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MODULES HAVING THE PURE INTERSECTION PROPERTY

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Abstract

R will be a ring with identity and module M will be unital left R-modules. In this paper, we introduce the concept of modules having the pure intersection property (the PIP). We investigate the properties of modules with the PIP. We give a characterization of modules with the PIP, among others and prove that for a flat module M, M has the PIP if and only if for any pure submodules N and L, $N + L$ is flat.

1. Introduction

In what follows R will denote a ring with identity and an R-module will mean unitary left R-module. Cohn in [2] defined a submodule N of an R-module M is a *pure submodule* in M the sequence $0 \to N \otimes L \to M \otimes L$ is exact for every R-module L. Anderson and Fuller in [1] called the *submodule* N a pure submodule if for every right ideal I of R, $IM \cap N = IN$. Ribenboim in [5] defined N to be pure in M if $rM \cap N = rN$ for each $r \in R$. Although the first condition implies the

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second (see [4, p. 158]) and the second condition implies the third. An example given by Lam in [4, p. 158], showed that these definitions are not equivalent. In this work, the definition of purity will be that of Anderson and Fuller. A module M is called *pure simple* if M and 0 are the only pure submodules of M. An R-module M is said to have *the pure intersection property* (briefly the *PIP*) if the intersection of any two pure submodules is again pure. The left annihilator of an element x in an R-module M will be denoted by $l(x)$. $N \leq M$ means that N is a submodule of M.

2. Pure Submodules

In this section, we recall some basic definitions of flat and pure submodules and list some of their important properties that are relevant to our work.

Let R be a ring with identity and let M be a left unitary R-module. An R-module M is called a *flat module* if for every short exact sequence of R-modules:

$$0 \to N \to K \to L \to 0$$

the sequence

$$0 \to N \otimes M \to K \otimes M \to L \otimes M \to 0$$

is also exact.

The following theorem gives some characterizations of flat modules [6].

Theorem 2.1. *Let M be an R-module. Then the following statements are equivalent.*

(1) *M is flat R-module.*

(2) *For each (finitely generated) right ideal I of R, and for each monomorphism $f : I \to R$ the map $f \otimes id_M : I \otimes M \to R \otimes M$ is a monomorphism.*

(3) *For every (finitely generated) right ideal I of R, $IM \cong I \otimes M$.*

Recall that a submodule N of an R-module M is a pure submodule if for every right ideal I of R, $IM \cap N = IN$, (see [1]).

Remark 2.2. (1) Let M be an R-module and let N be a direct summand of M. Then N is a pure submodule of M.

(2) Let M be an R-module and let N be a pure submodule of M. If H is a pure submodule of N, then H is a pure submodule of M.

(3) Let M be an R-module and let N be a pure submodule of M. If L is a submodule of M containing N, then N is a pure submodule of L.

In the following propositions we give sufficient conditions under which every pure submodule of an R-module is a direct summand.

Proposition 2.3. *Let M be a prime and injective R-module. Then every pure submodule of M is a direct summand.*

Proof. Let N be a pure submodule of M and I be an ideal of R. Let $0 \neq f : I \to N$ be an R-homomorphism, $i : N \to M$ be the inclusion map and $i \circ f : I \to M$. Since M is injective, there exists an $m \in M$ such that $f(a) = am$, for all $a \in I$. Now, $am \in aM \cap N = aN$, because N is pure in M. Thus $am = an$ for some $n \in N$. If $m \neq n$, then $a \in l(m - n)$. But M is prime, therefore $l(m - n) = l(m)$ which is a contradiction. So N is injective and N is a direct summand of M.

Proposition 2.4. *Let M be a divisible R-module. Then every pure submodule of M is divisible.*

Proof. Let N be a pure submodule of M. Let $0 \neq r \in R$ and $n \in N$. Since M is divisible, there exists an $m \in M$ such that $n = rm \in rM \cap N = rN$. Since N is pure in M, $n = rn_1$ for some $n_1 \in N$. Therefore N is divisible.

