find a solution to the complex system of ordinary differential equations:

$$\mathrm{d}C = C\eta. \tag{2.5.1}$$

Step 3. Now carry out a Iwasawa decomposition of C:

$$C = FV_+,$$

where $F = F(z, \bar{z}, \lambda) \in \Lambda(\mathrm{SU}_2)_\sigma$ and $V_+ \in \Lambda^+(\mathrm{SL}_2\mathbb{C})_\sigma$. If the Iwasawa decomposition shall be unique, then one needs to require additional properties, for example $V_+(0, \lambda) \in \mathbb{R}^+$. (Writing $V_+(0, \lambda) \in \mathbb{R}^+$ means that all matrix entries of $V_+(0, \lambda)$ are in $\mathbb{R}^+$.)

Then the following crucial theorem holds (both for the unique and non-unique decompositions):

Theorem 2.1 [10]. *F is the extended unitary frame of an immersion with constant mean curvature* $H \neq 0$.

Step 4. Insert F into the Sym-Bobenko formula

$$\Psi_\lambda(z) = -\frac{1}{2H}\left\{\left(i\lambda \frac{\partial}{\partial\lambda} F\right)F^{-1} + \frac{i}{2} F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}F^{-1}\right\}. \tag{2.5.2}$$

Then Ψ_λ is for every $\lambda \in \mathbb{S}^1$ a CMC-immersion from $\hat{\mathcal{M}}$ into $\mathfrak{su}_2$. The Gauss map of Ψ_λ is given by $N = -\frac{i}{2} F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}F^{-1}$. Speaking of the surface constructed by the Weierstraß representation usually means evaluating the Sym-Bobenko formula at $\lambda = 1$ and choosing the initial condition $C(0, \lambda) = I$ for the differential equation (2.5.1).

2.6. Dressing action

Next, we define the dressing action of $\Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$, $0 < r \leq 1$ on $\mathcal{F}$, the set of extended unitary frames of CMC-immersions. For $F(z, \lambda) \in \mathcal{F}$ and $h_+ \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$, we set

$$h_+(\lambda)F(z, \lambda) = (h_+ \cdot F)(z, \lambda)q_+(z, \lambda), \tag{2.6.1}$$

where the right hand side of (2.6.1) is defined by the Iwasawa decomposition in $\Lambda_r(\mathrm{SL}_2\mathbb{C})_\sigma$ of h_+F, i.e., $q : \hat{\mathcal{M}} \to \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$. In addition, $q_+(z, 0) \in \mathbb{R}^+$. It is easily proved (see, e.g., [4]), that $h_+ \cdot F$ is again in $\mathcal{F}$. Therefore, (2.6.1) defines an action on $\mathcal{F}$.

2.7. The Weierstraß representation on non-simply connected domains

In order to construct a CMC-immersion from a (non-simply connected) Riemann surface $\mathcal{M}$, we first construct an associated family of CMC-immersions from its universal cover $\hat{\mathcal{M}}$ and try to descend to $\mathcal{M}$ for some value λ_0, say $\lambda_0 = 1$.

According to [9], we can start with a potential η on $\hat{\mathcal{M}}$ being invariant under the fundamental group Γ of $\mathcal{M}$, that is $\gamma^*\eta = \eta$ for all $\gamma \in \Gamma$. (In fact η induces a holomorphic 1-form on $\mathcal{M}$, and we identify η with this induced 1-form.)

Furthermore, for every solution C of $\mathrm{d}C = C\eta$ on $\hat{\mathcal{M}}$ and every automorphism $\gamma \in \Gamma$, one has

$$C(\gamma(z), \lambda) = \varrho(\gamma, \lambda)C(z, \lambda), \tag{2.7.1}$$

where $\varrho \in \Lambda(\mathrm{SL}_2\mathbb{C})_\sigma$ does not depend on z. We call ϱ the *monodromy matrix* of the holomorphic frame C.

If the CMC-immersion $\hat{\Psi}_\lambda$ obtained from η on $\hat{\mathcal{M}}$ descends to $\mathcal{M}$ for some value of $\lambda \in \mathbb{S}^1$, then for all $g \in \mathrm{Aut}\hat{\mathcal{M}}$, the extended unitary frame F transforms like [8]:

$$F(g(z), \lambda) = \chi(g, \lambda)F(z, \lambda)k(\gamma, z), \tag{2.7.2}$$

where $\chi(g, \lambda) \in \Lambda(\mathrm{SU}_2)_\sigma$ and k is a unitary, λ-independent matrix. The matrix χ is called the *monodromy* of F.

Conversely, let η and C be as above. Let $C = FV_+$, where $F \in \Lambda(\mathrm{SU}_2)_\sigma$ is the extended unitary frame and $V_+ \in \Lambda^+(\mathrm{SL}_2\mathbb{C})_\sigma$. If $\varrho(\gamma, \lambda) \in \Lambda(\mathrm{SU}_2)_\sigma$, then for all $\gamma \in \Gamma$ we have

$$F(\gamma(z), \lambda) = \varrho(\gamma, \lambda)F(\gamma, \lambda)k(\gamma, z), \tag{2.7.3}$$

where $k(\gamma, z)$ is unitary and diagonal (and independent of λ). In this case, ϱ is not only the monodromy of C but also of F. This gives a chance that $\hat{\Psi}_\lambda$ will descend to $\mathcal{M}$ for some $\lambda_0 \in \mathbb{S}^1$. More precisely, the monodromy ϱ of C being unitary is a necessary condition for the CMC-immersion being invariant under the fundamental group Γ.

For all $\gamma \in \Gamma$ we obtain, via (2.7.3), inserted into (2.5.2)

$$\hat{\Psi}_\lambda(\gamma(z)) = \varrho^{-1}(\gamma, \lambda)\hat{\Psi}_\lambda \varrho(\gamma, \lambda) - \frac{1}{2H} i\lambda\partial_\lambda \varrho(\gamma, \lambda) \cdot \varrho^{-1}(\gamma, \lambda). \qquad (2.7.4)$$

Using formulas (2.7.2), (2.7.3) and (2.7.4), we obtain necessary and sufficient conditions for the immersion $\hat{\Psi}_\lambda$ to descend from $\hat{\mathcal{M}}$ to $\mathcal{M}$ ([9]):

1. $\varrho(\gamma, \lambda) \in \Lambda(\mathrm{SU}_2)_\sigma$ for all $\gamma \in \Gamma$,

2. $\varrho(\gamma, \lambda) = \pm I|_{\lambda=1}$ for all $\gamma \in \Gamma$,

3. $\partial_\lambda \varrho(\gamma, \lambda) \cdot \varrho^{-1}(\gamma, \lambda)|_{\lambda=1} = 0$ for all $\gamma \in \Gamma$.

In fact, permitting dressing, it suffices, that these conditions are satisfied for the monodromy matrices after conjugation with some matrix $h_+ \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})$. If there exists h_+, such that for $\varrho(\gamma_1, \lambda) := M_1(\lambda), \ldots, \varrho(\gamma_n, \lambda) := M_n(\lambda)$ we have $h_+ M_1 h_+^{-1}, \ldots, h_+ M_n h_+^{-1} \in \Lambda_r(\mathrm{SU}_2)_\sigma$, the matrices $M_1, \ldots, M_n$ are called *simultaneously r-unitarizable*. So one crucial point in the construction of k-noids is to find (sufficient) conditions on the simultaneous unitarizability of the monodromy matrices around the ends of the surface.

2.8. Gauging and its relation to dressing

With the usual notation, we consider (on a simply connected domain) a potential η, a solution C to $\mathrm{d}C = C\eta$, an extended unitary frame F obtained by Iwasawa-splitting $C = FV_+$ and a CMC-immersion Ψ obtained from F via the Sym-Bobenko formula.

As mentioned before, for $h_+ \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$ we consider $h_+F = \hat{F}\hat{W}_+$ and obtain a new CMC-immersion $\hat{\Psi}$ associated with $\hat{F}$. Note that a priori, there is no canonical way of defining a dressing action on the level of holomorphic frames: We can set $h_+C = \hat{C}$ or $h_+Ch_+^{-1} = \widetilde{C}$. Both are again holomorphic frames for $\hat{\Psi}$, since $\hat{C} = h_+C = h_+FV_+ = \hat{F}(W_+V_+)$ and $\widetilde{C} = h_+Ch_+^{-1} = h_+FV_+h_+^{-1} = \hat{F}(\hat{W}_+V_+h_+^{-1})$.

Let $L_+ = L_+(z, \lambda)$ be defined on $\widetilde{\mathcal{M}}$. Then we can consider $\hat{C} = CL_+$ and obtain from $\hat{C}$ the same immersion as from C, since $\hat{C} = CL_+ = F(V_+L_+)$ and $C = FV_+$. This operation is usually called "gauging". One can change/simplify the potential η this way, but not the resulting CMC-immersion, an easy calculation shows $\hat{\eta} = L_+^{-1}\eta L_+ + L_+^{-1}\mathrm{d}L_+$.

In some cases, however, dressing and gauging occur simultaneously. Consider, for example, an automorphism γ of $\widetilde{\mathcal{M}}$ and assume we know that we have

$$\gamma * \eta = L_+^{-1}\eta L_+ + L_+^{-1}\mathrm{d}L_+.$$

Then it is easy to verify that $\widetilde{C}(z, \lambda) = C(\gamma(z), \lambda)$ and $\hat{C}(z, \lambda) = C(z, \lambda)L_+(z, \lambda)$ satisfy the equation

$$\widetilde{C}^{-1}\mathrm{d}\widetilde{C} = \hat{C}^{-1}\mathrm{d}\hat{C}.$$

From this we can only conclude

$$\widetilde{C}(z, \lambda) = B(\lambda)\hat{C}(z, \lambda)$$

for some $B \in \Lambda(\mathrm{SL}_2\mathbb{C})_\sigma$. It will frequently be desirable to have some information about B.

To secure such information we choose a base point z_* and require all holomorphic frames and all extended unitary frames to attain the value I at z_* for all $\lambda \in \mathbb{S}^1$. We would like to point out that under this assumption a one-to-one relation between CMC-immersions and special potentials ("normalized potentials") can be proven. At any rate, the assumption $C(z_*, \lambda) = I$, $F(z_*, \overline{z_*}, \lambda) = I$ has a number of consequences:

1. Dressing by h_+ now needs to take the form $C \to h_+Ch_+^{-1}$.

2. Gauging by L_+ now needs to take the form $C \to h_+CL_+$, where $h_+ = L_+(z_*, \lambda)^{-1}$.

3. Every η yields a unique C and thus – if at all – a unique monodromy matrix.

3. Holomorphic Potentials and Monodromy

3.1.

Let $\mathcal{S}_n$ denote the Riemann sphere, with n points $z_1, \ldots, z_n$ removed, where we will always assume $z_n = \infty$. We consider a holomorphic potential η on $\mathcal{S}_n$ of the form

$$\eta = \begin{pmatrix} 0 & \nu(z, \lambda) \\ \tau(z, \lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.1.1}$$

that is, we restrict to off-diagonal potentials, see, e.g., [11] why this poses no restriction. Since η shall be a holomorphic potential, we can assume it is a holomorphic 1-form on $\mathcal{S}_n$, i.e., its entries ν, τ are holomorphic functions on $\mathcal{S}_n$. We will always assume that ν, τ are meromorphic functions on $\mathbb{S}^2$.

3.2.

We recall a lemma from [11] relating the differential equation $\mathrm{d}H = H\eta$ to a second order ODE:

Lemma 3.1. *The solutions H to the ODE $dH = H\eta$ are of the form*

$$H = \begin{pmatrix} y_1'/\nu & y_1 \\ y_2'/\nu & y_2 \end{pmatrix}, \tag{3.2.1}$$

where y_1 and y_2 are functions satisfying the scalar ordinary differential equation

$$y'' - \frac{\nu'}{\nu} y' - \nu\tau y = 0. \tag{3.2.2}$$

In the case of a trinoid, this equation will lead to a hypergeometric equation, whereas in the case of a 4-noid, this will lead to an equation of “Heun type”. With the exception of a discrete set of z-values, y_1 and y_2 form a fundamental system for (3.2.2).

3.3.

In this paper, similar to [11], we want equation (3.2.2) to be an equation of Fuchsian type. Thus we have [1]

$$-\frac{\nu'(z, \lambda)}{\nu(z, \lambda)} = \sum_{j=1}^{n-1} \frac{a_j(\lambda)}{z - z_j}, \tag{3.3.1}$$

$$-\nu(z, \lambda)\tau(z, \lambda) = \sum_{j=1}^{n-1} \left[\frac{b_j(\lambda)}{(z - z_j)^2} + \frac{c_j(\lambda)}{z - z_j} \right] \tag{3.3.2}$$

for some even functions $a_1, \ldots, a_{n-1}$, $b_1, \ldots, b_{n-1}$ and $c_1, \ldots, c_{n-1}$ of λ, which belong to the Wiener Algebra $\mathcal{A}$, and to which the following conditions apply:

$$\sum_{j=1}^{n-1} c_j(\lambda) = 0, \tag{3.3.3}$$

$$\sum_{j=1}^{n} a_j(\lambda) = 2, \tag{3.3.4}$$

$$\sum_{j=1}^{n-1} (b_j(\lambda) + c_j(\lambda) z_j) = b_n. \tag{3.3.5}$$

The parameters a_n and b_n correspond to the singularity at $z_n = \infty$. Since ν is meromorphic on $\mathbb{S}^2$, the a_j are integers and independent of λ. The indicial equations at each of the singular points z_j, $j = 1, \ldots, n$, are

$$r(r-1) + a_j r + b_j = 0, \tag{3.3.6}$$

whose solutions are

$$r_{j,\pm} = \frac{1}{2}[1 - a_j \pm \sqrt{(1 - a_j)^2 - 4b_j}]. \tag{3.3.7}$$

3.4.

Using (3.3.1) and the fact, that the a_j are integers, we obtain

$$\nu(z, \lambda) = \omega(\lambda) \prod_{j=1}^{n-1} (z - z_j)^{-a_j} \tag{3.4.1}$$

and

$$\tau(z, \lambda) = -\omega(\lambda)^{-1} \prod_{j=1}^{n-1} (z - z_j)^{a_j} \sum_{j=1}^{n-1} \left[\frac{b_j(\lambda)}{(z - z_j)^2} + \frac{c_j(\lambda)}{z - z_j} \right]. \tag{3.4.2}$$

The function ω may be removed by some r-dressing (see [11, Remark 3.5.1]), hence we can assume that ν and τ are of the form

$$\nu(z, \lambda) = \lambda^{-1} \prod_{j=1}^{n-1} (z - z_j)^{-a_j} \tag{3.4.3}$$

and

$$\tau(z, \lambda) = -\lambda \prod_{j=1}^{n-1} (z - z_j)^{a_j} \sum_{j=1}^{n-1} \left[\frac{b_j(\lambda)}{(z - z_j)^2} + \frac{c_j(\lambda)}{z - z - j} \right], \tag{3.4.4}$$

where b_j, c_j are as above.

3.5.

