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# ON A SET OF NUMBERS ARISING IN THE DYNAMICS OF UNIMODAL MAPS

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**Abstract**

In this paper we initiate the study of the arithmetical properties of a set of numbers which encode the dynamics of unimodal maps in a universal way along with that of the corresponding topological zeta function. Here we are concerned, in particular, with the Feigenbaum bifurcation.



## 1. Preliminaries

We start by reviewing some basic ideas of (a version of) the *kneading theory* for unimodal maps. For related approaches and/or more details see [4], [5], [7].

**Definition 1.1.** A smooth map $f : [0, 1] \to [0, 1]$ is called *unimodal* if it has exactly one critical point $0 < c_0 < 1$ and moreover $f(0) = f(1) = 0$.

For unimodal maps the orbit of the critical point $c_0$ determines in a sense the complexity of any other orbit. To be more precise, given $x \in [0, 1]$ we call *itinerary* of $x$ with $f$ the sequence $i(x) = s_1 s_2 s_3 \ldots,$ where $s_i = 0$ or 1 according to $f^{i-1}(x) < c_0$ or $f^{i-1}(x) \geq c_0$. An important point

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is that such symbolic representation is in fact ‘faithful’, that is, if $s(x) = s(x')$, then $x = x'$. Differently said, the partition of $[0, 1]$ in the two semi-intervals $P_0 = [0, c_0)$ and $P_1 = [c_0, 1)$ is *generating* for a unimodal map $f$ with critical point $c_0$.

It is clear that if $s = i(x)$ is a sequence obtained as above, then $i(f(x)) = \sigma(s)$, where $\sigma$ denotes the left-shift: if $s = s_1 s_2 s_3 \ldots$, then $\sigma(s) = s_2 s_3 s_4 \ldots$. The itinerary of the point $c_1 = f(c_0)$ is called *kneading sequence* $K(f)$ of $f$. We say moreover that a given sequence $s$ of 0 and 1 is *admissible* for $f$ if there is $x \in [0, 1]$ such that $i(x) = s$. A nice way to decide whether or not a given sequence is admissible amounts to establish an ordering on the itineraries which corresponds to ordering of the real line. In this way, the admissible sequences are those which never become greater than the kneading sequence when shifted. To this end, let us associate to a sequence $s = s_1 s_2 s_3 \ldots$ the number $\tau(s) \in [0, 1]$ defined as

$$\tau = 0.t_1 t_2 t_3 \ldots = \sum_{k=1}^{\infty} \frac{t_k}{2^k}, \quad t_k = \sum_{i=1}^{k} s_i \pmod 2. \tag{1.1}$$

Equivalently, if we set

$$\varepsilon_k = (-1)^{\sum_{i=1}^{k} s_i}, \tag{1.2}$$

then $t_k$ and $\varepsilon_k$ are related by

$$t_k = \frac{1 - \varepsilon_k}{2}, \qquad \varepsilon_k = 1 - 2t_k. \tag{1.3}$$

**Lemma 1.1.** *Given* $x, y \in [0, 1]$ *we have*

(1) *if* $\tau(i(x)) < \tau(i(y))$, *then* $x < y$;

(2) *if* $x < y$, *then* $\tau(i(x)) \leq \tau(i(y))$.

**Remark 1.** The equality in (2) cannot be removed. Indeed, the existence of an attracting periodic orbit typically implies the existence of an interval of points with the same itinerary. On the other hand, a theorem due to Guckenheimer (see [5]) says that a unimodal map $f$ has an attracting periodic orbit if and only if $K(f)$ is periodic. Vice versa, if $K(f)$ is not periodic, then implication (2) becomes: if $x < y$, then $\tau(i(x)) < \tau(i(y))$. This has important consequences. First of all: if $K(f)$ is not periodic and $K(f) = K(g)$, then $f$ and $g$ are topologically conjugated.