Corollary 2.5. *Let R be a principal ideal domain and let M be a divisible R-module. Then every pure submodule of M is a direct summand.*

Lemma 2.6. *Let M be an R-module and let N be a pure submodule of M. If L is a submodule of M containing N and L/N is pure in M/N, then L is a pure submodule of M.*

Proof. Let I be any right ideal in R and let $x \in IM \cap L$. Since L/N is pure in M/N, $I(M/N) \cap L/N = I(L/N)$. Thus $(IM + N)/N \cap (L/N) = (IL + N)/N$ and hence $IL + N = (IM + N) \cap L$. Since $x \in IM \cap L \leq (IM + N) \cap L$, $x \in IL + N$. Let $x = w + n$, where $w \in IL$ and $n \in N$. Now, consider $n = x - w \in IM \cap N = IN \leq IL$. Thus $x \in IL$ and L is pure in M.

Lemma 2.7. *Let M be an R-module and let N and L be submodules of M such that $N \cap L$ and $N + L$ are pure submodules of M. Then each of N and L is a pure submodule of M.*

Proof. Let M be an R-module and let N and L be submodules of M such that $N \cap L$ and $N + L$ are pure submodules of M. To show that N is a pure submodule of M. Let I be any right ideal in R. Now, $IM \cap N \leq IM \cap (N + L) = I(N + L) \leq IN + IL$. But $IM \cap N \leq N$, then $IM \cap N \leq (IN + IL) \cap N = IN + IL \cap N$. Since $N \cap L$ is pure in M, $N \cap L$ is pure in L and $IL \cap (N \cap L) = I(N \cap L) \leq IL$. On the other hand, $IL \cap (N \cap L) = (IL \cap L) \cap N = IL \cap N$. Therefore $IM \cap N \leq IN$ and N is a pure submodule of M.

The set $T(M) = \{x \in M / l(x) \neq 0\}$ is a submodule of M called *torsion submodule* of M. If $T(M) = 0$, then M is called *torsion-free*.

Proposition 2.8. *Let M be a module over a principal ideal domain R and N be a submodule of M. If M/N is a torsion-free R-module, then N is pure submodule in M.*

Proof. Assume that M/N is a torsion-free R-module where R is a principal ideal domain. To show that $IM \cap N = IN$ for some right ideal of R. Let $x \in IM \cap N$. Then $x = am \in N$, $a \in I$ and $m \in M$. Therefore $\overline{0} = \overline{x} = \overline{am} \in M/N$. But M/N is torsion-free, then $\overline{m} = \overline{0}$ and $m \in N$. Hence N is pure in M.

It is known that $T(M/T(M)) = 0$ and $M/T(M)$ is torsion-free, for any R-module M.

Corollary 2.9. *Let M be a module over a principal ideal domain R. Then $T(M)$ is a pure submodule of M.*

Proposition 2.10. *Let M be a torsion-free module over a principal ideal domain R and X be a submodule of M. Then there exists a smallest pure submodule in M containing X.*

Proof. Consider the following set

$$(X)_p = \{m \in M \text{ such that } rm \in X \text{ for some } 0 \neq r \in R\}.$$

It is clear that $X \subseteq (X)_p$ and $(X)_p$ is a submodule of M. To show that $(X)_p$ is pure in M. Let I be any right ideal of R. Then $I = aR$ for some $a \in R$. Let $0 \neq x \in IM \cap (X)_p$. If $x = rm \in (X)_p$, for some $r \in I$ and $m \in M$, then there exists $0 \neq s \in R$ such that $sx = srm \in X$. Therefore $m \in (X)_p$ and $x \in I(X)_p$. Now, if $x = \sum_{i=1}^{n} r_i m_i \in (X)_p$, $r_i \in I$ and $m_i \in M$, then $x = ay \in I(X)_p$. Hence $(X)_p$ is pure in M.

Remark 2.11. Let M be an R-module and let N be a pure submodule of M. If N_1 is a submodule of M such that $N_1 \cong N$, then it is not necessary that N_1 is a pure submodule of M. For example, consider $\mathbb{Z}$ as $\mathbb{Z}$-module, let $N = \mathbb{Z}$ and $N_1 = 2\mathbb{Z}$. It is clear that $\mathbb{Z} \cong 2\mathbb{Z}$ and $\mathbb{Z}$ is a pure submodule of $\mathbb{Z}$. But $2\mathbb{Z}$ is not pure in $\mathbb{Z}$. In fact $2 = 2 \cdot 1 \in (2)\mathbb{Z} \cap (2\mathbb{Z})$ but $2 \notin (2)(2\mathbb{Z})$.

The following theorem is needed in our subsequent results. It can be found with its proof in [3] and [6].