To describe the b_j in more detail, we consider for each $j \in \{1, ..., n\}$ some Delaunay matrix

$$D_j = \begin{pmatrix} e_j & X_j \\ \overline{X}_j & -e_j \end{pmatrix}, \tag{3.5.1}$$

where

$$X_j = s_j \lambda^{-1} + t_j \lambda, \quad \overline{X}_j = s_j \lambda + t_j \lambda^{-1}, \tag{3.5.2}$$

and

$$e_j \in \mathbb{R}, \quad s_j, t_j > 0, \tag{3.5.3}$$

$$e_j^2 + | s_j + t_j |^2 = \frac{1}{4}. \tag{3.5.4}$$

It is easy to verify that the eigenvalues of D_j are $\pm \mu_j$, where

$$\mu_j = \sqrt{e_j^2 + | s_j\lambda^{-1} + t_j\lambda |^2}. \tag{3.5.5}$$

Using this notation, we obtain for b_j the form (see (3.6.8) in [11])

$$b_j(\lambda) = \frac{1}{4}(1 - a_j)^2 - \mu_j^2. \tag{3.5.6}$$

Throughout this text, we will assume that the diagonal entries e_j vanish, in this case we have $\mu_j = | X_j | = \sqrt{X_j \overline{X}_j}$. For later use, we note

$$1 - 4\mu^2 = -4st\lambda^{-2}(1 - \lambda^2)^2. \tag{3.5.7}$$

3.6.

In the case $n = 3$, rewriting equation (3.2.2) via the substitution $y(z) = z^{r_{1,+}}(1 - z)^{r_{2,+}} w(z)$, leads to the hypergeometric equation

$$z(1 - z)w''(z) + [\gamma_1 - (\alpha + \beta + 1)z]w' - \alpha\beta w(z) = 0, \tag{3.6.1}$$

where

$$\gamma_1 = 1 + \sqrt{(1 - a_1)^2 - 4b_1} = 1 + 2\mu_1, \tag{3.6.2}$$

$$\alpha = \frac{1}{2}\{\gamma_1 + \sqrt{(1 - a_2)^2 - 4b_2} + \sqrt{(1 - a_3)^2 - 4b_3}\}$$

$$= \frac{1}{2}\{1 + 2\mu_1 + 2\mu_2 + 2\mu_3\}, \tag{3.6.3}$$

$$\beta = \frac{1}{2}\{\gamma_1 + \sqrt{(1 - a_2)^2 - 4b_2} - \sqrt{(1 - a_3)^2 - 4b_3}\}$$

$$= \frac{1}{2}\{1 + 2\mu_1 + 2\mu_2 - 2\mu_3\}. \tag{3.6.4}$$

By computing the monodromy matrices ϱ_0 and ϱ_1 for the solutions of this differential equation around $z = 0$ and $z = 1$ respectively, one can show [12]:

Theorem 3.2 [11]. *The monodromy matrices ϱ_0 and ϱ_1 are simultaneously unitarizable via r-dressing for some $r \in (0, 1)$ sufficiently close to* 1 *if and only if on* $\mathbb{S}^1$,

$$0 \leq \frac{\cos \pi(\mu_1 - \mu_2 - \mu_3) \cos \pi(\mu_1 - \mu_2 + \mu_3)}{\sin 2\pi(\mu_1) \sin 2\pi(\mu_2)} \leq 1. \tag{3.6.5}$$

Writing $a = 2\pi\mu_1$, $b = 2\pi\mu_2$, $c = 2\pi\mu_3$ this is equivalent to

$$0 \leq \cos(a - b) + \cos c \leq 2 \sin a \cdot \sin b. \tag{3.6.6}$$

And one can show that it is also equivalent to

$$0 \leq 1 - \cos^2 a - \cos^2 b - \cos^2 c - 2 \cos a \cdot \cos b \cdot \cos c. \tag{3.6.7}$$

The latter equation has been derived in [14].

3.7.

Next, we will discuss how the asymptotic behaviour of the holomorphic potential relates to the asymptotic behaviour of the corresponding end in the case of a perturbed Delaunay potential. In particular, we will give a sufficient condition on the embeddedness of an end.

We consider surfaces, which arise from perturbed Delaunay potentials, i.e., potentials of the form

$$\eta = \left(\frac{1}{z} D + \eta^h\right) dz = \frac{1}{z} D dz + \sum_{j=0}^{\infty} \eta_j z^j dz, \tag{3.7.1}$$

where D is a Delaunay matrix as in Section 3.5 and $\lambda\eta^h$ is holomorpic in λ for $|\lambda| < 1 + \varepsilon$ for some $\varepsilon > 0$, and holomorphic in z for $|z| < \hat{r}$ for some $0 < \hat{r}$. We are mainly interested in the question, under what conditions and for what λ, the differential equation $dC = C\eta$ has a solution in "EDP"-representation, that is in the form

$$C(z, \lambda) = e^{(\ln z) D} P(z, \lambda), \tag{3.7.2}$$

where

$$P(z, \lambda) = I + zP_1(\lambda) + z^2 P_2(\lambda) + \cdots, \tag{3.7.3}$$

with $|z| < \hat{r}$. It is easy to see that solving $\mathrm{d}C = C\eta$ with P as in (3.7.2) is equivalent to solving

$$\mathrm{d}P = P\eta - \left(\frac{1}{z} D\right) P \mathrm{d}z. \tag{3.7.4}$$

An important result is

Theorem 3.3 [11]. *Assume* η *as in* (3.7.1) *is holomorphic in* λ *in a neighbourhood of* $\mathbb{S}^1$, *then the coefficients* $P_{1,kl}$ *of* P_1 *are all contained in the Wiener Algebra* $\mathcal{A}$ *of* $\mathbb{S}^1$ *and are actually holomorphic in a neighbourhood of* $\mathbb{S}^1$ *if and only if* $\lambda = \pm 1$ *are zeros of order* ≥ 2 *of*

$$\eta_{0,11} + \frac{1}{1 + 2e} \overline{X} \eta_{0,12} - \frac{1}{1 - 2e} X \eta_{0,21}. \tag{3.7.5}$$

In this case, (3.7.4) *has a solution* P *of the form* (3.7.3) *such that* P *is holomorphic in* z *and* λ *is an open set containing* $\{z = 0\} \times \mathbb{S}^1$. *In particular*, $P \in \Lambda(\mathrm{SL}_2\mathbb{C})_\sigma$.

The condition expressed in Theorem 3.3 is referred to as "embedding condition", due to the results of the following theorems.

Since the monodromy for the solution (3.7.2) of $\mathrm{d}C = C\eta$ is $e^{(\ln z)D}$, and $e^{(\ln z)D}$ satisfies the closing conditions at $\lambda = 1$, the surface obtained from η closes at $\lambda = 1$, and we have

Theorem 3.4 [11]. *Let* η *be a holomorphic potential of the form* (3.7.1) *satisfying condition* (3.7.5). *Let* Φ *and* Φ_D *denote the immersions obtained from the perturbed Delaunay solution* C *to* $\mathrm{d}C = C\eta$ *in EDP representation* (3.7.2) *and the form* $e^{(\ln z)D}$, *respectively. Then there exist* $0 < r_0$ *and* $0 < K$ *such that for all* $\lambda \in \mathbb{S}^1$ *and all* $0 < |z| < r_0$, *we have*

$$\| \Phi(z, \lambda) - \Phi_D(z, \lambda) \| \leq K|z|, \quad \| \partial_z \Phi(z, \lambda) - \partial_z \Phi_D(z, \lambda) \| \leq K|z|. \tag{3.7.6}$$

More importantly for us, we obtain

Corollary 3.5. *Under the assumptions and with the notation of the theorem, the immersion* Φ *is an embedding for* $|z| < r_0$, *where* $0 < r_0$ *is chosen sufficiently small. In particular,* Φ *has for* $\lambda = 1$ *an embedded end at* $z = 0$.

4. Embedding and Existence Condition for k-noids

In this section, we continue the study of surfaces associated with the potentials introduced in Section 3. In particular, we investigate when each end is separately embedded after some dressing and when all ends will be simultaneously embedded.

4.1.

First we investigate the embedding condition (3.7.5) for the potentials under consideration. We would like to point out that for the discussion in this section, we still consider each end separately.

Actually, the embedding condition does not need to be satisfied by the given potential, since it may not have the form used in the formulation of the condition, but it suffices to verify this condition after some gauge.

Therefore, we only need to make sure that for each end z_j, there exists some gauge transformation $W_+^{(j)}(z, \lambda) \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$ such that

$$\hat{\eta} = W_+^{(j)^{-1}} \eta W_+^{(j)} + W_+^{(j)^{-1}} \mathrm{d} W_+^{(j)} \tag{4.1.1}$$

satisfies the condition in question at the end z_j.

We will carry out this program by finding a sequence of gauge transformations. First of all we gauge by $Q = \mathrm{diag}(A, A^{-1})$, where

$$A = \sqrt{\lambda \nu (z - z_j)}, \tag{4.1.2}$$

is independent of λ. With this choice of A, we obtain by conjugation with A the off-diagonal matrix entries $A^{-2}\nu = \lambda^{-1}(z - z_j)^{-1}$ and $A^2 \tau =$

$\lambda\nu\tau(z - z_j) = -\lambda b_j(\lambda)(z - z_j)^{-1} - \lambda c_j(\lambda) + \mathcal{O}(z - z_j)$. Moreover, $A^{-1}\mathrm{d}A = \frac{1}{2}\left(\sum_{k=1}^{n-1} \frac{-a_k}{z - z_k} + \frac{1}{z - z_j}\right)$. Therefore, developing $\tilde{\eta}$ into powers of $z - z_j$, we obtain

$$\tilde{\eta} = \frac{1}{z - z_j}\begin{pmatrix} \frac{1}{2}(1 - a_j) & \lambda^{-1} \\ -\lambda b_j(\lambda) & -\frac{1}{2}(1 - a_j) \end{pmatrix} + \begin{pmatrix} -\varepsilon_j & 0 \\ -\lambda c_j(\lambda) & \varepsilon_j \end{pmatrix} + \mathcal{O}(z - z_j), \tag{4.1.3}$$

where

$$\varepsilon_j = \frac{1}{2}\sum_{k \neq j} \frac{a_k}{z_j - z_k}. \tag{4.1.4}$$

Gauging with the lower triangular matrix $R_+ = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$ with $p = \lambda\left(e_j - \frac{1}{2}(1 - a_j)\right)$ gives

$$\tilde{\tilde{\eta}} = \frac{1}{z - z_j}\begin{pmatrix} e_j & \lambda^{-1} \\ -\lambda|X_j|^2 & -e_j \end{pmatrix} + \begin{pmatrix} -\varepsilon_j & 0 \\ \tilde{\tilde{B}}_j & \varepsilon_j \end{pmatrix} + \mathcal{O}(z - z_j), \tag{4.1.5}$$

where X_j is as in (3.5.2) and

$$\tilde{\tilde{B}}_j = 2\lambda\left(e_j - \frac{1}{2}(1 - a_j)\right)\varepsilon_j - \lambda c_j. \tag{4.1.6}$$

A final gauge by $T_+ = \mathrm{diag}(\sqrt{\lambda X_j}^{-1}, \sqrt{\lambda X_j})$ then produces

$$\hat{\eta} = \frac{1}{z - z_j}\begin{pmatrix} e_j & X_j \\ \overline{X_j} & -e_j \end{pmatrix} + \begin{pmatrix} -\varepsilon_j & 0 \\ \hat{B}_j & \varepsilon_j \end{pmatrix} + \mathcal{O}(z - z_j), \tag{4.1.7}$$

where

$$\hat{B}_j = \lambda[\lambda X_j]^{-1}\{(2e_j - (1 - a_j))\varepsilon_j - c_j\}. \tag{4.1.8}$$

Note that indeed $\sqrt{\lambda X_j} \in \mathcal{A}_+$. Assume first $t_j < s_j$, then we rewrite $\lambda X_j = s_j + \lambda^2 t_j = s_j\left(1 + \frac{t_j}{s_j}\lambda^2\right)$. Clearly now, $\lambda X_j \in \mathcal{A}_+$ and also $\sqrt{\lambda X_j} \in \mathcal{A}_+$, since we can use the series expansion for $\sqrt{1+\zeta}$, where $|\zeta| < 1$. If $s_j < t_j$, then in the last gauge we interchange the roles of X_j and $\overline{X_j}$. If $t_j = s_j$, then one would need to carry out the gauge on any r-circle sufficiently close to $r = 1$. In the next sections, we will only consider non-cylindrical ends, and in this case $s_j \neq t_j$.

This way we have transformed the original potential η via gauging into the potential $\hat{\eta}$, which is in the form (3.7.1). Since we want to construct surfaces with embedded ends, we require $\hat{\eta}$ to satisfy the assumptions of Theorem 3.3. For the Fuchsian potentials introduced in 3.4 the condition can be expressed in the following way:

Proposition 4.1 [11]**.** *The embedding condition of Theorem* 3.3 *is satisfied for* $\hat{\eta}$ *if and only if*

$$\frac{1}{1-2e_j}\cdot\frac{c_j(\lambda) - a_j\varepsilon_j}{1 - 4|\mu_j(\lambda)|^2} \in \mathcal{A}_+. \tag{4.1.9}$$

Remark 4.1. (1) Note, if one replaces A by $\operatorname{diag}(\sqrt{\lambda\nu}, \sqrt{\lambda\nu}^{-1})$ and p in R_+ by $p = -\lambda\varepsilon_j$ in the series of gauge matrices above, one actually gauges η to a potential of the form (3.1.1), where $a_1 = \cdots = a_{n-1} = 0$ in the formula (3.4.3) for $\nu(z, \lambda)$.

(2) Since the solution $\hat{C}$ of $\mathrm{d}\hat{C} = \hat{C}\hat{\eta}$ in EDP representation (3.7.2) has monodromy $e^{\ln(z-z_j)D_j}$, the monodromy of the solution C to $\mathrm{d}C = C\eta$ with η as in Section 3.4 is given by $\chi = e^{\ln(z-z_j)D_j}\chi_A$, where $\chi_A = \pm I$ denotes the monodromy of the first gauge matrix $Q = \operatorname{diag}(\sqrt{\lambda\nu(z-z_j)}, \sqrt{\lambda\nu(z-z_j)}^{-1})$. If $a_1 = \cdots = a_n = 0$, then $\chi_A = -I$ and the monodromy is $\chi = -e^{\ln(z-z_j)}$.

4.2.

We will need the following result:

Theorem 4.1 [11]. *Assume the holomorphic potential* η *satisfies the assumptions of Section* 3 *as well as* (4.1.9). *Assume moreover that there exists some* $T_+ \in \Lambda_r^+ \mathrm{SL}_2\mathbb{C}_\sigma$ *such that for every* $j = 1, \ldots, n$ *the conjugated monodromy matrix* $T_+ M_j T_+^{-1}$ *is r-unitary. Then the surface associated with the dressed r-potential* $T_+ \eta T_+^{-1}$ *yields for* $\lambda = \pm 1$ *a k-noid of constant mean curvature with embedded ends that are asymptotically Delaunay surfaces.*

So it turns out that the key requirement (besides the embedding condition) is the simultaneous unitarizability of the monodromy matrices. In the next sections, we will find conditions, which yield the simultaneous unitarizability of the monodromy matrices of planar 4-noids. This is done with the help of the results for trinoids.