**Proof of Lemma 1.1.** Let us show the first part. Set $i(x) = s_1 s_2 \ldots$, $i(y) = s'_1 s'_2 \ldots$ and let $n = \min\{i \geq 1 : s_i \neq s'_i\}$ be the *discrepancy* between $i(x)$ and $i(y)$. We proceed by induction in $n$. If $n = 1$, then the result is clear. Suppose it is true for sequences with discrepancy $n - 1$. We have $i(f(x)) = s_2 s_3 \ldots$ and $i(f(y)) = s'_2 s'_3 \ldots$. Two cases are possible: either $s_1 = 0$ or $s_1 = 1$. If $s_1 = 0$, then $\tau(i(f(x))) < \tau(i(f(y)))$ because applying $f$ we do not modify the number of 1's before the discrepancy. Using the induction we then have that $f(x) < f(y)$. But since $f$ is increasing on $[0, c_0)$ we also have $x < y$. If $s_1 = 1$, then $\tau(i(f(x))) > \tau(i(f(y)))$ because there is a 1 less among the symbols $s_2 \ldots s_n$. Therefore, by the induction, we get $f(x) > f(y)$ and since $f$ is decreasing on $(c_0, 1]$ we see that $x < y$. The second assertion follows similarly.

An immediate consequence is the following:

**Theorem 1.1.** *Every sequence s such that*

$$\tau(\sigma(K(f))) \leq \tau(\sigma^m(s)) \leq \tau(K(f)), \quad m \geq 0$$

*is admissible and is the itinerary of a point in* $[f(c_1), c_1]$.

In particular,

$$\tau(\sigma^m(K(f))) \leq \tau(K(f)), \quad m \geq 0.$$

A sequence $K$ with this property is said *maximal*. If moreover we consider a one-parameter family of unimodal maps $f_r$ so that $r \to f_r$ is continuous on some real interval with respect to the $\mathcal{C}^1$ topology, then we can reformulate a theorem of Metropolis et al. (see [5]) by saying that every maximal sequence $K$ such that

$$\tau(K(f_{r_a})) \leq \tau(K) \leq \tau(K(f_{r_b}))$$

is the kneading sequence of $f_r$ for some $r_a \le r \le r_b$. Notice that for $f_r([0, 1]) \subseteq [0, 1]$ one needs that $f_r(c_0) \le 1$. In particular, if $r = r_b$, then $f_{r_b}(c_0) = 1$ and $K(f_{r_b}) = 1\overline{0}$ (where $\overline{s_1 \ldots s_l}$ indicates the unended repetition of the word $s_1 \ldots s_l$), to which it corresponds the number $\tau(K(f_{r_b})) = 1$. At the other endpoint we have $\tau(K(f_{r_a})) = 0$ (when $f^n(c_0)$ converges monotonically to zero). We finally observe that given $q \in [0, 1]$ we have

$$\tau(s) = q \Rightarrow \tau(\sigma(s)) = T(q),$$

where $T : [0, 1] \to [0, 1]$ is the *tent map* given by

$$T(x) = \begin{cases} 2x & \text{if } x < 1/2, \\ 2(1 - x) & \text{if } x \ge 1/2. \end{cases}$$

Putting together these observations we obtain the following representation [9]:

- The subset $\Lambda \subset [0, 1]$ defined as

$$\Lambda = \{\tau \in [0, 1] : T^m(\tau) \le \tau, \ \forall m \ge 0\}$$

represents a *universal encoding* for the dynamics of unimodal maps: those having the same parameter $\tau$ have identical topological properties. In particular, every 0 in the binary expansion of $\tau \in \Lambda$ corresponds to a ‘forbidden word’ in the associated dynamics: let $K(f) = s_1 s_2 \ldots$ and $\tau(K(f)) = 0.t_1 t_2 \ldots$, then if $t_j = 0$ the word $s_1 \ldots \hat{s}_j$ (with $\hat{s}_j = 1 - s_j$) is a forbidden word. Let $A = \{0, 1\}$ be the alphabet and $A^* = \bigcup_{n \in \mathbb{N}} A^n$ be the set of all possible finite words written in the alphabet $A$. A word $u \in A^*$ of length $|u| = n$ is said *f-admissible* if there is $x \in [0, 1]$ whose itinerary with $f$ up to the $n$-th letter coincides with $u$. The set $\mathcal{L} \subseteq A^*$ defined as