Lemma 2.12. *Let M be an R-module and let P be a submodule of M.*

(1) *If M/P is a flat R-module, then P is a pure submodule of M.*

(2) *If M is a flat R-module, then M/P is a flat R-module if and only if P is a pure submodule of M.*

(3) *If M is a flat R-module and P is a pure submodule of M, then P is a flat R-module.*

3. Modules with the Pure Intersection Property

In this section we give the definition of modules having the pure intersection property with some examples and basic properties.

Definition 3.1. An R-module M is said to have the *pure intersection property* (briefly the PIP) if the intersection of any two pure submodules is again pure.

Remark 3.2. (1) Recall that an R-module M is called *pure simple* if M and 0 are the only pure submodules of M. It is clear that every pure simple R-module has the PIP. For example $\mathbb{Z}$ as $\mathbb{Z}$-module is pure simple. To see this, for every non trivial submodule $n\mathbb{Z}$ of $\mathbb{Z}$, $n = n \cdot 1 \in (n)\mathbb{Z} \cap n\mathbb{Z}$, but $n \notin (n)n\mathbb{Z}$.

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Let $N = \mathbb{Z}_4 \oplus 0$ and $L = \mathbb{Z}(\overline{1}, \overline{1})$, the submodule generated by $(\overline{1}, \overline{1})$. It is clear that each of N and L is a direct summand of M and hence N and L are pure submodules of M. But $N \cap L = \{(\overline{0}, \overline{0}), (\overline{2}, \overline{0})\}$ is not pure in M. In fact, consider the element $(\overline{2}, \overline{0}) = 2(\overline{1}, \overline{0}) \in (2)(\mathbb{Z}_4 \oplus \mathbb{Z}_2) \cap (N \cap L)$, but $(\overline{2}, \overline{0}) \notin (2)(N \cap L) = 0$.

Proposition 3.3. *Every torsion-free module M over a principal ideal domain has the PIP.*

Proof. Assume that M is a torsion-free module over a principal ideal domain R. Let N and L be two pure submodules of M. Let I be any ideal of R. Then $I = (a)$ for some $a \in R$. Now,

$$IM \cap (N \cap L) = (IM \cap N) \cap (IM \cap L) = IN \cap IL.$$

Let $x \in IM \cap (N \cap L)$. Thus $x = an = al$ for some $n \in N$ and $l \in L$. Therefore $a(n - l) = 0$ and $l = n$. Hence $x \in I(N \cap N)$ and M has the PIP.

Proposition 3.4. (1) *If an R-module M has the PIP, then every pure submodule of M has the PIP.*

(2) *If an R-module M has the PIP and N is a pure submodule of M, then M/N has the PIP.*

Proof. (1) Let M be an R-module with the PIP and N be a pure submodule of M. Let A and B be two pure submodules of N and I be any right ideal of R. Now,

$$\begin{aligned} IN \cap (A \cap B) &= (IM \cap N) \cap (A \cap B) \\ &= IM \cap (A \cap B) \\ &= I(A \cap B). \end{aligned}$$

Thus, N has the PIP.

(2) Let A/N and B/N be pure submodules of M/N and let K be a right ideal in R. We want to show that

$$(K(M/N)) \cap ((A/N) \cap (B/N)) = K((A/N) \cap (B/N)).$$

Each of A and B is pure in M, by Lemma 2.6. Since M has the PIP, $A \cap B$ is pure in M. Thus, $K(A \cap B) = KM \cap (A \cap B)$. It is clear that $K((A/N) \cap (B/N)) = K((A \cap B)/N) = (K(A \cap B) + N)/N$. Now,

$$\begin{aligned} K(M/N) \cap ((A/N) \cap (B/N)) &= (KM + N)/N \cap (A \cap B)/N \\ &= ((KM + N) \cap (A \cap B))/N \\ &= (KM \cap (A \cap B) + N)/N \\ &= (K(A \cap B) + N)/N \\ &= K((A/N) \cap (B/N)). \end{aligned}$$

Therefore M/N has the PIP.

4. Characterization of Modules with the Pure Intersection Property

In this section we give some characterization of modules with the pure intersection property.