5. Transformations of the Heun Equation

5.1.

In the rest of this paper, we consider 4-noids. We choose a basepoint z_* and we normalize the holomorphic frame and the extended unitary frame by $C(z_*, \lambda) = I$ and $F(z_*, \overline{z_*}, \lambda) = I$, respectively. We will use a potential η of the form

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ \tau(z, \lambda) & 0 \end{pmatrix} dz \tag{5.1.1}$$

as in Section 3. Note this way we assume $a_1 = \cdots = a_{n-1} = 0$ in the formula (3.4.3) for $\nu(z, \lambda)$, see Remark 4.1. Furthermore, we will assume

$$\tau = -\lambda\left(\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{b_2(\lambda)}{(z-a)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1} + \frac{c_2(\lambda)}{z-a}\right), \tag{5.1.2}$$

where $a > 1$ (in particular $a \in \mathbb{R}$), and the b_j, c_j are real-valued functions (hence $\overline{\eta(\bar{z}, \lambda^{-1})} = \eta(z, \lambda)$, which we conjecture is one way to express the reflective symmetry of the k-noid about the x-y-plane, also see [5]). Eventually we will take a sufficiently large.

Remark 5.1. As has been seen in Section 4.1, we may gauge η to a "perturbed Delaunay potential":

$$\tilde{\eta} = \frac{1}{z - z_j} D_j + R_j,$$

where $D_j = \begin{pmatrix} e_j & X_j \\ \overline{X}_j & -e_j \end{pmatrix}$ is a Delaunay matrix, and the matrix R_j is holomorphic in z in a neighbourhood of z_j. In addition, it is easy to see that the embedding condition (4.1.9) is satisfied. Furthermore, as all ends are lying in the x-y-plane, we choose the diagonal entries e_j of the matrix D_j to vanish for all j. So the eigenvalues μ_j of D_j are given by $\pm\mu_j = \pm| X_j |$, where $X_j = s_j\lambda^{-1} + t_j\lambda$.

For unitarizability questions we will always compare the ends at $z = 0$ and $z = 1$. To obtain results for other ends as well, we apply certain fractional linear transformations. We will only consider such fractional linear transformations, for which the set of singularities is changed from $\{0, 1, \infty, a\}$ to $\{0, 1, \infty, a_*\}$.

Remark 5.2. From Section 3.2 we know that the solutions H of the differential equation $dH = H\eta$ with η as in (5.1.2) are of the form $H = \begin{pmatrix} \lambda y_1' & y_1 \\ \lambda y_2' & y_2 \end{pmatrix}$ with y_1, y_2 being a fundamental system for the ordinary differential equation:

$$y'' - \nu\tau y = 0. \tag{5.1.3}$$

(Since $\nu = \lambda^{-1}$, we have $\frac{\nu'}{\nu} = 0$.) So the fractional linear transformations that need to be considered, are the transformations which change the singularities of the differential equation (5.1.3) in the desired manner.

According to [20, Sec. A.2] there are exactly 24 such transformations, but we will make use only of the four transformations listed hereafter.

Below we list the fractional linear transformations, the singularities after the transformation and what monodromy matrices will be compared after the transformation. The singularities are listed in order so that the first is associated (after the transformation) with the coefficients b_0 and $\hat{c}_0$, the second with b_1 and $\hat{c}_1$, etc. The monodromy matrices are also given this way, that is the monodromy matrix ϱ_j corresponds to the end associated with b_j.

Transformation	Sing. at b_0	Sing. at b_1	Sing. at b_∞	Sing. at b_2	Monodromies
$z \mapsto z$	0	1	∞	a	ϱ_0, ϱ_1
$z \mapsto 1 - z$	1	0	∞	$1 - a$	ϱ_1, ϱ_0
$z \mapsto az$	0	$\frac{1}{a}$	∞	1	ϱ_0, ϱ_2
$z \mapsto (a-1)z + 1$	$\frac{1}{1-a}$	0	∞	1	ϱ_1, ϱ_2
$z \mapsto \frac{a}{z}$	∞	a	0	1	$\varrho_2, \varrho_\infty$

(5.1.4)

More precisely, we do the following: First we compute $\gamma^*\eta(z)$ for $\gamma(z) = \dfrac{Az + B}{Cz + D}$, which gives an off-diagonal matrix, with $\gamma' \cdot \lambda^{-1}$ as $(1, 2)$-entry. Then we gauge with an appropriate matrix, such that the $(1, 2)$-entry becomes λ^{-1} again and we obtain a new potential η^*. For the first three cases, this gauge matrix is just a constant diagonal matrix, for $\gamma(z) = \dfrac{a}{z}$ we first gauge with the matrix $\mathrm{diag}\left(z, \dfrac{1}{z}\right)$ which leaves $\dfrac{1}{z}$-entries on the diagonal, gauging with $R = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$, where $p = \lambda \cdot \dfrac{1}{z}$ gives again off-diagonal form with λ^{-1} as $(1, 2)$-entry.

In all cases, the gauge is balanced by a dressing with some matrix, if we normalize $C(z_*, \lambda) = I = F(z_*, \overline{z_*}, \lambda)$, then this dressing matrix is just the inverse of the gauge matrix at the basepoint z_* (see Section 2.8). As a consequence, since k-noids have umbilics, the monodromy matrices transform via conjugation by this matrix (cf. [7]). But since we are only interested in whether the monodromy matrices corresponding to the two ends are simultaneously unitarizable, this has no effect on our investigations.

For the $(2, 1)$-entry of this new potential η^*, which we denote again by τ (and the $(1, 2)$-entry is again $\nu = \lambda^{-1}$) we thus obtain, respectively:

- the original expression is

$$-\nu\tau = \frac{b_0}{z^2} + \frac{b_1}{(z-1)^2} + \frac{b_2}{(z-a)^2} + \frac{c_0}{z} + \frac{c_1}{z-1} + \frac{c_2}{z-a}, \tag{5.1.5}$$

- applying $z \mapsto 1 - z$ gives

$$-\nu\tau = \frac{b_1}{z^2} + \frac{b_0}{(z-1)^2} + \frac{b_2}{(z-(1-a))^2} + \frac{-c_1}{z} + \frac{-c_0}{z-1} + \frac{-c_2}{z-(1-a)}, \tag{5.1.6}$$

- applying $z \mapsto az$ yields

$$-\nu\tau = \frac{b_0}{z^2} + \frac{b_2}{(z-1)^2} + \frac{b_1}{\left(z-\frac{1}{a}\right)^2} + \frac{ac_0}{z} + \frac{ac_2}{z-1} + \frac{ac_1}{z-\frac{1}{a}}, \tag{5.1.7}$$

- applying $z \mapsto (a-1)z + 1$ results in

$$-\nu\tau = -\frac{b_1}{z^2} + \frac{b_2}{(z-1)^2} + \frac{b_0}{\left(z-\frac{1}{1-a}\right)^2} + \frac{(a-1)c_1}{z} + \frac{(a-1)c_2}{z-1} + \frac{(a-1)c_0}{z-\frac{1}{1-a}}, \tag{5.1.8}$$

- and applying $z \mapsto \frac{a}{z}$, we obtain (after a straightforward computation):

$$-\nu\tau = -\frac{b_\infty}{z^2} + \frac{b_2}{(z-1)^2} + \frac{b_1}{(z-a)^2}$$

$$+\frac{1}{az}(2b_1 + 2b_2 a + c_1 + a^2 c_2) + \frac{1}{z-1}(-2b_2 - ac_2)$$

$$+\frac{1}{a(z-a)}(-2b_1 - c_1), \tag{5.1.9}$$

where $b_\infty = b_0 + b_1 + b_2 + c_1 + ac_2$.

In the cases (5.1.5)-(5.1.8) the calculations are trivial, in the case (5.1.9) we have the following: First, calculating $\gamma^*\eta =: \tilde{\eta}$ gives

$$\tilde{\eta} = \begin{pmatrix} 0 & -\frac{a}{z^2} \\ -\frac{b_0}{a} + \frac{-ab_1}{(z-a)^2} + \frac{-b_2}{a(z-1)^2} + \frac{-c_0}{z} + \frac{c_2}{z(z-1)} + \frac{ac_1}{z(z-a)} & 0 \end{pmatrix} \mathrm{d}z$$

$$= \begin{pmatrix} 0 & -\frac{a}{z^2} \\ -\frac{b_0}{a} + \frac{-ab_1}{(z-a)^2} + \frac{-b_2}{a(z-1)^2} + \frac{-c_0}{z} + \frac{-c_2}{z} + \frac{c_2}{z-1} + \frac{-c_1}{z} + \frac{c_1}{z-a} & 0 \end{pmatrix} \mathrm{d}z$$

$$\overset{\sum c_i = 0}{=} \begin{pmatrix} 0 & -\frac{a}{z^2} \\ \frac{-b_0}{a} + \frac{-ab_1}{(z-a)^2} + \frac{-b_2}{a(z-1)^2} + \frac{c_2}{z-1} + \frac{c_1}{z-a} & 0 \end{pmatrix} \mathrm{d}z.$$

Then, gauging with $P(z) := \mathrm{diag}\left(\sqrt{-\frac{a}{z^2}}, \sqrt{-\frac{a}{z^2}}^{\,-1}\right)$ gives

$$P^{-1}\tilde{\eta}P + P^{-1}\mathrm{d}P$$

$$= \begin{pmatrix} \frac{-1}{z} & \lambda^{-1} \\ \frac{b_0}{z^2} + \frac{a^2 b_1}{z^2(z-a)^2} + \frac{b_2}{z^2(z-1)^2} + \frac{-ac_2}{z^2(z-1)} + \frac{-ac_1}{(z-a)z^2} & \frac{1}{z} \end{pmatrix} \mathrm{d}z$$

and finally gauging with $R = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$, where $p = \lambda \cdot \frac{1}{z}$ results in:

$\begin{pmatrix} 0 & \nu \\ \tau & 0 \end{pmatrix}$ with $\nu = \lambda^{-1}$ and

$$\tau = \frac{b_0}{z^2} + \frac{\frac{2b_1}{a} z + b_1}{z^2} + \frac{-\frac{2b_1}{a} z + 3b_1}{(z-a)^2} + \frac{2b_2 z + 3b_2}{(z-1)^2} + \frac{ac_2 z + ac_2}{z^2}$$

$$+ \frac{-ac_2}{z-1} + \frac{\frac{c_1}{a} z + c_1}{z^2} + \frac{-\frac{c_1}{a}}{z-a}$$

$$= \frac{1}{z^2} \cdot (b_0 + b_1 + b_2 + ac_2 + c_1) + \frac{b_2}{(z-1)^2} + \frac{b_1}{(z-a)^2}$$

$$+ \frac{1}{z}\left(\frac{2b_1}{a} + 2b_2 - \frac{c_1}{a} + ac_2\right) + \frac{1}{z-1} \cdot (-2b_2 - ac_2)$$

$$+ \frac{1}{z-a}\left(-\frac{2b_1}{a} - \frac{c_1}{a}\right)$$

$$= \frac{1}{z^2} \cdot (b_0 + b_1 + b_2 + c_1 + ac_2) + \frac{b_2}{(z-1)^2} + \frac{b_1}{(z-a)^2}$$

$$+ \frac{1}{az}(2b_1 + 2b_2 a + c_1 + a^2 c_2) + \frac{1}{z-1}(-2b_2 - ac_2)$$

$$+ \frac{1}{a(z-a)}(-2b_1 - c_1).$$

5.2.

In the case of trinoids it turned out to be useful to convert everything to a setting, where one exponent of each finite singularity in the equation (3.2.2) is 0. While in the trinoid case this transforms everything into a hypergeometric equation, in the 4-noid case we obtain a Heun equation. We set

$$y = z^{r_0}(z-1)^{r_1}(z-a)^{r_2} w, \tag{5.2.1}$$

where $r_j = \frac{1}{2}(1 + 2\mu_j)$ in view of (3.3.7) and (3.5.6). Here μ_j denotes the eigenvalue of the Delaunay matrix, associated to the end z_j (see, e.g., Remark 5.1).

A somewhat lengthy but straightforward computation leads to the differential equation

$$w'' + \left(\frac{1+2\mu_0}{z} + \frac{1+2\mu_1}{z-1} + \frac{1+2\mu_2}{z-a}\right)w'$$

$$+ \left\{\frac{1}{z}\left[-\frac{1}{2}(1+2\mu_0)(1+2\mu_1) - \frac{1}{2a}(1+2\mu_0)(1+2\mu_2) + c_0\right]\right.$$

$$+ \frac{1}{z-1}\left[\frac{1}{2}(1+2\mu_0)(1+2\mu_1) + \frac{1}{2(1-a)}(1+2\mu_1)(1+2\mu_2) + c_1\right]$$

$$\left. + \frac{1}{z-a}\left[\frac{1}{2a}(1+2\mu_0)(1+2\mu_2) - \frac{1}{2(1-a)}(1+2\mu_1)(1+2\mu_2) + c_2\right]\right\}w = 0. \tag{5.2.2}$$

Remark 5.3. Comparing (5.2.2) to [22], we see that the λ_j's of [22, (3.1)] are exactly the coefficients we call c_j. We will make use of the results obtained in [22] later.

It will be convenient to rewrite the coefficient Ω at w. Again, a somewhat lengthy but otherwise straightforward computation yields:

$$\Omega = \frac{1}{z(z-1)}\left(\frac{1}{2}(1+2\mu_0)(1+2\mu_1) + T_{01}\right)$$

$$+ \frac{1}{z(z-a)}\left(\frac{1}{2}(1+2\mu_0)(1+2\mu_2) + T_{02}\right)$$

$$+ \frac{1}{(z-1)(z-a)}\left(\frac{1}{2}(1+2\mu_1)(1+2\mu_2) + T_{12}\right), \tag{5.2.3}$$

where

$$-T_{01} - \frac{1}{a}T_{02} = c_0,$$

$$T_{01} - \frac{1}{a-1}T_{12} = c_1$$

and

$$\frac{1}{a} T_{02} + \frac{1}{a-1} T_{12} = c_2. \tag{5.2.4}$$

Note that, since $c_0 + c_1 + c_2 = 0$, this is an under-determined linear system for the T_{ij}. For given a, c_0, c_1, c_2, one of the T_{ij} can be chosen arbitrarily.