$$\mathcal{L} = \{u \in A^*, \ u \text{ is } f\text{-admissible}\}$$

is the *language* generated by $f$. The function

$$p(n) = \#\{u \in \mathcal{L}, |u| = n\} \tag{1.4}$$

is called the *complexity function* of $\mathcal{L}$ and the limit

$$h = \lim_{n \to \infty} \frac{1}{n} \log p(n) \tag{1.5}$$

is the *topological entropy*. To summarize, the parameter $\tau$ furnishes a universal encoding in the sense that all unimodal maps with the same $\tau$ determine the same language $\mathcal{L} = \mathcal{L}(\tau)$ and, in particular, have the same topological entropy $h = h(\tau)$.

**Remark 2.** It is plain that the extremal situation in which $T^m(\tau) = \tau$ for some $m > 0$ is that in which $\tau$ is a periodic point for the tent map $T$. In this case the kneading sequence $K$ is periodic and so is the corresponding attractor. This suggests that isolated points as well as 'holes' in $\Lambda$ have to be related to periodic attractors. In particular, there is a one-to-one correspondence between the holes in $\Lambda$ and the periodic windows in the bifurcation diagram of unimodal maps, namely intervals in parameter space where the topological entropy is constant [9].



### 1.1. Topological zeta function

A great deal of information on the set of periodic points of a given map $f : [0, 1] \to [0, 1]$ can be stored into the *topological zeta function* of Artin and Mazur, defined as

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \# \operatorname{Per}_n(f). \tag{1.6}$$

For a unimodal map $f$ with parameter $\tau$ the numbers $\# \operatorname{Per}_n(f)$ are uniquely determined by the value of $\tau$ and we therefore write $\zeta(\tau, z)$. The series converges absolutely and uniformly for $|z| < e^{-h}$ and $z = e^{-h}$ is a singular point (e.g., a pole) of $\zeta(\tau, z)$. This function can also be written as an Euler product noting that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \# \operatorname{Per}_n(f) = \sum_{p=1}^{\infty} N(p) \sum_{k=1}^{\infty} \frac{z^{kp}}{k} = \log \prod_{p=1}^{\infty} (1 - z^p)^{-N(p)},$$

where $N(p)$ is the number of distinct periodic orbits of *prime period* $p$. Hence we have

$$\zeta(\tau, z) = \prod_{p=1}^{\infty} (1 - z^p)^{-N(p)}. \tag{1.7}$$

The combinatorial features of the set of periodic orbits of a given map $f$ reflect onto the analytic properties of $\zeta(\tau, z)$ in the complex plane.

More specifically, from the work of Milnor and Thurston ([12], Lemma 4.5 and Corollary 10.7) one can extract the following result:

**Proposition 1.1.** *Let* $\Lambda \ni \tau = 0.t_1 t_2 t_3 \ldots$. *If the sequence* $t_1 t_2 t_3 \ldots$ *is eventually periodic or aperiodic, then*

$$\zeta(\tau, z) = \frac{1}{(1 - z)\left(1 + \sum_{k=1}^{\infty} \varepsilon_k z^k\right)}, \tag{1.8}$$

*where the numbers* $\varepsilon_k$ *are defined in* (1.2)-(1.3). *If instead the kneading sequence is periodic and* $\tau = 0.\overline{t_1 \ldots t_n}$, *then*

$$\zeta(\tau, z) = \frac{1}{(1 - z)\left(1 + \sum_{k=1}^{n-1} \varepsilon_k z^k\right)}. \tag{1.9}$$

We now list some examples in which the zeta function can be written in closed form by means of Lemma 1.1.