Theorem 4.1. *Let M be an R-module. Then M has the PIP if and only if $I(N \cap L) = IN \cap IL$ for every right ideal I in R and for every pure submodules N and L of M.*

Proof. Suppose that M has the PIP and each of N and L is a pure submodule of M. Then $N \cap L$ is pure. Let I be any right ideal in R. Then $I(N \cap L) = IM \cap (N \cap L)$. Now,

$$\begin{aligned} IN \cap IL &= (IM \cap N) \cap (IM \cap L) \\ &= (IM \cap (N \cap L)) \\ &= I(N \cap L). \end{aligned}$$

Conversely, let N and L be pure submodules of M and I be a right ideal in R. Therefore

$$IM \cap (N \cap L) = (IM \cap N) \cap (IM \cap L) = IN \cap IL = I(N \cap L).$$

Thus $N \cap L$ is pure in M and hence M has the PIP.

As application of Theorem 4.1, we give the following corollary.

Corollary 4.2. *Every prime module M over a principal ideal domain has the PIP.*

Proof. Let I be an ideal in R and let N and H be pure submodules of M. Since R is a principal ideal domain, $I = (a)$ for some $a \in R$. We show that $a(N \cap H) = aN \cap aH$. Let $0 \neq x \in aN \cap aH$, hence $x = an = ah$, $n \in N$, $h \in H$, so $a(n - h) = 0$. Assume that $n \neq h$. Since $a \in l(n - h)$ and M is prime, $a \in l(n)$ and $x = 0$, which is a contradiction. Thus $n = h$ and $x \in a(N \cap H)$. So, by Theorem 4.1, M has the PIP.

The following theorem gives another characterization for modules with the PIP.

Theorem 4.3. *Let M be an R-module. Then M has the PIP if and only if for every pure submodules N and L of M and for every R-homomorphism $f : N \cap L \to M$ such that $N \cap Im\, f = 0$ and $N + Im\, f$ is pure in M, $Ker\, f$ is pure in M.*

Proof. Assume that M has the PIP. Let N and L be pure submodules of M and $f : N \cap L \to M$ be an R-homomorphism such that $N \cap Im\, f = 0$ and $N + Im\, f$ is pure in M. Let $T = \{x + f(x);\ x \in N \cap L\}$. Then, it is clear that T is pure in M. Let I be a right ideal in R and $y = \sum_{i=1}^{n} r_i m_i \in IM \cap T$, $r_i \in I$, $m_i \in M$, $i = 1, \ldots, n$. Hence, $y = x + f(x)$, for some $x \in N \cap L$. Since, $y = \sum_{i=1}^{n} r_i m_i = x + f(x) \in N \cap L + Im\, f \leq N + Im\, f$ and $N + Im\, f$ is pure in M, $y = \sum_{i=1}^{n} r_i m_i \in IM \cap (N + Im\, f) = I(N + Im\, f)$. Therefore, $y = \sum_{i=1}^{m} s_i(x_i + y_i)$, $x_i \in N$, $y_i \in Im\, f$, $s_i \in I$, $i = 1, \ldots, m$. Thus

$$y = \sum_{i=1}^{m} s_i x_i + \sum_{i=1}^{m} s_i y_i = x + f(x).$$

Hence, $x - \sum_{i=1}^{m} s_i x_i = \sum_{i=1}^{m} s_i y_i - f(x) \in N \cap Im\, f = 0$. Therefore, $x = \sum_{i=1}^{m} s_i x_i \in IN \cap (N \cap L)$. But $N \cap L$ is pure in M, hence it is pure in N and $IN \cap (N \cap L) = I(N \cap L)$. Thus $x \in I(N \cap L)$. Let $x = \sum_{i=1}^{k} h_i w_i$, $w_i \in N \cap L$, $h_i \in I$, $i = 1, \ldots, k$. Then $f(x) = \sum_{i=1}^{k} h_i f(w_i)$.

Now,

$$y = x + f(x) = \sum_{i=1}^{k} h_i w_i + \sum_{i=1}^{k} h_i f(w_i)$$

$$= \sum_{i=1}^{k} h_i (w_i + f(w_i)) \in IT.$$

Thus $IM \cap T = IT$ and T is pure in M. Next, we show that $Ker\, f = (N \cap L) \cap T$. Let $x \in Ker\, f$. Then $x \in N \cap L$ and $f(x) = 0$, hence $x \in T$. Now, let $x \in (N \cap L) \cap T$. Then $x = y + f(y)$, $y \in N \cap L$. Thus $x - y = f(y) \in N \cap Im\, f = 0$. Therefore $f(x) = f(y) = 0$ and $x \in Ker\, f$. Since M has the PIP, $(N \cap L) \cap T = Ker\, f$ is pure in M.