In the case of trinoids, all the terms involving the number "a" are not present. In this case, we have the hypergeometric equation in the form

$$v'' + \left(\frac{1+2\mu_0}{z} + \frac{1+2\mu_1}{z-1}\right)v' + \frac{A_{01}}{z(z-1)} v = 0. \tag{5.2.5}$$

Here the coefficient at v' is frequently written in the form

$$\frac{1+2\mu_0}{z} + \frac{1+2\mu_1}{z-1} = \frac{(\alpha+\beta+1)z-\gamma}{z(z-1)} \tag{5.2.6}$$

and

$$A_{01} = \alpha\beta = \frac{1}{2}(1+2\mu_0)(1+2\mu_1) + T_{01}. \tag{5.2.7}$$

It turns out, see Section 3.6, that

$$\gamma = 1 + 2\mu_0, \tag{5.2.8}$$

$$\alpha = \frac{1}{2}(1 + 2\mu_0 + 2\mu_1 + 2\mu_\infty^{(0,1)}), \tag{5.2.9}$$

$$\beta = \frac{1}{2}(1 + 2\mu_0 + 2\mu_1 - 2\mu_\infty^{(0,1)}) \tag{5.2.10}$$

satisfy (5.2.6) and (5.2.7) in the case of trinoids. Here $\mu_\infty^{(0,1)}$ is an expression corresponding to μ_3 in (3.6.1)-(3.6.4), i.e., representing the eigenvalue of the Delaunay matrix at the end $z = \infty$, with respect to the differential equation (5.2.5), which represents a trinoid with ends at $z = 0, 1, \infty$.

In the case of 4-noids, we define A_{01} by exactly these expressions:

$$A_{01} = \frac{1}{4}[(1 + 2\mu_0 + 2\mu_1)^2 - 4(\mu_\infty^{(0,1)})^2], \tag{5.2.11}$$

where we choose $\mu_\infty^{(0,1)}$ such that (5.2.8)-(5.2.10) define a CMC-trinoid *without cylindrical ends*.

For this to work, we need to define T_{01} so that we have

$$\begin{aligned} A_{01} &= \frac{1}{4}[(1 + 2\mu_0 + 2\mu_1)^2 - 4(\mu_\infty^{(0,1)})^2] \\ &= \frac{1}{2}(1 + 2\mu_0)(1 + 2\mu_1) + T_{01}. \end{aligned} \tag{5.2.12}$$

A straightforward computation, using only (5.2.12), shows

$$T_{01} = \left(\frac{1}{4} - (\mu_\infty^{(0,1)})^2\right) - \left(\frac{1}{4} - \mu_1^2\right) - \left(\frac{1}{4} - \mu_0^2\right). \tag{5.2.13}$$

From the derivation of α, β, γ_1 in Sections 3.5 and 3.6, we know that $\mu_\infty^{(0,1)}$ is an expression like μ_j, derived from some Delaunay matrix. Therefore, using (3.5.7), we see that T_{01} is of the form

$$T_{01} = (\lambda - \lambda^{-1})^2 t_{01}, \tag{5.2.14}$$

with $t_{01} \in \mathbb{R}$.

Similarly, we consider the differential equations resulting from (5.2.2), if the terms involving the end at $z = 1$ and $z = 0$ respectively, are not present, this yields

$$\begin{aligned} A_{02} &= \frac{1}{4}[(1 + 2\mu_0 + 2\mu_2)^2 - 4(\mu_\infty^{(0,2)})^2] \\ &= \frac{1}{2}(1 + 2\mu_0)(1 + 2\mu_2) + T_{02}, \end{aligned} \tag{5.2.15}$$

$$T_{02} = \left(\frac{1}{4} - (\mu_\infty^{(0,2)})^2\right) - \left(\frac{1}{4} - \mu_2^2\right) - \left(\frac{1}{4} - \mu_0^2\right) \tag{5.2.16}$$

and

$$A_{12} = \frac{1}{4}[(1 + 2\mu_1 + 2\mu_2)^2 - 4(\mu_\infty^{(1,2)})^2]$$

$$= \frac{1}{2}(1 + 2\mu_1)(1 + 2\mu_2) + T_{12}, \tag{5.2.17}$$

$$T_{12} = \left(\frac{1}{4} - (\mu_\infty^{(1,2)})^2\right) - \left(\frac{1}{4} - \mu_1^2\right) - \left(\frac{1}{4} - \mu_2^2\right). \tag{5.2.18}$$

We note that $(\mu_\infty^{(i,j)})^2$ are "Delaunay-like expressions", representing freedom in the choice of parameters. In particular

$$\frac{1}{4} - (\mu_\infty^{(i,j)})^2 = (\lambda - \lambda^{-1})^2 m_{ij}, \quad m_{ij} \in \mathbb{R}, \tag{5.2.19}$$

see equation (3.5.7).

5.3.

For the construction of 4-noids it will be necessary to know the range of the values of

$$\frac{\cos\pi(|X_1| - |X_2| - |X_3|)\cdot\cos\pi(|X_1| - |X_2| + |X_3|)}{\sin 2\pi|X_1|\cdot\sin 2\pi|X_2|} \tag{5.3.1}$$

more precisely. We note that the denominator vanishes, since $0 \le |X_j| \le \frac{1}{2}$, if and only if $|X_1| = 0, \frac{1}{2}$ or $|X_2| = 0, \frac{1}{2}$. But we consider non-cylindrical ends only. Therefore, the denominator vanishes if and only if $|X_1| = \frac{1}{2}$ or $|X_2| = \frac{1}{2}$, which is equivalent with $\lambda = \pm 1$.

If we consider a non-cylindrical end, then $|X_j|$ is an even function of λ. To see this, we first note, that being non-cylindrical is equivalent to $s_j \ne t_j$ for $X_j = s_j\lambda + t_j\lambda^{-1}$. Without restriction, we may assume $t_j > s_j > 0$, this gives

$$|X_j| = \sqrt{(t_j\lambda + s_j\lambda^{-1})\cdot(t_j\lambda^{-1} + s_j\lambda)} = \sqrt{t_j\lambda\left(1 + \frac{s_j}{t_j}\lambda^{-2}\right)\cdot t_j\lambda^{-1}\left(1 + \frac{s_j}{t_j}\lambda^2\right)}$$

$$= t_j \cdot \sqrt{1 + \frac{s_j}{t_j}\lambda^{-2}} \cdot \sqrt{1 + \frac{s_j}{t_j}\lambda^{2}}.$$

Now we use the series expansion for $\sqrt{1+z}$, where $|z| < 1$. We know $|\lambda^2| = |\lambda^{-2}| = 1$, so $\left|\frac{s_j}{t_j}\lambda^{\pm 2}\right| < 1$, consequently we have

$$|X_j| = t_j \cdot \left(1 + \frac{1}{2}\frac{s_j}{t_j}\lambda^{-2} + \cdots\right) \cdot \left(1 + \frac{1}{2}\frac{s_j}{t_j}\lambda^{2} + \cdots\right).$$

Hence, it suffices to consider the case $\lambda = 1$. First we rewrite

$$\sin(2\pi|X_j|) = -\sin(\pi(2|X_j| - 1)) \tag{5.3.2}$$

and

$$\cos(\pi(|X_1| - |X_2| - |X_3|))$$

$$= -\sin\left(\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + \frac{1}{2} - |X_3|\right)\right), \tag{5.3.3}$$

$$\cos(\pi(|X_1| - |X_2| + |X_3|))$$

$$= \sin\left(\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + |X_3| - \frac{1}{2}\right)\right). \tag{5.3.4}$$

Using (5.3.2), (5.3.3) and (5.3.4), we see that the fraction (5.3.1) is equal to

$$-\frac{\pi(2|X_1| - 1)}{\sin\pi(2|X_1| - 1)} \cdot \frac{\pi(2|X_2| - 1)}{\sin\pi(2|X_2| - 1)}$$

$$\times \frac{\sin\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + \frac{1}{2} - |X_3|\right)}{\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + \frac{1}{2} - |X_3|\right)}$$

$$\cdot \frac{\sin\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + |X_3| - \frac{1}{2}\right)}{\pi\left(|X_1| - \frac{1}{2} + \frac{1}{2} - |X_2| + |X_3| - \frac{1}{2}\right)}$$

$$\times \frac{1}{4} \cdot \frac{(2|X_1|-1+1-2|X_2|+1-2|X_3|) \cdot (2|X_1|-1+1-2|X_2|+2|X_3|-1)}{(2|X_1|-1)(2|X_2|-1)}. \tag{5.3.5}$$

We observe that the expression above consists (naturally) of five factors.

We know that none of the first two factors vanishes, since $0 \leq \pi(1-2|X_j|) < \pi$. (Remember that $|X_j| = \mu_j$ and $0 < \mu_j \leq \frac{1}{2}$ for the Delaunay matrices we consider.)

Lemma 5.1. *For* $\lambda = \pm 1$ *the expression* (5.3.5) *is equal to*

$$\frac{1}{4} \cdot \frac{(s_3t_3)^2 - (s_1t_1 - s_2t_2)^2}{s_1t_1 \cdot s_2t_2}, \tag{5.3.6}$$

where $|X_j| = \sqrt{(s_j\lambda^{-1} + t_j\lambda)(s_j\lambda + t_j\lambda^{-1})}$ *for* $j = 1, 2, 3$.

Proof. Since $2|X_j| - 1 = 0$ for $\lambda = \pm 1$, the first four factors in (5.3.5) are equal to 1 for $\lambda = \pm 1$. We also note that in view of (3.5.7) we have $1 - 2|X_j| = \frac{1-4|X_j|^2}{1+2|X_j|} = -4s_jt_j\lambda^{-2}(1-\lambda^2)^2(1+2|X_j|)^{-1}$ for any $\lambda \in \mathbb{S}^1$. Near $\lambda = 1$ we thus have the expansion $1 - 2|X_j| = (1-\lambda^2)^2(-2s_jt_j + O(\varepsilon))$, where $\lambda = 1 \pm \varepsilon$. (Note that $|X_j| = \frac{1}{2}$ for $\lambda = 1$.)

For the last factor in (5.3.5) this yields near $\lambda = 1$:

$$\frac{1}{4} \cdot (1-\lambda^2)^{-2}(-2s_1t_1 + O(\varepsilon))^{-1} \cdot (1-\lambda^2)^{-2}(-2s_2t_2 + O(\varepsilon))^{-1}$$

$$\times (-(1-\lambda^2)^2(-2s_1t_1 + O(\varepsilon)) + (1-\lambda^2)^2(-2s_2t_2 + O(\varepsilon))$$

$$+ (1-\lambda^2)^2(-2s_3t_3 + O(\varepsilon)))$$

$$\times (-(1-\lambda^2)^2(-2s_1t_1 + O(\varepsilon)) + (1-\lambda^2)^2(-2s_2t_2 + O(\varepsilon))$$

$$- (1-\lambda^2)^2(-2s_3t_3 + O(\varepsilon)))$$

$$= \frac{1}{4} \cdot (-2s_1t_1 + O(\varepsilon))^{-1} \cdot (-2s_2t_2 + O(\varepsilon))^{-1}$$

$$\times (2s_1t_1 - O(\varepsilon) - 2s_2t_2 + O(\varepsilon) - 2s_3t_3 + O(\varepsilon))$$

$$\cdot (2s_1t_1 - O(\varepsilon) - 2s_2t_2 + O(\varepsilon) + 2s_3t_3 + O(\varepsilon)).$$

So taking the limit $\varepsilon \to 0$ gives

$$\frac{1}{4} \cdot (4s_1t_1 \cdot s_2t_2)^{-1} \cdot (2s_1t_1 - 2s_2t_2 - 2s_3t_3)(2s_1t_1 - 2s_2t_2 + 2s_3t_3)$$

$$= \frac{1}{4s_1t_1s_2t_2} \cdot ((s_1t_1 - s_2t_2)^2 - s_3^2t_3^2).$$

Since all expressions are even in λ, the same argument applies to $\lambda = -1$. Altogether, we thus obtain at $\lambda = \pm 1$ the expression (5.3.6).

Corollary 5.2. *Equation* (5.3.6) *has the value* 0 *if and only if*

$$s_1t_1 = s_2t_2 + s_3t_3 \text{ or } s_2t_2 = s_3t_3 + s_1t_1. \tag{5.3.7}$$

Moreover, the value 1 *is attained if and only if*

$$s_3t_3 = s_1t_1 + s_2t_2. \tag{5.3.8}$$

Proof. Note that $s_jt_j > 0$, since we have embedded ends, that is we consider asymptotically unduloids.

6. Spherical Polygons and Unitarization

6.1.

In this chapter, we compare our setting to the one used in [3]. We consider the quantity:

$$\nu_j := \frac{1}{2} - |X_j| = \frac{1}{2} - \mu_j. \tag{6.1.1}$$

In [11, Sec. 3.4 and Sec. 3.6], it is shown that the eigenvalues of the monodromy matrix χ_j about the end $z = z_j$ are $-e^{\pm 2\pi i|X_j|}$ and hence we have $\frac{1}{2}\operatorname{tr}\chi_j = -\frac{1}{2}(e^{2\pi i|X_j|} + e^{-2\pi i|X_j|}) = -\cos(2\pi i\mu_j)$. Consequently, we obtain

$$\mathrm{tr}\chi_j = \cos(2\pi\nu_j),$$

since $\cos(\pi - 2\pi|X_j|) = -\cos(2\pi|X_j|)$. (Those results can also be found in [17] and [23] or follow from the considerations in Sections 3 and 4.) A consideration of the eigenvalues at $\lambda = 1$ shows $\prod_{j=1}^{n} \chi_j = I$.

In the rest of this section, we will present geometric interpretations of the quantities ν_j as well as conditions on the ν_j, such that the χ_j are all simultaneously unitarizable.

6.2.

In this section we consider spherical n-gons.

Definition 6.1. A spherical n-gon is a simply closed continuous curve consisting of n geodesic segments on $\mathbb{S}^2$ with lengths in $[0, \pi]$.

Please note, no further constraints are put on a spherical n-gon; in particular, it may be non-convex, self-intersecting, or it may fail to bound an immersed disk.

For the characterization of trinoids of genus $g = 0$ the following inequalities are of crucial importance:

Definition 6.2 (Spherical n-gon inequalities)**.** Let $n \geq 2$ and $\nu_1, \ldots, \nu_n \in \left[0, \frac{1}{2}\right]$. Let $P \subset \{1, \ldots, n\}$ with $|P|$ odd and let $P' = \{1, \ldots, n\} \setminus P$. Then for such a P, the inequality

$$\sum_{i \in P} \nu_i - \sum_{i \in P'} \nu_i - \frac{|P| - 1}{2} \leq 0 \tag{6.2.1}$$

is called an *n-gon inequality*.

The inequalities (6.2.1) were found by Biswas [2]. Of special interest for us is the case $n = 3$. The "spherical triangle inequalities" (case $n = 3$) are:

$$\nu_1 \leq \nu_2 + \nu_3, \tag{6.2.2}$$

$$\nu_2 \leq \nu_1 + \nu_3, \tag{6.2.3}$$

$$\nu_3 \leq \nu_1 + \nu_2, \tag{6.2.4}$$

$$\nu_1 + \nu_2 + \nu_3 \leq 1. \tag{6.2.5}$$

The following basic result has been proven in [2] (see also [3]).

Theorem 6.1 (Spherical n-gon theorem)**.** *Let* $n \geq 2$ *and* $\nu_1, \ldots, \nu_n \in \left[0, \frac{1}{2}\right]$. *The following are equivalent*:

(i) *there exists an* n*-gon on* $\mathbb{S}^2$, *whose sides have lengths* $(2\pi\nu_1, \ldots, 2\pi\nu_n)$,

(ii) $\nu_1, \ldots, \nu_n$ *satisfy the* n*-gon inequalities* (6.2.1).