- The number $\tau = 1 = 0.1\overline{1}$ corresponds to the situation where the critical point gets mapped to the origin in two steps, and yields

$$\zeta(1, z) = \frac{1}{1 - 2z}, \quad h(1) = \log 2.$$

- The number $\tau = 5/6 = 0.1\overline{10}$ corresponds to the situation where the critical point gets mapped to the fixed point in three steps (*band* *merging*). In this case we find

$$\zeta(5/6,\, z) = \frac{1+z}{(1-z)(1-2z^2)}, \quad h(5/6) = \log\sqrt{2}.$$

- The number $\tau = 6/7 = 0.\overline{110}$ corresponds to the opening of the period three window. The last orbit in the Sarkovskii order settles down and thus there are periodic orbits of any period. Here we get

$$\zeta(6/7,\, z) = \frac{1}{(1-z)(1-z-z^2)}, \quad h(6/7) = \log\frac{\sqrt{5}-1}{2}.$$

In the examples above the number $\tau$ was always rational. In the next section we show a situation leading to a transcendental irrational $\tau$. A systematic study of the arithmetical properties of the numbers in $\Lambda$, along wit their relation with the dynamics, is far from being reached. In particular, the question of what is the most irrational $\tau$ (and to which chaotic state it corresponds) is open. In Section 2 we shall study the above quantities for the Feigenbaum bifurcation but in order to get a self-contained exposition we first recall some standard notions (for details see [7]).



### 1.2. Kneading theory and renormalization

Let $f : [0, 1] \to [0, 1]$ be a unimodal map with a unique fixed point $b$ in the interval $(c_0, 1)$, so that $f'(b) < 0$. Let $a$ be the (unique) point in $(0, c_0)$ such that $f(a) = b$ and set $J = [a, b]$. Consider the linear map $L$ defined by

$$L(x) = \frac{1}{a-b}(x-b). \tag{1.10}$$

It expands $J$ to $[0, 1]$ reversing its orientation. The inverse map is

$$L^{-1}(x) = (a-b)x + b. \tag{1.11}$$

The renormalization operator $\mathcal{R}$ is thus defined as

$$(\mathcal{R}f)(x) = L \circ f^2 \circ L^{-1}(x). \tag{1.12}$$

Plainly $(\mathcal{R}f)(0) = (\mathcal{R}f)(1) = 0$ and $c_0$ is the only critical point of $\mathcal{R}f$. Moreover, 2-periodic points for $f$ become fixed points of $\mathcal{R}f$.

Now let $K(f) = s_1 s_2 s_3 \ldots$ be the kneading sequence of $f$. The following properties are easily verified (see [7]):

1. If $\mathcal{R}f$ is defined and unimodal, then $s_{2k+1} = 1$, $\forall k \geq 0$;

2. $K(\mathcal{R}f) = \hat{s}_2 \hat{s}_4 \hat{s}_6 \ldots$. In other words, one can define a renormalization operator on sequences acting as (with slight abuse we keep using the same symbol):

$$\mathcal{R}(s_1 s_2 s_3 \ldots) = \hat{s}_2 \hat{s}_4 \hat{s}_6 \ldots; \tag{1.13}$$

3. if both $\mathcal{R}f$ and $\mathcal{R}^2 f$ are unimodal, then $s_{4k+2} = 0$;

4. if $\mathcal{R}^l f$ is unimodal for $l \leq n$, then all symbols $s_j$ with $j = 2^n k + 2^{n-1}$ are determined;

5. since the numbers $2^n k + 2^{n-1}$ exhaust all even numbers as $n$ varies in $\mathbb{N}$ it follows that if $\mathcal{R}^n f$ is unimodal for each $n \geq 1$, then all symbols of $K(f)$ are determined.

How $K(f)$ looks like for an infinitely renormalizable unimodal map, that is a map $f$ such that $\mathcal{R}f = f$?

Set

$$K_1 = \overline{1}$$

$$K_2 = \overline{10}$$

$$K_3 = \overline{1011}$$

$$K_4 = \overline{10111010}$$

$$K_5 = \overline{1011101010111011}$$

and more generally $K_{j+1}$ is obtained from $K_j$ by applying one of the following equivalent procedures:

- duplicating the repeating sequence and reversing the last symbol;
- doubling all indices, reversing the resulting symbols (all with even index) and inserting a 1 at each position with odd index;
- applying the *Feigenbaum substitution* $1 \to 10$ and $0 \to 11$ (the symbol 1 being the *prefix*) to the repeating sequence.