For the converse, let N and L be pure submodules of M. Define the R-homomorphism $f : N \cap L \to M$ by $f(x) = 0$, $\forall x \in N \cap L$. It is clear that $N \cap Im\, f = 0$ and $N + Im\, f$ is pure in M. Then $Ker\, f = N \cap L$ is pure in M. Hence, M has the PIP.

By the same argument, we can prove the following theorem.

Theorem 4.4. *Let M be an R-module. Then M has the PIP if and only if every pure submodules N and L of M and for every R-homomorphism* $f : N \cap L \to H$, *where H is a submodule of M such that* $N \cap H = 0$ *and* $N + H$ *is pure in M, Ker f is pure in M.*

The following corollary follows immediately from Theorem 4.4.

Corollary 4.5. *Let M be an R-module with the PIP. Let N and L be pure submodules of M such that* $N \cap L = 0$ *and* $N + L$ *is pure in M. Then for every R-homomorphism* $f : N \to L$, *Ker f is pure in M.*

The following corollary is the main tool for our subsequent results.

Corollary 4.6. *Let M be an R-module with the PIP. Then for every decomposition* $M = N \oplus L$ *and for every R-homomorphism* $f : N \to L$, *Ker f is pure in M.*

Proof. Since $N \cap L = 0$, $N + L = M$ is pure in M and $N = N \cap M$, by Theorem 4.4, *Ker f* is pure in M.

Remark 4.7. Let N be a pure submodule of an R-module M. Then there exists a pure submodule $\overline{N}$ in M such that $\overline{N}$ is maximal with respect to the property $N + \overline{N}$ is pure in M and $N \cap \overline{N} = 0$.

Proof. Consider the following set:

$$F = \{L;\ L \text{ is pure in } M \text{ such that } N \cap L = 0 \text{ and } N + L \text{ is pure in } M\}.$$

It is clear that $0 \in F$ and hence $F \neq \varnothing$. Let $\{H_\alpha\}_{\alpha\in\Lambda}$ be a chain in F. Then, it is clear that $\bigcup_{\alpha\in\Lambda} H_\alpha$ is a submodule of M and since $H_\alpha \cap N = 0$, for all $\alpha \in \Lambda$, $\left(\bigcup_{\alpha\in\Lambda} H_\alpha\right) \cap N = 0$. By Exercise 19.11 (1) in [1], $\bigcup_{\alpha\in\Lambda} H_\alpha$ is pure in M. To show that $N + \bigcup_{\alpha\in\Lambda} H_\alpha$ is pure in M. Let $\sum_{i=1}^{n} r_i m_i \in IM \cap \left(N + \left(\bigcup_{\alpha\in\Lambda} H_\alpha\right)\right)$. Thus $\sum_{i=1}^{n} r_i m_i \in N + \left(\bigcup_{\alpha\in\Lambda} H_\alpha\right)$. Therefore $\sum_{i=1}^{n} r_i m_i \in N + H_{\alpha_0}$, for some $\alpha_0 \in \Lambda$ and hence $\sum_{i=1}^{n} r_i m_i \in I(N + H_{\alpha_0})$. Thus $\sum_{i=1}^{n} r_i m_i \in I\left(N + \bigcup_{\alpha \in \Lambda} H_\alpha\right)$. By Zorn's lemma, F has a maximal element say $\overline{N}$.

We call a pure submodule satisfying the condition in Remark 4.7, a pure complement of N and we denote it by $\overline{N}$.

Theorem 4.8. *Let M be an R-module such that for every pure submodules N and L in M either $N \leq L \oplus \overline{L}$ or $L \leq N \oplus \overline{N}$. M has the PIP if and only if for every R-homomorphism $f : N \cap (L \oplus \overline{L}) \to \overline{N}$, $Ker\, f$ is pure in M.*

Proof. Suppose that M has the PIP and N and L are pure submodules of M. Let $f : N \cap (L \oplus \overline{L}) \to \overline{N}$ be an R-homomorphism. Then, by Theorem 4.4, $Ker\, f$ is pure in M.