Note that for a nondegenerate spherical triangle, the inequalities (6.2.2)-(6.2.4) are strict.

The relation between the results of Section 3.6 and the result above is

Theorem 6.2. *Let* ν_i *be defined as in* (6.1.1). *Then the inequalities in* (3.6.5) *are satisfied if and only if the* ν_i *satisfy the spherical triangle inequalities* (6.2.2)-(6.2.5). *Hence, the monodromy matrices are simultaneously unitarizable via* r*-dressing for some* $r \in (0, 1)$ *sufficiently close to* 1 *if and only if the* ν_i *satisfy* (6.2.2)-(6.2.5).

Proof. Since $|X_i| = \frac{1}{2} - \nu_i$, expression (5.3.1) becomes

$$\frac{\cos\pi(|X_1| - |X_2| - |X_3|) \cdot \cos\pi(|X_1| - |X_2| + |X_3|)}{\sin 2\pi|X_1| \cdot \sin 2\pi|X_2|}$$

$$= \frac{\cos\pi\left(\frac{1}{2} - \nu_1 - \left(\frac{1}{2} - \nu_2\right) - \left(\frac{1}{2} - \nu_3\right)\right) \cdot \cos\pi\left(\frac{1}{2} - \nu_1 - \left(\frac{1}{2} - \nu_2\right) + \frac{1}{2} - \nu_3\right)}{\sin 2\pi\left(\frac{1}{2} - \nu_1\right) \cdot \sin 2\pi\left(\frac{1}{2} - \nu_2\right)}$$

$$= \frac{\cos \pi\left(-\frac{1}{2} - \nu_1 + \nu_2 + \nu_3\right) \cdot \cos \pi\left(\frac{1}{2} - \nu_1 + \nu_2 - \nu_3\right)}{\sin(-2\pi\nu_1 + \pi) \cdot \sin(-2\pi\nu_2 + \pi)}$$

$$= \frac{\cos \pi\left(\nu_2 + \nu_3 - \nu_1 - \frac{1}{2}\right) \cdot \cos \pi\left(\nu_2 - \nu_1 - \nu_3 + \frac{1}{2}\right)}{\sin(-2\pi\nu_1 + \pi) \cdot \sin(-2\pi\nu_2 + \pi)}.$$

Using the identities $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$ and $\sin(x + \pi) = -\sin(x) = \sin(x - \pi)$, we obtain

$$\frac{\cos \pi(|X_1| - |X_2| - |X_3|) \cdot \cos \pi(|X_1| - |X_2| + |X_3|)}{\sin 2\pi|X_1| \cdot \sin 2\pi|X_2|}$$

$$= \frac{\sin \pi(\nu_2 + \nu_3 - \nu_1) \cdot (-\sin \pi(\nu_2 - \nu_1 - \nu_3))}{-\sin(-2\pi\nu_1) \cdot (-\sin(-2\pi\nu_2))}$$

$$= \frac{\sin \pi(\nu_2 + \nu_3 - \nu_1)}{\sin 2\pi\nu_2} \cdot \frac{\sin \pi(\nu_1 + \nu_3 - \nu_2)}{\sin 2\pi\nu_1}. \tag{6.2.6}$$

• **Case 1.** $\sin(2\pi\nu_1) \neq 0$ and $\sin(2\pi\nu_2) \neq 0$.

In this case, we have $\nu_1, \nu_2 \neq 0, \frac{1}{2}$. Therefore, $0 < \nu_1, \nu_2 < \frac{1}{2}$ and no denominator is negative. Consequently the term (6.2.6) is nonnegative if and only if one of the following four cases occurs:

(a) $\nu_1 < \nu_2 + \nu_3$ and $\nu_2 < \nu_1 + \nu_3$,

(b) $\nu_1 > \nu_2 + \nu_3$ and $\nu_2 > \nu_1 + \nu_3$,

(c) $\nu_1 = \nu_2 + \nu_3$,

(d) $\nu_2 = \nu_1 + \nu_3$.

But in case (b), we derive $\nu_1 > \nu_3 + \nu_2 > \nu_3 + \nu_1 + \nu_3 = \nu_1 + 2\nu_3$, a contradiction (because $\nu_3 \geq 0$).

In case (c), i.e., $\nu_1 = \nu_2 + \nu_3$, we also have $\nu_1 + \nu_3 - \nu_2 = \nu_2 + \nu_3 + \nu_3 - \nu_2 = 2\nu_3 \geq 0$, i.e., $\nu_2 \leq \nu_1 + \nu_3$.

In case (d), i.e., $\nu_2 = \nu_1 + \nu_3$, we obtain $\nu_2 + \nu_3 - \nu_1 = 2\nu_3 \geq 0$, that is $\nu_1 \leq \nu_2 + \nu_3$. Altogether, we have shown that the expression (6.2.6) is nonnegative if and only if $\nu_1 \leq \nu_2 + \nu_3$ and $\nu_2 \leq \nu_1 + \nu_3$, i.e., if and only if (6.2.2) and (6.2.3) are satisfied.

Next we characterize, when (6.2.6) is smaller or equal to one. Note that

$$\frac{\sin \pi(\nu_2 + \nu_3 - \nu_1) \cdot \sin \pi(\nu_1 + \nu_3 - \nu_2)}{\sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2} \leq 1$$

is equivalent to

$$\sin \pi(\nu_2 + \nu_3 - \nu_1) \cdot \sin \pi(\nu_1 + \nu_3 - \nu) \leq \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2.$$

Using standard trigonometric identities, we obtain

$$\begin{aligned}&\frac{1}{2}[\cos \pi(\nu_2 + \nu_3 - \nu_1 - (\nu_1 + \nu_3 - \nu_2))\\ &\quad - \cos \pi(\nu_2 + \nu_3 - \nu_1 + (\nu_1 + \nu_3 - \nu_2))]\\ &\leq \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2,\end{aligned}$$

whence we have

$$\cos(2\pi\nu_2 - 2\pi\nu_1) - \cos 2\pi\nu_3 \leq 2 \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2.$$

Using again standard trigonometric identities, we derive

$$\cos 2\pi\nu_1 \cdot \cos 2\pi\nu_2 + \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2 - \cos 2\pi\nu_3 \leq 2 \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2,$$

thus

$$\cos 2\pi\nu_1 \cdot \cos 2\pi\nu_2 - \sin 2\pi\nu_1 \cdot \sin 2\pi\nu_2 \leq \cos 2\pi\nu_3,$$

which is equivalent to

$$\cos(2\pi\nu_1 + 2\pi\nu_2) \leq \cos 2\pi\nu_3. \tag{6.2.7}$$

Clearly $2\pi(\nu_1 + \nu_2) \in (0, 2\pi)$ and $2\pi\nu_3 \in (0, \pi)$. Since the cosine function is strictly decreasing in $(0, \pi)$, strictly increasing in $(\pi, 2\pi)$ and has reflective symmetry about $x = \pi$, condition (6.2.7) is satisfied if and only if either

(a) $\nu_1 + \nu_2 \leq \frac{1}{2}$ and $\nu_3 \leq \nu_1 + \nu_2$ or (b) $\nu_1 + \nu_2 \geq \frac{1}{2}$ and $\nu_3 \leq \frac{1}{2} - \left(\nu_1 + \nu_2 - \frac{1}{2}\right) = 1 - (\nu_1 + \nu_2)$, that is, $\nu_1 + \nu_2 + \nu_3 \leq 1$.

So condition (6.2.7) is satisfied, iff (6.2.4) and (6.2.5) are satisfied.

- **Case 2.** $\sin(2\pi\nu_1) = 0$ or $\sin(2\pi\nu_2) = 0$.

In this limiting case (3.6.5) remains true, if also a factor of the nominator vanishes. It suffices to consider the case $\nu_1 = 0, \frac{1}{2}$, since (6.2.6) is symmetric in ν_1, ν_2.

First examine what happens if $\nu_1 = 0$. Assume (3.6.5) is valid, then we have either $\nu_2 + \nu_3 = 0$, or $\nu_3 - \nu_2 = 0$. Since $\nu_2, \nu_3 \geq 0$, the first equation can only be true if $\nu_2 = 0 = \nu_3$, i.e., we only need to consider $\nu_3 = \nu_2$, in this case, clearly the triangle inequalities (6.2.2)-(6.2.5) are satisfied.

Conversely, if the triangle inequalities are satisfied for $\nu_1 = 0$, we obtain $\nu_2 = \nu_3$, hence (3.6.5) is valid.

Next consider $\nu_1 = \frac{1}{2}$, if (3.6.5) shall be satisfied, then either $\nu_2 + \nu_3 = \frac{1}{2}$ or $\nu_2 = \nu_3 + \frac{1}{2}$. Since $\nu_2, \nu_3 \leq \frac{1}{2}$ the second equality implies $\nu_3 = 0$, thus only the first equality needs to be considered. But then one easily checks that the triangle inequalities are satisfied.

Conversely, if $\nu_1 = \frac{1}{2}$ and the triangle inequalities are satisfied, then one obtains from (6.2.2) and (6.2.5), that $\frac{1}{2} \leq \nu_2 + \nu_3$ and $\nu_2 + \nu_3 \leq \frac{1}{2}$, hence $\nu_2 + \nu_3 = \frac{1}{2}$.

6.3.

In the last section, we have proven, that the spherical triangle inequalities for ν_i are equivalent to the simultaneous unitarizibility of matrices $M_i \in \mathrm{SL}_2\mathbb{C}$ with $\mathrm{tr} M_i = \cos 2\pi\nu_i$ and $\prod M_i = I$. There is another, more direct approach for this proof, which we will sketch now (see, e.g., [3], or [23]). In the case of three matrices, the ν_i can be interpreted geometrically.

First we note:

Lemma 6.3 [23]. *Let* $M \in \mathrm{SU}_2$ *with eigenvalues* $\exp(\pm 2\pi i\nu)$, *where* $0 \leq \nu \leq \frac{1}{2}$. *Then* M *can be written as* $M = \cos(2\pi\nu)I + \sin(2\pi\nu)A = \exp(2\pi\nu A)$, *with a uniquely determined* $A \in \mathfrak{su}_2$, *such that* $\det A = 1$.

Conjugation with M describes a rotation about A by an angle of $2\pi\nu$, *therefore we call A the axis of M.*

Proof. Let $M = \begin{pmatrix} p_1 + ip_2 & q_1 + iq_2 \\ -q_1 + iq_2 & p_1 - ip_2 \end{pmatrix}$. Since the eigenvalues of M are $\exp(\pm 2\pi i\nu)$, we have $2\mathrm{tr}M = 2p_1 = 2\cos(2\pi\nu)$, and therefore $p_1 = \cos(2\pi\nu)$. Consequently, $M - \cos(2\pi\nu)I = \begin{pmatrix} ip_2 & q_1 + iq_2 \\ -q_1 + iq_2 & -ip_2 \end{pmatrix}$, which obviously is in $\mathfrak{su}_2$ and $\det(M - \cos(2\pi\nu)I) = p_2^2 + q_1^2 + q_2^2$. On the other hand, we have $1 = \det(M_k) = \cos^2(2\pi\nu) + p_2^2 + q_1^2 + q_2^2$, thus $\sin^2(2\pi\nu) = 1 - \cos^2(2\pi\nu) = p_2^2 + q_1^2 + q_2^2 = \det(M - \cos(2\pi\nu)I)$, therefore, we have $M = \cos(2\pi\nu)I + \sin(2\pi\nu)A$, where $\det A = 1$, and $A \in \mathfrak{su}_2$. The uniqueness of A is obvious.

Since $\det A = 1$ and $\mathrm{tr}A = 0$, for the characteristic polynomial we obtain $A^2 - \mathrm{tr}A \cdot A + \det A \cdot I = A^2 + I = 0$, so $A^2 = -I$. An easy calculation then shows $\exp(2\pi\nu A) = \cos(2\pi\nu)I + \sin(2\pi\nu)A = M$.

Every rigid motion can be expressed as a conjugation with a matrix $M \in \mathrm{SU}_2$. Since clearly $MAM^{-1} = \exp(2\pi\nu A)A\exp(-2\pi\nu A) = I$, the axis A is left invariant, hence M describes a rotation about A, the rotation angle can be calculated explicitly (cf. [13, Section 2.4]), and is given by $2\pi\nu$.

Before we give a geometric interpretation of the ν_i, we state

Lemma 6.4. *Let* $\pm I \neq M_1, M_2, M_3 \in \mathrm{SU}_2$ *such that* $M_1 M_2 M_3 = I$, *and let* A_k *be the axis of* M_k, $k = 1, 2, 3$. *Then the axes* A_k, $k = 1, 2, 3$, *are linearly independent iff* $[M_i, M_j] \neq 0$ *for* $i \neq j$. *In this case, we call* M_1, M_2, M_3 *nondegenerate.*

Proof. First assume that, e.g., $[M_1, M_2] = 0$. An easy computation shows that this is equivalent to $[A_1, A_2] = 0$, since $\sin(2\pi\nu_1)\sin(2\pi\nu_2) \neq 0$ (otherwise $M_1 = \pm I$ or $M_2 = \pm I$). But since $A_k \neq 0$ (otherwise $M_k = \pm I$), $A_1 A_2 = A_2 A_1$ is equivalent to $A_2 = r_1 I + r_2 A_1$, so $A_1 A_2 = A_1(r_1 I + r_2 A_1) = r_1 A_1 - r_2 I$, and $I = r_1^{-1} A_2 - \frac{r_2}{r_1} A_1$ follows. Therefore

$$A_1 A_2 = r_1 A_1 - r_2 \cdot \left(\frac{1}{r_1} A_2 - \frac{r_2}{r_1} A_1\right) = \left(r_1 + \frac{r_2^2}{r_1}\right) A_1 - \frac{r_2}{r_1} A_2.$$

Since multiplying out $M_1 M_2 M_3 = I$ yields

$$\begin{aligned} A_3 &= (\cos 2\pi\nu_1 \cos 2\pi\nu_2 - \cos 2\pi\nu_3) I - \cos 2\pi\nu_2 \sin 2\pi\nu_1 A_1 \\ &\quad - \cos 2\pi\nu_1 \sin 2\pi\nu_2 A_2 + \sin 2\pi\nu_1 \sin 2\pi\nu_2 A_2 A_1, \end{aligned}$$

the argument above shows $A_3 \in \mathrm{span}(A_1, A_2)$.