By construction $K_j$ has period $2^j$ with an odd number of 1's. We also have that

$$\mathcal{R}(K_{j+1}) = K_j, \quad j \geq 1. \tag{1.14}$$

Therefore the limit sequence

$$K_\infty = \lim_{j \to \infty} K_j = 1011\,1010\,1011\,1011\,1011\,1010\,1011\,1010... \tag{1.15}$$

is aperiodic and invariant under renormalization (the latter can be interpreted as a self-similarity property):

$$\mathcal{R}(K_\infty) = K_\infty. \tag{1.16}$$

For all $j \geq 0$, $K_j$ is a prefix of $K_\infty$. Finally, one easily verifies that $K_j$ is the kneading sequence of a unimodal map having a periodic attractor of period $2^j$, whereas $K_\infty$ is that of an infinitely renormalizable map.

Inspection of the sequences $K_n$ suggests that the asymptotic frequencies of the symbols 0 and 1 appearing in $K_\infty$ are 1/3 and 2/3 respectively. To check this, we shall use a standard technique in the theory of substitution (see [13]): let $\phi$ be the substitution $\phi(1) = 10$ and $\phi(0) = 11$ considered above and $N_i(\phi(j))$ be the number of occurrences of the symbol $i = 0, 1$ in the word $\phi(j)$. The asymptotic frequency of $i$ in $K_\infty$ is then given by

$$f_i = \lim_{n \to \infty} \frac{N_i(\phi^n(1))}{2^n}, \quad i = 0, 1, \tag{1.17}$$

where we have used the fact that $|K_n| = 2^n$. To compute $f_i$ we construct the matrix

$$M = [N_i(\phi(j))]_{i,\, j\in\{0,1\}}. \tag{1.18}$$

A short reflection yields

$$M^n = [N_i(\phi^n(j))]_{i,\, j=0,1}, \tag{1.19}$$

and thus, setting $u = (0, 1)$, we get

$$f_i = \lim_{n\to\infty} \frac{(M^n u)_i}{2^n}. \tag{1.20}$$

From Perron-Frobenius theorem we have that $M$ has a simple positive eigenvalue of maximal modulus $\lambda$ to which it corresponds an eigenvector with strictly positive components. In our case we find

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \tag{1.21}$$

whose eigenvalues are 2 and $-1$. The normalized eigenvector corresponding to the leading eigenvalue is $v = (1/3, 2/3)$. From (1.20) one deduces that $f_i = v_i$, $i = 0, 1$, which are the claimed frequencies.

**Remark 3.** One may consider the sequence $K_\infty$ as an element of $\{0, 1\}^{\mathbb{N}}$ and observe that the continuous injective map $T : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ defined as follows: if $\omega = 111\ldots$, then $T\omega = 000\ldots$; if $\omega = 1\ldots10\ldots$, then $T\omega = 0\ldots01\ldots$; if $\omega = 0\ldots$, then $T\omega = 1\ldots$, acts a (right) translation on $K_\infty$. Therefore $T$ leaves invariant the space $X = \overline{\{T^j K_\infty\}_{j\geq 0}}$. The map $T : X \to X$ is called *dyadic adding machine*.

## 2. Arithmetics of the Feigenbaum Bifurcation

We now look at the values of the parameter $\tau$ corresponding to the kneading sequences arising in the period doubling scenario discussed in the preceding section.

Set $\tau_j = \tau(K_j)$. We find

$$\tau_1 = 0.\overline{10}$$

$$\tau_2 = 0.\overline{1100}$$

$$\tau_3 = 0.\overline{11010010}$$

$$\tau_4 = 0.\overline{1101001100101100}$$

$$\tau_5 = 0.\overline{11010011001011010010110011010010}$$

and $\tau_{j+1}$ is obtained from $\tau_j$ by applying the rule

$$\tau_j = 0.\overline{t_1 \ldots t_{2^j}} \Rightarrow \tau_{j+1} = 0.\overline{t_1 \ldots t_{2^j-1} \hat{t}_{2^j} \hat{t}_1 \ldots \hat{t}_{2^j-1} t_{2^j}}, \tag{2.22}$$

or, alternatively, by the following substitution: let

$$a = 00, \quad b = 01, \quad c = 10, \quad d = 11.$$

Then

$$a \to ac, \quad b \to ad, \quad c \to da, \quad d \to db. \tag{2.23}$$

It is easy to check that $\tau_j \in \Lambda$, $\forall j \geq 1$. They form an increasing sequence:

$$\tau_1 < \tau_2 < \tau_3 < \cdots$$

and satisfy

$$\widetilde{\mathcal{R}}(\tau_{j+1}) = \tau_j, \quad \text{where} \quad \widetilde{\mathcal{R}}(0.t_1t_2t_3\ldots) := 0.t_2t_4t_6\ldots. \tag{2.24}$$