Conversely, let $A_3 \in \mathrm{span}(A_1, A_2)$. Without restriction, we may assume that A_1 and A_2 both lie in the x-y-plane, hence they are off-diagonal matrices. Using $M_3 = M_2^{-1} M_1^{-1}$, we derive

$$\begin{aligned} \sin(2\pi\nu_3) A_3 &= (\cos(2\pi\nu_1)\cos(2\pi\nu_2) - \cos(2\pi\nu_3)) I \\ &\quad + \sin(2\pi\nu_1)\sin(2\pi\nu_2) A_2 A_1 - \cos(2\pi\nu_2)\sin(2\pi\nu_1) A_1 \\ &\quad - \cos(2\pi\nu_1)\sin(2\pi\nu_2) A_2. \end{aligned} \tag{6.3.1}$$

Since A_1 and A_2 are off-diagonal, the product $A_2 A_1$ is a diagonal matrix. Furthermore, since A_3 is supposed to be a linear combination of A_1 and A_2, it must also be an off-diagonal matrix. Hence, the diagonal entries on the right hand side of equation (6.3.1) must vanish. This gives $(\cos(2\pi\nu_1)\cos(2\pi\nu_2) - \cos(2\pi\nu_3))I + \sin(2\pi\nu_1)\sin(2\pi\nu_2)A_2A_1 = 0$, thus

$$A_2A_1 = \frac{\cos(2\pi\nu_3) - \cos(2\pi\nu_2)\cos(2\pi\nu_2)}{\sin(2\pi\nu_1)\sin(2\pi\nu_2)} I. \tag{6.3.2}$$

On the other hand, the relation $M_3^{-1} = M_1M_2$ shows

$$\begin{aligned}\sin(2\pi\nu_3)A_3 = {} & (\cos(2\pi\nu_3) - \cos(2\pi\nu_1)\cos(2\pi\nu_2))I \\ & - \sin(2\pi\nu_1)\sin(2\pi\nu_2)A_1A_2 - \cos(2\pi\nu_2)\sin(2\pi\nu_1)A_1 \\ & - \cos(2\pi\nu_2)\sin(2\pi\nu_1)A_2. \end{aligned} \tag{6.3.3}$$

Again, the diagonal entries in (6.3.3) must be zero, thus we obtain

$$A_2A_1 = \frac{\cos(2\pi\nu_3) - \cos(2\pi\nu_2)\cos(2\pi\nu_2)}{\sin(2\pi\nu_1)\sin(2\pi\nu_2)} I = A_1A_2. \tag{6.3.4}$$

And consequently $M_1M_2 = M_2M_1$.

Theorem 6.5 [3]. *Let $M_1, M_2, M_3 \in \mathrm{SU}_2$ be nondegenerate, such that $M_1M_2M_3 = I$. Let A_k be the axes of M_k and $x_k = \cos(2\pi\nu_k)$. Let P_k denote the planes perpendicular to A_k and containing the center of $\mathbb{S}^2$. Then the side lengths of the triangle formed by P_{ij} with angles $\frac{1}{2}\mathrm{tr}A_iA_j$ are $2\pi\nu_k$.*

Consequently, the ν_i satisfy the (strict) triangle-inequalities (6.2.2)-(6.2.5).

Proof. First note that a spherical triangle is completely determined by the three inbound angles, i.e., by $\frac{1}{2}\mathrm{tr}A_1A_2$, $\frac{1}{2}\mathrm{tr}A_1A_3$ and $\frac{1}{2}\mathrm{tr}A_2A_3$ and the triangle can be constructed using the law of cosines for the angles of a spherical triangle, which is a consequence of the law of cosines for the sides of a spherical triangle, see [24, Kap. 10]. Thus it suffices to show that $2\pi\nu_1$, $2\pi\nu_2$ and $2\pi\nu_3$ all satisfy the law of cosines for the sides of a spherical triangle. Also note that M_1, M_2, M_3 are nondegenerate, hence

A_1, A_2, A_3 are linearly independent and span a nondegenerate spherical triangle. Therefore, the side lengths fulfill the strict spherical triangle inequalities.

Write $M_k = \cos(2\pi\nu_k)I + \sin(2\pi\nu_k)A_k$. Then $M_i^{-1} = M_j M_k$, so

$$\begin{aligned}&\cos(2\pi\nu_i)I - \sin(2\pi\nu_i)A_i\\ &= (\cos(2\pi\nu_j)I + \sin(2\pi\nu_j)A_j)(\cos(2\pi\nu_k)I + \sin(2\pi\nu_k)A_k).\end{aligned}$$

Multiplying yields

$$\begin{aligned}&\cos(2\pi\nu_i)I - \sin(2\pi\nu_i)A_i\\ &= \cos(2\pi\nu_j)\cos(2\pi\nu_k)I + \cos(2\pi\nu_j)\sin(2\pi\nu_k)A_k\\ &\quad + \sin(2\pi\nu_j)\cos(2\pi\nu_k)A_j + \sin(2\pi\nu_j)\sin(2\pi\nu_k)A_jA_k.\end{aligned}$$

Since $\mathrm{tr}A_j = \mathrm{tr}A_k = 0$, taking the half-trace now gives

$$\cos(2\pi\nu_i) = \cos(2\pi\nu_j)\cos(2\pi\nu_k) + \frac{1}{2}\sin(2\pi\nu_j)\sin(2\pi\nu_k)\mathrm{tr}A_jA_k. \tag{6.3.5}$$

This is just the law of cosines for the sides of spherical triangle, finishing the proof.

Theorem 6.5 immediately shows:

Corollary 6.6. *Let $M_1, M_2, M_3 \in \mathrm{SL}_2\mathbb{C}$ be simultaneously unitarizable with $M_1M_2M_3 = I$. Let ν_k be defined by*

$$\frac{1}{2}\mathrm{tr}M_k = \cos 2\pi\nu_k.$$

Then ν_1, ν_2, ν_3 satisfy the triangle inequalities (6.2.2)-(6.2.5).

Proof. Assume that M_1, M_2, M_3 are simultaneously unitarizable. Let P be a unitarizer of the matrices M_1, M_2, M_3, that is $PM_iP^{-1} \in \mathrm{SU}_2$. Let $B_i = PM_iP^{-1}$. The trace is invariant under conjugation of a matrix, so

$$\frac{1}{2}\operatorname{tr}B_k = \frac{1}{2}\operatorname{tr}M_k = \cos(2\pi\nu_k).$$

By Theorem 6.5 the ν_k satisfy the triangle inequalities.

Remark 6.1. (1) Theorem 6.5 together with Lemma 6.4 suggests that equality in the triangle inequalities corresponds exactly to the case of commuting matrices (regarding equality as a type of limiting process of the construction carried out in the proof of Theorem 6.5). Indeed, this is the case, those results (using completely different techniques) can be found in [25].

(2) Note, given a spherical triangle, the "reverse construction" of the proof of Theorem 6.5 does not lead to unitary matrices M_i with $M_1M_2M_3 = I$, more precisely: Take a spherical triangle with sides a_i, whose side lengths are $2\pi\nu_i$. Let A_k be the unit vectors perpendicular to the plane containing a_k. Now consider the following map M: First, rotate about A_3 by the angle $2\pi\nu_3$, then about A_2 with angle $2\pi\nu_2$ and lastly, rotate about A_1 with angle $2\pi\nu_1$. This map is described by the matrix $M_1M_2M_3 \in \mathrm{SU}_2$, where $M_k = \cos 2\pi\nu_k + \sin 2\pi\nu_k A_k$, see Lemma 6.3.

In general, M is not the identity map, but a rotation, whose axis is given by the vector through the vertex p_3 opposite to a_3. Hence in general $M_1M_2M_3 \neq I$. For example, consider a spherical triangle, where $\nu_1 = \nu_2 = \frac{1}{4}$, and $\nu_3 \neq \frac{1}{2}$, i.e., where p_3 is the pole of the geodesic a_3. Then p_2 (the vertex opposite of ν_2) is fixed by M, but $M(p_3) \neq p_3$! (Since p_3 is the pole of M_3, it is invariant under this motion, M_2 moves p_3 along the geodesic containing a_2, and clearly $M_2(p_3)$ does not lie in the plane containing a_1, hence by M_1 the point $M_2(p_3)$ is moved in a plane parallel to the plane containing a_1, thus $M_1(M_2(p_3)) \notin a_1$, so $p_3 \neq M_1(M_2(p_3)) = M(p_3)$, because $p_3 \in a_1 \cap a_2$.)

It would be interesting to know whether there are additional conditions on the spherical triangle, such that the above construction yields $M_1M_2M_3 = I$.

Unfortunately, in the case of n-gons for $n > 3$, the n-gon inequalities do not provide a sufficient condition for the unitarizability of the monodromy matrices, but at least we have (Cor. 3.9 in [3]):

Theorem 6.7. *Let* $M_1, \ldots, M_n \in \mathrm{SL}_2\mathbb{C}$ *with* $\prod M_k = I$. *Suppose that* $M_1, \ldots, M_n$ *are simultaneously unitarizable. Let* ν_k *be defined by* $\frac{1}{2}\mathrm{tr} M_k = \cos 2\pi\nu_k$. *Then* $\nu_1, \ldots, \nu_n$ *satisfy the n-gon inequalities* (6.2.1).

The proof is by induction on n, where Theorem 6.5 provides the starting point for the induction.

Up to now, no additional conditions on the ν_i are known, that would characterize equivalently the simultaneous unitarizability of the monodromy matrices.

7. Unitarizability in the Asymptotic Limit

7.1.

Since for $n \geq 4$ the spherical inequalities do not provide sufficient conditions for the simultaneous unitarizability of the monodromy matrices and we do not know other conditions for the simultaneous unitarizability, it does not seem advisable to work with the monodromy matrices of the 4-noid directly. Instead, we consider the trinoid, which is created in the limit $a \to \infty$. For this trinoid, we are able to control the behaviour of the monodromy matrices by the ν_i. If in (3.6.5) we could ensure the strict inequalities, then we could use the results of the trinoid also in the case of any a large enough. So we seek conditions on the ν_i that provide strict inequalities. With the help of Corollary 5.2, we can show

Corollary 7.1. *Assume* $\nu_1 + \nu_2 + \nu_3 < 1$ *for all* $\lambda \in \mathbb{S}^1$, $\nu_1 < \nu_2 + \nu_3$, $\nu_2 < \nu_1 + \nu_3$ *and* $\nu_3 < \nu_1 + \nu_2$ *for all* $\lambda \in \mathbb{S}^1 \setminus \{\pm 1\}$. *Then in* (3.6.5) *we have strict inequality from above and from below for* $\lambda \in \mathbb{S}^1$.

Note, since $\nu_j = \frac{1}{2} - |X_j|$, we have $\nu_j = 0$ for $\lambda = \pm 1$. So the strict inequalities $\nu_i < \nu_j + \nu_k$, $\{i, j, k\} = \{1, 2, 3\}$ can only be satisfied for $\lambda \neq \pm 1$. We would also like to reiterate that we consider exclusively non-cylindrical ends, i.e., $|X_j| \neq 0$ for all $\lambda \in \mathbb{S}^1$.

Proof. (1) First we show that equality in (3.6.5) can only occur if equality occurs in the triangle inequalities (6.2.2)-(6.2.5).

In (3.6.5), we have equality from below if and only if

$$\cos \pi(|X_1| - |X_2| - |X_3|) = 0 \tag{7.1.1}$$

or

$$\cos \pi(|X_1| - |X_2| + |X_3|) = 0. \tag{7.1.2}$$

Since $0 < |X_j| \leq \frac{1}{2}$, we have $-1 < |X_1| - |X_2| - |X_3| < \frac{1}{2}$, i.e., it is sufficient to consider the interval $\left]-\pi, \frac{\pi}{2}\right[$ in (7.1.1). Hence, (7.1.1) is equivalent to $\pi(|X_1| - |X_2| - |X_3|) = -\frac{\pi}{2}$ and thus $\nu_1 = \nu_2 + \nu_3$.

Since $-\frac{1}{2} < |X_1| - |X_2| + |X_3| < 1$, equation (7.1.2) is equivalent to $\pi(|X_1| - |X_2| + |X_3|) = \frac{\pi}{2}$, hence $\nu_2 = \nu_1 + \nu_3$.

If however, we have equality from above, this implies, compare (3.6.6), $\cos(2\pi|X_1| - 2\pi|X_2|) + \cos(2\pi|X_3|) = 2\sin(2\pi|X_1|)\sin(2\pi|X_2|)$, which is equivalent to $\cos(2\pi|X_1| - 2\pi|X_2|) + \cos(2\pi|X_3|) = \cos(2\pi|X_1| - 2\pi|X_2|) - \cos(2\pi|X_1| + 2\pi|X_2|)$, hence $\cos(2\pi|X_1| - 2\pi|X_2|) + \cos 2\pi|X_3| = 0$. From this, we obtain $2\pi|X_1| + 2\pi|X_2| = \pi - 2\pi|X_3|$, whence $-2\nu_1 + 1 - 2\nu_2 + 1 = 1 + 2\nu_3 - 1$, i.e., $\nu_1 + \nu_2 + \nu_3 = 1$.

(2) Now we show that the equalities for $\lambda = \pm 1$ in Corollary 5.2 imply that the inequalities in (6.2.2)-(6.2.5) are not strict for all $\lambda \in \mathbb{S}^1 \setminus \{\pm 1\}$.

By equation (3.5.7), we have $\frac{1}{4} - |X_i|^2 = -s_i t_i (\lambda^{-1} - \lambda)^2$, hence $|X_i| = \sqrt{\frac{1}{4} + s_i t_i (\lambda^{-1} - \lambda)^2}$. For ν_i, we thus obtain

$$\nu_i = \frac{1}{2} - |X_i| = \frac{1}{2} - \sqrt{\frac{1}{4} + s_i t_i (\lambda^{-1} - \lambda)^2} = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4 s_i t_i (\lambda^{-1} - \lambda)^2}.$$

Since $s_i t_i \leq \frac{1}{16}$ and $|(\lambda^{-1} - \lambda)^2| = |2i \cdot \mathrm{Im}(\lambda)|^2 < 4$ for $\lambda \in \mathbb{S}^1$ sufficiently close to ± 1, we have $|s_i t_i (\lambda^{-1} - \lambda)^2| < 4$ and thus we may use the series expansion $\sqrt{1 + \varepsilon} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots$, where $\varepsilon = 4 s_i t_i (\lambda^{-1} - \lambda)^2$, this gives

$$\begin{aligned}\nu_i &= \frac{1}{2} - \frac{1}{2} \cdot \left(1 + \frac{1}{2}(4 s_i t_i (\lambda^{-1} - \lambda)^2) - \frac{1}{8} \cdot (4 s_i t_i (\lambda^{-1} - \lambda)^2)^2\right) + \mathcal{O}((\lambda^{-1} - \lambda)^6) \\ &= -\frac{1}{4}(4 s_i t_i (\lambda^{-1} - \lambda)^2) + \frac{1}{16}(4 s_i t_i (\lambda^{-1} - \lambda)^2)^2 + \mathcal{O}((\lambda^{-1} - \lambda)^6) \\ &= -s_i t_i (\lambda^{-1} - \lambda)^2 + s_i^2 t_i^2 (\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6).\end{aligned} \tag{7.1.3}$$

Note that $s_i t_i > 0$ (we only consider embedded ends, i.e., asymptotically unduloids) and $(\lambda^{-1} - \lambda)^2 < 0$. For example, the inequality $\nu_1 < \nu_2 + \nu_3$ then reads

$$\begin{aligned}&-s_1 t_1 (\lambda^{-1} - \lambda)^2 + s_1^2 t_1^2 (\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6) \\ &< -s_2 t_2 (\lambda^{-1} - \lambda)^2 + s_2^2 t_2^2 (\lambda^{-1} - \lambda)^4 - s_3 t_3 (\lambda^{-1} - \lambda)^2 \\ &\quad + s_3^2 t_3^2 (\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6).\end{aligned} \tag{7.1.4}$$

Hence we obtain

$$\begin{aligned}&(s_1^2 t_1^2 - s_2^2 t_2^2 - s_3^2 t_3^2)(\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6) \\ &< (s_1 t_1 - s_2 t_2 - s_3 t_3) \cdot (\lambda^{-1} - \lambda)^2 + \mathcal{O}((\lambda^{-1} - \lambda)^6).\end{aligned} \tag{7.1.5}$$

Suppose now, we had $s_1 t_1 = s_2 t_2 + s_3 t_3$, then

$$((s_2 t_2 + s_3 t_3)^2 - s_2^2 t_2^2 - s_3^2 t_3^2)(\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6)$$

$$< 0 + \mathcal{O}((\lambda^{-1} - \lambda)^6) \tag{7.1.6}$$

follows. Hence we would obtain

$$(s_2^2 t_2^2 + s_3^2 t_3^2 + 2 s_2 t_2 s_3 t_3 - s_2^2 t_2^2 - s_3^2 t_3^2)(\lambda^{-1} - \lambda)^4$$

$$< 0 + \mathcal{O}((\lambda^{-1} - \lambda)^6), \tag{7.1.7}$$

that is

$$2 s_2 t_2 s_3 t_3 (\lambda^{-1} - \lambda)^4 + \mathcal{O}((\lambda^{-1} - \lambda)^6) < 0. \tag{7.1.8}$$

But $s_2 t_2 s_3 t_3$ as well as $(\lambda^{-1} - \lambda)^4$ are positive, so for $|(\lambda^{-1} - \lambda)|$ small enough, we obtain a contradiction. The other cases follow accordingly.