For each $j \geq 1$, $\tau_j = 0.\overline{t_1 \ldots t_{2^j}}$ is the rational number given by

$$\tau_j = \frac{2^{2^j}}{2^{2^j} - 1} \sum_{k=1}^{2^j} \frac{t_k}{2^k}. \tag{2.25}$$

We have

$$\tau_1 = \frac{2}{3}, \quad \tau_2 = \frac{4}{5}, \quad \tau_3 = \frac{14}{17}, \quad \tau_4 = \frac{212}{257}, \quad \tau_5 = \frac{54062}{65537},$$

$$\tau_6 = \frac{3542953172}{4294967297}, \quad \tau_7 = \frac{15216868001456509742}{18446744073709551617}.$$

By (2.22) and (2.25) the following recursive law is in force:

$$\tau_j = \frac{p_j}{q_j} \Rightarrow \tau_{j+1} = \frac{p_{j+1}}{q_{j+1}} = \frac{2 + p_j(q_j - 2)}{2 + q_j(q_j - 2)}, \tag{2.26}$$

where all fractions are in lowest terms. From the above it follows $q_{j+1} - 1 = (q_j - 1)^2$ and thus $q_j = 2^{2^{j-1}} + 1$. Note that the above recursion can be written in the form

$$p_1 = 2, \quad q_1 = 3, \quad p_{j+1} = 2 + (2^{2^{j-1}} - 1)p_j,$$

$$q_{j+1} = 2 + (2^{2^{j-1}} - 1)q_j, \quad j \geq 1. \tag{2.27}$$

This yields

$$q_j - p_j = (2^{2^{j-2}} - 1)(q_{j-1} - p_{j-1}) = \cdots = \prod_{k=0}^{j-2}(2^{2^k} - 1) \tag{2.28}$$

and recalling that $q_j = 2^{2^{j-1}} + 1$ we get $p_j = 2^{2^{j-1}} + 1 - \prod_{k=0}^{j-2}(2^{2^k} - 1)$. We thus find the expression

$$\tau_j = 1 - \frac{\prod_{k=0}^{j-2}(2^{2^k} - 1)}{2^{2^{j-1}} + 1} = 1 - \frac{\prod_{k=0}^{j-1}(1 - 2^{-2^k})}{2(1 - 2^{-2^j})} \tag{2.29}$$

and

$$\tau_{j+1} - \tau_j = \left(\frac{2}{2^{2^j} + 1}\right)\tau_j. \tag{2.30}$$

The number

$$\tau_\infty = \lim_{j\to\infty} \tau_j = 1 - \frac{1}{2}\prod_{k=0}^{\infty}(1 - 2^{-2^k})$$

$$= 0.11010011001011010010110011010011... \tag{2.31}$$

satisfies $\tau_\infty = \tau(K_\infty)$ and is plainly irrational (since $K_\infty$ is aperiodic). One easily recognizes the Thue-Morse sequence beginning with 0, that is, the fixed point of the substitution $0 \to 01$ and $1 \to 10$ with prefix 0[1]. It enjoys the invariance property

$$\widetilde{\mathcal{R}}(\tau_\infty) = \tau_\infty \tag{2.32}$$

which can also be expressed in the form

$$\tau_\infty = \sum_{k=1}^{\infty} \frac{t_k}{2^k} = \sum_{k=1}^{\infty} \frac{t_{2^l k}}{2^k}, \quad \forall l \geq 0. \tag{2.33}$$

Thus, for instance, $t_k = 1$ whenever $k = 2^\ell$ for some $\ell \geq 0$. More specifically, we have

**Proposition 2.1.** *For an integer $p \geq 1$ set*

$$s(p) = \sum_{i \geq 0} n_i (\text{mod } 2) \ \ \textit{if} \ \ p = \sum_{i \geq 0} n_i 2^i, \ n_i \in \{0, 1\}.$$

*Let $\tau_\infty = 0.t_1 t_2 \ldots$. Then $t_k = s(p)$ whenever $k = p \cdot 2^\ell$ for some $\ell \geq 0$ and $p \geq 1$ odd.*

**Proof.** Due to (2.33) it will suffice to show by induction over $r$ the following property: $P_r = \{p \text{ odd and } p \leq 2^r \Rightarrow t_k = s(p)\}$. Note that $P_0$ is obvious. Consider an odd $p'$ such that $2^r < p' \leq 2^{r+1}$. Then $p' = 2^r + p$ with $1 \leq p \leq 2^r$ and $p$ odd. Then $s(p') = s(p) + 1 (\text{mod } 2)$ and by the above $P_r \Rightarrow P_{r+1}$.

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[1]By the way, we have shown the following result:

**Proposition 2.2.** *Let $\xi : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the map defined as $(\xi s)_k = \sum_{i=1}^{k} s_i (\text{mod } 2)$. Let $u$ be the fixed point of the Feigenbaum substitution $1 \to 10$ and $0 \to 11$ with prefix 1 and $w$ be the fixed point of the Thue-Morse substitution $0 \to 01$ and $1 \to 10$ with prefix 0. Then $0\xi(u) = \xi(0u) = w$.*

After this paper was finished I became aware of the work of Allouche and Cosnard [1] where some of the results presented here, in particular, the proposition above, were previously obtained.

Furthermore, from (2.23) we see at once that the symbols 0 and 1 both appear in $\tau_\infty$ with frequency 1/2. One may wonder if $\tau_\infty$ is a *normal* number, in the sense of Borel. That means that in its dyadic expansion (2.31) the asymptotic frequency of any word of length $n$ is $2^{-n}$. On the other hand, reasoning as for the sequence $K_\infty$ (and using the substitutions (2.23)) it is not difficult to verify that the frequency of the pairs 00, 01, 10 and 11 are $\frac{1}{3}, \frac{1}{6}, \frac{1}{6}$ and $\frac{1}{3}$. Therefore $\tau_\infty$ is not a normal number. In addition $\tau_\infty$ is transcendental, as is shown by Mahler in [11] (see also [2], [6], [8]).

We end this digression with some partial insight into the structure of the continued fraction expansion of $\tau_\infty$.

Recall that any number $\tau \in [0, 1]$ can be expanded as

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \ldots], \tag{2.34}$$

where $a_i$'s are integers. Successive truncations of this expansion yield a sequence of rational numbers

$$\frac{r_n}{s_n} = [a_1, a_2, a_3, \ldots, a_n] \tag{2.35}$$

which are called *convergents* of $\tau$ (see [10]).



Now, the problem we are interested in is the following: are the continued fraction expansions of the numbers $\tau_j$ predictable (i.e., have a definite pattern) as their binary expansions do? The expansions of the first eight $\tau_j$'s are

$\tau_1 = [1, 2]$

$\tau_2 = [1, 4]$

$\tau_3 = [1, 4, 1, 2]$

$\tau_4 = [1, 4, 1, 2, 2, 6]$

$\tau_5 = [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2]$

$\tau_6 = [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 1, 21, 1, 10, 2, 1, 1, 1, 5]$

$\tau_7 = [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 1, 21, 1, 10, 2, 1, 1, 1,$
$4, 1, 2, 29, 1, 24, 1, 1, 7, 11, 3, 2, 5, 1, 1, 1, 89]$

$\tau_8 = [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 1, 21, 1, 10, 2, 1, 1, 1, 4, 1,$
$2, 29, 1, 24, 1, 1, 7, 11, 3, 2, 5, 1, 1, 1, 88, 1, 1, 1, 6, 1, 1, 33, 2, 6, 1,$
$24, 1, 5, 212, 2, 1, 10, 1, 3, 11, 2, 1, 2, 1, 10, 1, 1, 2, 3, 2549, 1, 2].$