7.2.

In Section 5.2, we have determined T_{01} by $(\mu_\infty^{(0,1)})^2$, T_{02} by $(\mu_\infty^{(0,2)})^2$ and T_{12} by $(\mu_\infty^{(1,2)})^2$, respectively. In view of (5.2.19), each of these parameters represents one real degree of freedom. In this section, we discuss restrictions, which are necessary for the existence of 4-noids.

First we consider T_{01}. In the limit $a \to \infty$, the coefficients of (5.2.2) reduce to give a hypergeometric equation. To this equation we can apply (3.6.5), where $|X_j|$ are to be replaced by μ_j (compare Remark 5.1).

We know from Corollary 7.1, that, for $\lambda \in \mathbb{S}^1$, we have strict inequalities in (3.6.5), if for μ_0, μ_1 and $\mu_\infty^{(0,1)}$ and the corresponding ν_0, ν_1 and $\nu_\infty^{(0,1)} = \frac{1}{2} - \mu_\infty^{(0,1)}$, we have

$$\nu_0 < \nu_1 + \nu_\infty^{(0,1)}, \quad \nu_1 < \nu_0 + \nu_\infty^{(0,1)}, \quad \nu_\infty^{(0,1)} < \nu_0 + \nu_1, \quad \text{and}$$

$$\nu_0 + \nu_1 + \nu_\infty^{(0,1)} < 1 \tag{7.2.1}$$

for $\lambda \neq \pm 1$.

In view of [22], which inter alia states, that the so-called connection coefficients (coefficients which specify how the power series solutions around different singularities are related) depend holomorphically on the parameters of the Heun or Hypergeometric equation, we obtain that the μ_i depend holomorphically on the parameter a. Hence, taking $a \to \infty$ transforms the Heun equation (with four singularities) holomorphically to a hypergeometric equation (with three singularities) with parameters μ_0, μ_1, $\mu_\infty^{(0,1)}$.

In particular, for a sufficiently large, the inequality (3.6.5) for the parameters μ_0, μ_1, $\mu_\infty^{(0,1)}$ in place of $|X_1|$, $|X_2|$, $|X_3|$ is still satisfied. Hence, we obtain

Theorem 7.2. *If* (7.2.1) *is satisfied, then the monodromy matrices for the singularities at* $z = 0$ *and* $z = 1$ *are simultaneously unitarizable via r-dressing for some* $r \in (0, 1)$ *sufficiently close to* 1 *if a is sufficiently large.*

7.3.

Next we apply the transformation $z \mapsto az$. Then in the original Fuchsian equation we have the coefficients b_0, ac_0, b_2, ac_2, and b_1, ac_1 at the singularities $z = 0, 1$ and $\frac{1}{a}$ respectively (see Section 5.1). The transformation (5.2.1) then yields the Heun equation

$$w'' + \left(\frac{1 + 2\mu_0}{z} + \frac{1 + 2\mu_2}{z - 1} + \frac{1 + 2\mu_1}{z - \frac{1}{a}} \right) w' + \widetilde{\Omega} w = 0, \tag{7.3.1}$$

where

$$\widetilde{\Omega} = \frac{1}{z}\left[-\frac{1}{2}(1 + 2\mu_0)(1 + 2\mu_2) - \frac{a}{2}(1 + 2\mu_0)(1 + 2\mu_1) + ac_0 \right]$$

$$+\frac{1}{z-1}\left[\frac{1}{2}(1+2\mu_0)(1+2\mu_2)+\frac{1}{2\left(1-\frac{1}{a}\right)}(1+2\mu_2)(1+2\mu_1)+ac_2\right]$$

$$+\frac{1}{z-\frac{1}{a}}\left[\frac{a}{2}(1+2\mu_0)(1+2\mu_1)-\frac{1}{2\left(1-\frac{1}{a}\right)}(1+2\mu_2)(1+2\mu_1)+ac_1\right]. \tag{7.3.2}$$

Combining terms as before we rewrite $\widetilde{\Omega}$ in the form

$$\widetilde{\Omega} = \frac{\widetilde{A}_{01}}{z(z-1)} + \frac{\widetilde{A}_{02}}{z\left(z-\frac{1}{a}\right)} + \frac{\widetilde{A}_{12}}{(z-1)\left(z-\frac{1}{a}\right)}, \tag{7.3.3}$$

where

$$\widetilde{A}_{01} = \frac{1}{2}(1+2\mu_0)(1+2\mu_2)+\widetilde{T}_{01}, \tag{7.3.4}$$

$$\widetilde{A}_{02} = \frac{1}{2}(1+2\mu_0)(1+2\mu_1)+\widetilde{T}_{02}, \tag{7.3.5}$$

$$\widetilde{A}_{12} = \frac{1}{2}(1+2\mu_2)(1+2\mu_1)+\widetilde{T}_{12}, \tag{7.3.6}$$

and

$$-\widetilde{T}_{01} - a\widetilde{T}_{02} = ac_0,$$

$$\widetilde{T}_{01} - \frac{1}{\frac{1}{a}-1}\widetilde{T}_{12} = ac_2,$$

$$a\widetilde{T}_{02} + \frac{1}{\frac{1}{a}-1}\widetilde{T}_{12} = ac_1. \tag{7.3.7}$$

Choosing

$$\widetilde{T}_{01} = T_{02}, \quad \widetilde{T}_{02} = T_{01}, \text{ and } \widetilde{T}_{12} = T_{12} \tag{7.3.8}$$

it is easy, in view of (5.2.4), to verify that (7.3.7) is satisfied. Here again, one of the $\widetilde{T}_{ij}$ can be chosen arbitrarily for given a, c_0, c_1, c_2, cf. remark after equation (5.2.4).

Defining analogously $\widetilde{\mu}_\infty^{(0,1)}$ etc. via the equations corresponding to (5.2.12), (5.2.15) and (5.2.17) it is clear now that we have

$$\widetilde{\mu}_\infty^{(0,1)} = \mu_\infty^{(0,2)}, \quad \widetilde{\mu}_\infty^{(0,2)} = \mu_\infty^{(0,1)}, \quad \widetilde{\mu}_\infty^{(1,2)} = \mu_\infty^{(1,2)} \text{ and}$$

$$\widetilde{A}_{01} = A_{02}, \quad \widetilde{A}_{02} = A_{01}, \quad \widetilde{A}_{12} = A_{12}. \tag{7.3.9}$$

Next we consider the limit $a \to \infty$: Carrying out this limit for the coefficients, we obtain the equation

$$v'' + \left(\frac{2 + 2\mu_0 + 2\mu_1}{z} + \frac{1 + 2\mu_2}{z - 1}\right)v' + \left(\frac{A_{02} + A_{12}}{z(z-1)} + \frac{A_{01}}{z^2}\right)v = 0. \tag{7.3.10}$$

In view of (5.2.12), we obtain for this differential equation at $z = 0$ the exponents

$$rs_{\varnothing} = -\frac{1}{2} \cdot (1 + 2\mu_0 + 2\mu_1) \pm \mu_\infty^{(0,1)}. \tag{7.3.11}$$

Using the substitution $v = z^s u$ with $s = s_+$ yields the hypergeometric equation

$$u'' + \left(\frac{1 + 2\mu_\infty^{(0,1)}}{z} + \frac{1 + 2\mu_2}{z - 1}\right)u' + \frac{A}{z(z-1)}u = 0, \tag{7.3.12}$$

where $A = s(1 + 2\mu_2) + A_{02} + A_{12}$. Writing

$$A = \frac{1}{4}[(1 + 2\mu_2 + 2\mu_\infty^{(0,1)})^2 - 4Y^2] \tag{7.3.13}$$

we obtain

Lemma 7.3. *Retaining the notation introduced above, we have*

$$\mu_0^2 + \mu_1^2 + \mu_2^2 - (\mu_\infty^{(0,1)})^2 - (\mu_\infty^{(0,2)})^2 - (\mu_\infty^{(1,2)})^2 = \frac{1}{4} - Y^2. \tag{7.3.14}$$

Proof. Using formulas (5.2.15) and (5.2.17) for A_{02}, A_{12}, we obtain

$$\begin{aligned} A &= \frac{1}{2}(-(1+2\mu_0+2\mu_1)+2\mu_\infty^{(0,1)})(1+2\mu_2) \\ &\quad + \frac{1}{4}[(1+2\mu_0+2\mu_2)^2 - 4(\mu_\infty^{(0,2)})^2] \\ &\quad + \frac{1}{4}[(1+2\mu_1+2\mu_2)^2 - 4(\mu_\infty^{(1,2)})^2] \\ &= \frac{1}{4}[(1+2\mu_2+2\mu_\infty^{(0,1)})^2 - 4Y^2], \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\mu_\infty^{(0,1)}(1+2\mu_2) - \frac{1}{2}(1+2\mu_2) - (\mu_0+\mu_1)(1+2\mu_2) \\ &+ \frac{1}{4} + \mu_0^2 + \mu_2^2 + \mu_0 + \mu_2 + 2\mu_0\mu_2 - (\mu_\infty^{(0,2)})^2 \\ &+ \frac{1}{4} + \mu_1^2 + \mu_2^2 + \mu_1 + \mu_2 + 2\mu_1\mu_2 - (\mu_\infty^{(1,2)})^2 \\ = &\ \frac{1}{4} + (\mu_\infty^{(0,1)})^2 + \mu_2^2 + \mu_\infty^{(0,1)} + \mu_2 + 2\mu_\infty^{(0,1)}\mu_2 - Y^2, \end{aligned}$$

and this is equivalent to

$$\frac{1}{4} - Y^2 = \mu_0^2 + \mu_1^2 + \mu_2^2 - (\mu_\infty^{(0,1)})^2 - (\mu_\infty^{(0,2)})^2 - (\mu_\infty^{(1,2)})^2.$$

7.4.

In order to determine Y of the last section, we consider yet another transformation.

From Section 5.1, we obtain that the second order differential equation under consideration is transformed by $\hat{y}(z) = y\left(\frac{a}{z}\right)$ into

$$\hat{y}'' + \left(\frac{b_\infty}{z^2} + \frac{b_2}{(z-1)^2} + \frac{b_1}{(z-a)^2} + \frac{\hat{c}_0}{z} + \frac{\hat{c}_1}{z-1} + \frac{\hat{c}_2}{z-a}\right)\hat{y} = 0, \qquad (7.4.1)$$

where

$$\begin{aligned}\hat{c}_0 &= \frac{1}{a}(2b_1 + c_1) + 2b_2 + ac_2,\\ \hat{c}_1 &= -2b_2 - ac_2,\\ \hat{c}_2 &= -\frac{1}{a}(2b_1 + c_1).\end{aligned} \tag{7.4.2}$$

Equation (5.2.2) thus yields

$$w'' + \left(\frac{1+2\mu_\infty}{z} + \frac{1+2\mu_2}{z-1} + \frac{1+2\mu_1}{z-a}\right) w' + \left(\frac{\hat{A}_{\infty 2}}{z(z-1)} + \frac{\hat{A}_{\infty 1}}{z(z-a)} + \frac{\hat{A}_{21}}{(z-1)(z-a)}\right) w = 0, \tag{7.4.3}$$

where

$$\begin{aligned}\hat{A}_{\infty 2} &= \frac{1}{2}(1+2\mu_\infty)(1+2\mu_2) + \hat{T}_{\infty 2},\\ \hat{A}_{\infty 1} &= \frac{1}{2}(1+2\mu_\infty)(1+2\mu_1) + \hat{T}_{\infty 1},\\ \hat{A}_{21} &= \frac{1}{2}(1+2\mu_2)(1+2\mu_1) + \hat{T}_{12}.\end{aligned} \tag{7.4.4}$$

Moreover, by (5.2.4), we have

$$\begin{aligned}-\hat{T}_{\infty 2} - \frac{1}{a}\hat{T}_{\infty 1} &= \hat{c}_0,\\ \hat{T}_{\infty 2} - \frac{1}{a-1}\hat{T}_{21} &= \hat{c}_1,\\ \frac{1}{a}\hat{T}_{\infty 1} + \frac{1}{a-1}\hat{T}_{21} &= \hat{c}_2.\end{aligned} \tag{7.4.5}$$

We claim that

$$\begin{aligned}\hat{T}_{\infty 2} &= -2b_2 - T_{02} - T_{12},\\ \hat{T}_{\infty 1} &= -2b_1 - T_{01} - T_{12},\end{aligned}$$

$$\hat{T}_{21} = T_{12} \tag{7.4.6}$$

satisfy (7.4.5). To verify these relations we insert into (7.4.2) the relations (5.2.4):

$$\begin{aligned}
\hat{c}_0 &= \frac{1}{a}(2b_1 + c_1) + 2b_2 + ac_2 \\
&= \frac{1}{a}\left(2b_1 + T_{01} - \frac{1}{a-1}T_{12}\right) + 2b_2 + a\left(\frac{1}{a}T_{02} + \frac{1}{a-1}T_{12}\right) \\
&= 2b_2 + T_{02} + T_{12} + \frac{1}{a}(2b_1 + T_{01} + T_{12}) = -\hat{T}_{\infty 2} - \frac{1}{a}\hat{T}_{\infty 1}.
\end{aligned}$$

Similarly

$$\hat{c}_1 = -2b_2 - ac_2 = -2b_2 - T_{02} - T_{12} - \frac{1}{a-1}T_{12} = \hat{T}_{\infty 2} - \frac{1}{a-1}\hat{T}_{21},$$

and

$$\begin{aligned}
\hat{c}_2 &= -\frac{1}{a}(2b_1 + c_1) = -\frac{1}{a}2b_1 - \frac{1}{a}T_{01} + \frac{1}{a}\cdot\frac{1}{a-1}T_{12} \\
&= \frac{1}{a}(-2b_1 - T_{01} - T_{12}) + \frac{1}{a-1}T_{12} \\
&= \frac{1}{a}\hat{T}_{\infty 1} + \frac{1}{a-1}\hat{T}_{21}.
\end{aligned}$$

We have seen in Section 5.2 that the T_{ij} are independent of a. Hence, in view of (7.4.6) the $\hat{T}_{ij}$ are independent of a. Thus the limit $a \to \infty$ yields

$$v'' + \left(\frac{1 + 2\mu_\infty}{z} + \frac{1 + 2\mu_2}{z-1}\right) + \frac{\hat{A}_{\infty 2}}{z(z-1)}v = 0. \tag{7.4.7}$$

We therefore write

$$\hat{A}_{\infty 2} = \frac{1}{4}[(1 + 2\mu_\infty + 2\mu_2)^2 - 4Z^2] = \frac{1}{2}(1 + 2\mu_\infty)(1 + 2\mu_2) + \hat{T}_{\infty 2}. \tag{7.4.8}$$

Substituting T_{02} and T_{12} from equations (5.2.16) and (5.2.18) respectively, and using $b_2 = \frac{1}{4} - \mu_2^2$, we obtain

$$\frac{1}{4}(1 + 4\mu_\infty^2 + 4\mu_2^2 + 4\mu_\infty + 4\mu_2 + 8\mu_\infty\mu_2) - Z^2$$

$$= \frac{1}{2}(1 + 2\mu_2 + 2\mu_\infty + 4\mu_\infty\mu_2) - 2b_2 - T_{02} - T_{12},$$

which is equivalent to

$$\mu_\infty^2 + \mu_2^2 = \frac{1}{4} + Z^2 - \left(\frac{1}{2} - 2\mu_2^2\right)$$
$$- \left(\left(\frac{1}{4} - (\mu_\infty^{(0,2)})^2\right) - \left(\frac{1}{4} - \mu_2^2\right) - \left(\frac{1}{4} - \mu_0^2\right)\right)$$
$$- \left(\left(\frac{1}{4} - (\mu_\infty^{(1,2)})^2\right) - \left(\frac{1}{4} - \mu_1^2\right) - \left(\frac{1}{4} - \mu_2^2\right)\right),$$

and this can be written as

$$\mu_\infty^2 + \mu_2^2 = -\frac{1}{4} + Z^2 + 2\mu_2^2 - \left(-\frac{1}{4} - (\mu_\infty^{(0,2)})^2 + \mu_2^2 + \mu_0^2\right)$$
$$- \left(-\frac{1}{4} - (\mu_\infty^{(1,2)})^2 + \mu_1^2 + \mu_2^2\right).$$

Consequently, we have

$$\mu_\infty^2 + \mu_2^2 = \frac{1}{4} + Z^2 + (\mu_\infty^{(0,2)})^2 - \mu_0^2 + (\mu_\infty^{(1,2)})^2 - \mu_1^2,$$

and hence

$$\frac{1}{4} + Z^2 = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_\infty^2 - (\mu_\infty^{(0,2)})^2 - (\mu_\infty^{(1,2)})^2. \tag{7.4.9}$$

Finally, we evaluate the relation $b_\infty = b_0 + b_1 + b_2 + 0 \cdot c_0 + 1 \cdot c_1 + a \cdot c_2$, where we substitute T_{ij} for c_k and use the formulas (5.2.13), (5.2.16) and (5.2.18) for T_{ij}. A straightforward computation now yields

$$\frac{1}{4} = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_\infty^2 - (\mu_\infty^{(0,1)})^2 - (\mu_\infty^{(0,2)})^2 - (\mu_\infty^{(1,2)})^2. \tag{7.4.10}$$

A comparison with Lemma 7.3 yields $Y = \mu_\infty$ and a comparison with (7.4.9) shows

$$Z = \mu_\infty^{(0,1)}. \tag{7.4.11}$$

In particular $Y = \mu_\infty$ is an eigenvalue of some planar Delaunay matrix. We thus can consider μ_2, $\mu_\infty^{(0,1)}$, Y as eigenvalues of three Delaunay matrices. Thus, we obtain, using the results of the last four sections:

Corollary 7.4. *The monodromy matrices at $z = 0$ and $z = a$ of the original 4-noid potential are simultaneously unitarizable for some $r \in (0, 1)$ sufficiently close to 1 for sufficiently large a if and only if μ_2, $\mu_\infty^{(0,1)}$ and Y satisfy the trinoid inequality* (3.6.5).

8. Existence of Planar 4-noids

8.1.

We are now in a position to construct planar 4-noids by the method of this paper.

Let ν_j and $\nu_\infty^{(i,j)}$ be defined as in Section 7.2, and denote by ϱ_0, ϱ_1, ϱ_2, ϱ_∞ the monodromy matrices at $z = 0, 1, a, \infty$, respectively.

We have seen in Theorem 7.2 that ϱ_0, ϱ_1 are simultaneously unitarizable for a large enough if and only if $(\nu_0, \nu_1, \nu_\infty^{(0,1)})$ satisfy (7.2.1). Since $Y = \mu_\infty$, we obtain from (7.3.1), (7.3.12) and (7.3.13) for sufficiently large a:

The monodromy matrices ϱ_0 and ϱ_2 are simultaneously unitarizable for some $r \in (0, 1)$ sufficiently close to 1 if and only if $(\nu_\infty^{(0,1)}, \nu_2, \nu_\infty)$ satisfy (7.2.1). (8.1.1)

If one combines the two transformations $z \mapsto 1 - z$ and $z \mapsto az$, then one obtains for a large enough:

The monodromy matrices ϱ_1 and ϱ_2 are simultaneously unitarizable for some $r \in (0, 1)$ sufficiently close to 1 if and only if $(\nu_\infty^{(0,1)}, \nu_2, \nu_\infty)$ satisfy (7.2.1). (8.1.2)

Note it is not surprising that (8.1.1) and (8.1.2) hold under the same condition, because both conditions come from the reduction of the Heun equation to a hypergeometric equation by "eliminating" the coefficient a.

The proof of the existence of planar 4-noids will consist of several steps:

• **Step 0.** Show that $(\nu_0, \nu_1, \nu_\infty^{(0,1)})$, $(\nu_\infty^{(0,1)}, \nu_2, \nu_\infty)$ satisfy (7.2.1). (This implies that the pairs of monodromy matrices, ϱ_0, ϱ_1, and ϱ_0, ϱ_2 and ϱ_1, ϱ_2 can be unitarized simultaneously.)

• **Step 1.** Since ϱ_0, ϱ_1 and ϱ_0, ϱ_2 and ϱ_1, ϱ_2 are pairwise simultaneously unitarizable, also ϱ_0, ϱ_1, ϱ_2, ϱ_∞ are simultaneously unitarizable.

• **Step 2.** ϱ_0, ϱ_1, ϱ_2, ϱ_∞ satisfy the closing conditions for ends.

• **Step 3.** The ends at $z = 0, 1, a, \infty$ are all embedded (whence asymptotically Delaunay).

Before we address Step 1, we note that the result of Step 1 already implies Steps 2 and 3. By Step 1, the monodromy matrices ϱ_0, ϱ_1, ϱ_2 and ϱ_∞ are simultaneously unitarizable. By Theorem 5.3.1 of [11], the monodromy matrices also satisfy the closing conditions for ends. And using Theorem 5.4.1 of [11], we obtain that the ends at $z = 0, 1, a, \infty$ are all embedded (also compare Theorem 4.1).

8.2.

In this section, we will address Step 1.

Theorem 8.1. *Retain the notation of the sections before. Let* ϱ_0, ϱ_1, ϱ_2, $\varrho_\infty \in \Lambda(\mathrm{SL}_2\mathbb{C})_\sigma$ *be the monodromy matrices around the ends at*

$z = 0, 1, a, \infty$, *in particular we have*

$$\varrho_0 \varrho_1 \varrho_2 \varrho_\infty = \pm I. \tag{8.2.1}$$

Assume that the monodromy matrices are pairwise simultaneously unitarizable for some $r \in (0, 1)$ *sufficiently close to* 1. *Then* ϱ_0, ϱ_1, ϱ_2, ϱ_∞ *are in fact simultaneously unitarizable for some* $r \in (0, 1)$ *sufficiently close to* 1.

Proof. Clearly, because of (8.2.1), it suffices to show that ϱ_0, ϱ_1, ϱ_2 are simultaneously unitarizable.

Since ϱ_0 is unitarizable and diagonalizable, we may assume, see Section 2 in [12] (especially Proposition 2.2 and the following) that ϱ_0 is given by $\varrho_0 = \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \alpha^{-1}(\lambda) \end{pmatrix}$, where $\alpha(\lambda) \in \mathbb{S}^1$ and $\alpha(\lambda) \neq \pm 1$ except for finitely many values of λ (otherwise ϱ_0 would be the identity map, and the simultaneous unitarizability of ϱ_1, ϱ_2 would just be equivalent to the simultaneous unitarizability of ϱ_0, ϱ_1, ϱ_2).

Since ϱ_0, ϱ_1 are simultaneously unitarizable, we may assume that ϱ_1 is already unitary, i.e., $\varrho_1 = \begin{pmatrix} p & \tau \\ -\bar{\tau} & p \end{pmatrix}$. Instead of ϱ_0, ϱ_1, we may of course consider the matrices $D\varrho_0 D^{-1} = \varrho_0$ and $D\varrho_1 D^{-1}$, where $D = \operatorname{diag}(s, s^{-1})$ is some diagonal matrix in $\Lambda(\mathrm{SL}_2\mathbb{C})_\sigma$, and $D\varrho_1 D^{-1} = \begin{pmatrix} p & \tau s^2 \\ -\bar{\tau} s^{-2} & \bar{p} \end{pmatrix}$. Write τ in polar coordinates as $\tau = qe^{i\varphi}$. Setting $s = e^{-i\varphi/2}$ then gives $D\varrho_1 D^{-1} = \begin{pmatrix} p & q \\ -q & \bar{p} \end{pmatrix}$, so w.r.g. we may assume

$$\varrho_0 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \varrho_1 = \begin{pmatrix} p & q \\ -q & \bar{p} \end{pmatrix}, \quad \alpha \in \mathbb{S}^1,\ p \in \mathbb{C},\ q \in \mathbb{R},\ q > 0.$$

Now we consider ϱ_2. By assumption, there exists $S \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$ such that $S\varrho_0 S^{-1}$ and $S\varrho_2 S^{-1}$ are both in $\Lambda\mathrm{SU}_2$. According to [12, Theorem 2.16], any $S \in \Lambda_r^+(\mathrm{SL}_2\mathbb{C})_\sigma$ with this property may be written as $S = DT$ where D is some diagonal matrix and $T\varrho_0 T^{-1}$ is a diagonal matrix in $\Lambda\mathrm{SU}_2$. Since ϱ_0 is already diagonal, T is either diagonal or off-diagonal. We only discuss the diagonal case (the off-diagonal case is very similar, in principal, in this case, conjugation of an diagonal matrix with T just switches the diagonal entries). So S is in fact a diagonal matrix, say $S = \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}$, where $r \in \mathbb{C}$. Let $\varrho_2 = \begin{pmatrix} f & p \\ q & g \end{pmatrix}$. Then

$$S\varrho_2 S^{-1} = \begin{pmatrix} f & p \cdot r^{-2} \\ q \cdot r^2 & g \end{pmatrix} =: \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in \Lambda\mathrm{SU}_2,$$

thus we have $f = u$, $g = \overline{u}$ as well as $p = vr$ and $q = -\overline{v}r^{-2}$. Hence, we have

$$\varrho_2 = \begin{pmatrix} u & r^2 v \\ -r^{-2}\overline{v} & \overline{u} \end{pmatrix}.$$

Clearly, since ϱ_1, ϱ_2 are simultaneously unitarizable, also $\varrho_1\varrho_2 = \begin{pmatrix} p & q \\ -q & \overline{p} \end{pmatrix}\begin{pmatrix} u & r^2 v \\ -r^{-2}\overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} pu - q\overline{v}r^{-2} & pvr^2 + q\overline{u} \\ -qu - \overline{pv}r^{-2} & -qvr^2 + \overline{pu} \end{pmatrix}$ is unitarizable. Hence $\mathrm{tr}(\varrho_0\varrho_2) = pu - q\overline{v}r^{-2} - qvr^2 + \overline{pu}$ must be real (see, for example, Theorem 3.5 in [9]).

Obviously $pu + \overline{pu} = \mathrm{Re}(pu) \in \mathbb{R}$, thus $q \cdot (\overline{v}r^{-2} + vr^2)$ needs to be in $\mathbb{R}$. This gives $r^{-2}\overline{v} + r^2 v = r^{-2}v + r^2\overline{v}$, hence $r^2(v - \overline{v}) = r^{-2}(v - \overline{v})$, i.e.,

$$(r^2 - r^{-2}) \cdot (v - \overline{v}) = 0.$$

Now we just need to consider two cases:

- **Case 1.** $r^2 - r^{-2} = 0$, i.e., $r = \pm 1$. Then ϱ_2 already is unitary and we are done.

• **Case 2.** $r \neq \pm 1$. Then $v = \overline{v} \neq 0$, that is $v \in \mathbb{R}$.

On the other hand, for $a \to \infty$, we have $\varrho_2 \to (\varrho_0 \varrho_1)^{-1} = \begin{pmatrix} \alpha^{-1}\overline{p} & -\alpha q \\ -q\alpha^{-1} & p\alpha \end{pmatrix}$. Clearly $\alpha \notin \mathbb{R}$ except for finitely many values of λ. Therefore, if a is large enough, $v \notin \mathbb{R}$, a contradiction. Consequently, we have $r = \pm 1$. This concludes the proof.

8.3.

Finally, using all the results from the previous sections, we are able to state our main theorem

Theorem 8.2. *Let* $\mu_0,\ \mu_1,\ \mu_2,\ \mu_\infty^{(0,1)},\ \mu_\infty^{(0,2)},\ \mu_\infty^{(1,2)}$ *be given Delaunay-type expressions. Let* $b_j = \frac{1}{4} - \mu_j^2$ *and let* c_j *be given by* (5.2.4), *where the* T_{ij} *are defined by* (5.2.13), (5.2.16), *and* (5.2.18). *We assume*

$$\frac{1}{4} = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_\infty^2 - (\mu_\infty^{(0,1)})^2 - (\mu_\infty^{(0,2)})^2 - (\mu_\infty^{(1,2)})^2 \tag{8.3.1}$$

and

The triples $(\nu_0,\ \nu_1,\ \nu_\infty^{(0,1)})$ *and* $(\nu_2,\ \nu_\infty,\ \nu_\infty^{(0,1)})$ *satisfy the strict spherical inequalities* (7.2.1). (8.3.2)

Then the potential η, *given in* (3.1.1) *with* ν *and* τ *as in* (3.4.1) *and* (3.4.2), *yields for all sufficiently large* $a \in \mathbb{R}$ *a planar* 4*-noid with embedded ends.*

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