A direct inspection suggests that there is a subsequence $n_j$ of the integers so that if $\tau_j = [a_1, ..., a_{n_j}]$, then

$$\tau_{j+1} = \begin{cases} [a_1, ..., a_{n_j} - 1, b_{n_j+1}, ..., b_{n_{j+1}}] & \text{if } n_j \text{ is odd,} \\ [a_1, ..., a_{n_j}, b_{n_j+1}, ..., b_{n_{j+1}}] & \text{if } n_j \text{ is even,} \end{cases} \tag{2.36}$$

for some $b_{n_j+1}, ..., b_{n_{j+1}}$. The sequence $n_j$ for $1 \le j \le 12$ is

$$2, 2, 4, 6, 12, 23, 39, 71, 121, 253, 528, 1129.$$

Unfortunately we are not able to say much more. In particular it is not clear what kind of relation could be established between the $\tau_j$'s and the convergents of $\tau_\infty$. Note that Shallit obtained in [14] a rather complete description of the patterns arising for irrational numbers of the type $\sum_{k \ge 0} u^{-2^k}$, $u$ is an integer. On the other hand, a high-temperature-like expansion of the product appearing in (2.31) yields the expression

$$\tau_\infty = 1 - \frac{1}{2}\left[1 - \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \sum_{\substack{k_1 \ne k_2 \ne \cdots \ne k_\ell, \\ k_i \ge 0}} 2^{-\sum_{i=1}^{\ell} 2^{k_i}}\right] \tag{2.37}$$

of which the numbers studied by Shallit are just the first order $(\ell = 1)$

term with $u = 2$. We conclude with a brief description of the topological zeta functions arising in this situation.

**Zeta functions.** For the values $\tau = \tau_j$ considered above, we get the polynomial zeta function

$$1/\zeta(\tau_j, z) = (1 - z)\prod_{n=0}^{j}(1 - z^{2^n}), \tag{2.38}$$

whose zeroes are all on the unit circle $|z| = 1$. Moreover we have $\zeta(\tau_j, z) \to \zeta(\tau_\infty, z)$ when $j \to \infty$, where

$$1/\zeta(\tau_\infty, z) = (1 - z)\prod_{n=0}^{\infty}(1 - z^{2^n}). \tag{2.39}$$

From Sarkovskii theorem (see [3]) it follows that $h(\tau_j) = 0$ for all $j \geq 0$. Also $h(\tau_\infty) = 0$ (but for any $\tau \in \Lambda$ with $\tau > \tau_\infty$ we have $h(\tau) > 0$). Put

$$\Xi(z) = \prod_{n=0}^{\infty}(1 - z^{2^n}). \tag{2.40}$$



This function satisfies the functional equation

$$\Xi(z) = (1 - z)\Xi(z^2) \tag{2.41}$$

from which we see that if $\Xi(z) = 0$, then $|z| = 1$. In particular, given $m \geq 1$ and $k = 0, 1, \ldots, 2^l - 1$ all factors of the product defining $\Xi(z)$ corresponding to $n \geq m$ vanish at $z = e^{2\pi ik/2^l}$. Therefore the zeroes of $\Xi(z)$ are dense on the unit circle. We then have that the radius of convergence of $\zeta(\tau_\infty, z)$ is equal to 1 and that the unit circle is a (opaque) natural boundary for this function.

Finally, expanding the product (2.40) we get

$$\Xi(z) = 1 - z - z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + z^9$$
$$+ z^{10} - z^{11} + z^{12} - z^{13} - z^{14} + z^{15} - z^{16} + \ldots$$

from which we see that if $\tau_\infty = 0.t_1t_2t_3\ldots$, then the coefficient of $z^k$ with $k \geq 1$ in the above expansion is but the number $\varepsilon_k = 1 - 2t_k$ defined in (1.3), in agreement with Proposition 1.1. In turn, we notice that the number $\tau_\infty$ can be written as

$$\tau_\infty = 1 - \frac{1}{2}\,\Xi\left(\frac{1}{2}\right). \tag{2.42}$$

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