For the converse, let N and L be pure submodules of M such that $N \leq L \oplus \overline{L}$. Let $\pi_1 : N \oplus \overline{N} \to N$ and $\pi_2 : L \oplus \overline{L} \to \overline{L}$ be the natural projections. Let $h = \pi_2 \circ \pi_1|_{L \cap (N \oplus \overline{N})}$. Then we show that $Ker\, h = (L \cap N) \oplus (L \cap \overline{N})$. Let $x \in Ker\, h$. Then $x \in L \cap (N \oplus \overline{N})$ and $x = n + \overline{n}$, $n \in N$ and $\overline{n} \in \overline{N}$. Now, $0 = h(x) = \pi_2 \circ \pi_1(n + \overline{n}) = \pi_2(n)$. So $n \in L$ and $\overline{n} \in L$. Thus $x \in (L \cap N) \oplus (L \cap \overline{N})$. Now, let $x \in (L \cap N) \oplus (L \cap \overline{N})$. Then $x = n + \overline{n}$, $n \in L \cap N$ and $\overline{n} \in L \cap \overline{N}$. Thus $h(x) = \pi_2 \circ \pi_1(n + \overline{n}) = \pi_2(n) = 0$. Therefore, $Ker\, h = (L \cap N) \oplus (L \cap \overline{N})$ is pure. Since $N \cap L$ is pure in $Ker\, h$, $N \cap L$ is pure in M and M has the PIP.

Now, we give an example of modules that do not have the PIP.

Example 4.9. Let R be an integral domain and let Q be the quotient field of R considered as R-module. Then for any $0 \neq N$ proper submodule of Q, $Q \oplus Q/N$ does not satisfy the PIP. In fact, let $f : Q \to Q/N$ be the natural epimorphism. Then $Ker\, f = N$. It is known that Q is pure simple. Thus, N is not pure in Q. Hence, by Corollary 4.6, $Q \oplus Q/N$ does not have the PIP.

Proposition 4.10. *Let M be an R-module such that if for any two pure submodules N and L of M, $N+L$ is flat R-module. Then M has the PIP.*

Proof. Let N and L be pure submodules of M. By the second isomorphism theorem, $N/(N \cap L) \cong (N+L)/L$. Since $N+L$ is flat R-module and L is pure in M, by Remark 2.2(3), L is pure in $N+L$ and, by Lemma 2.12(2), $(N+L)/L$ is flat R-module. Thus $N/(N \cap L)$ is flat and hence $N \cap L$ is pure in N (Lemma 2.12(1)). But N is pure in M, so by Remark 2.2(2), $N \cap L$ is pure in M and M has the PIP.

The converse of Proposition 4.10 is not true in general as the following example shows. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$. Since M is semisimple, M has the PIP. Let $N = \langle \overline{0}, \overline{2}, \overline{4} \rangle$ and $L = \langle \overline{0}, \overline{3} \rangle$. Then N and L are pure in M, but $N + L = M = \mathbb{Z}_6$ is not flat.

Theorem 4.11. *Let M be a flat R-module. Then M has the PIP if and only if for any two pure submodules N and L of M, $N+L$ is flat R-module.*

Proof. Assume that M has the PIP. Let I be a right ideal in R and N, L are pure submodules of M. Consider the following short exact sequence

$$0 \to N \cap L \xrightarrow{f} N \oplus L \xrightarrow{g} N + L \to 0,$$

where $f(x) = (x, -x)$ for each $x \in N \cap L$ and $g(n, l) = n + l$ for each $n \in N$ and $l \in L$.

Now, we construct the following diagram:

$$\begin{array}{ccccccccc}
 & & I \otimes (N \cap L) & \xrightarrow{1 \otimes f} & I \otimes (N \oplus L) & \xrightarrow{1 \otimes g} & I \otimes (N + L) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & IN \cap IL & \xrightarrow[\bar{f}]{} & IN \oplus IL & \xrightarrow[\bar{g}]{} & IN + IL & \longrightarrow & 0
\end{array}$$

where $\bar{f}(x) = (x, -x)$ for each $x \in IN \cap IL$, $\bar{g}(n, l) = n + l$ for each $n \in IN$ and $l \in IL$, $\alpha(r \otimes x) = rx$ for each $r \in I$ and $x \in N \cap L$, $\beta(r \otimes (n, l)) = (rn, rl)$ for each $r \in I$, $n \in N$ and $l \in L$ and $\gamma(r \otimes (n + l)) = rn + rl$ for each $r \in I$, $n \in N$ and $l \in L$.

It can be easily checked that the diagram is commutative. Since N and L are pure in M and M is flat R-module, by Lemma 2.12(3), N and L are flat R-modules and hence $N \oplus L$ is flat R-module. By Theorem 2.1(3),

$$I \otimes (N \oplus L) \cong I(N \oplus L) = IN \oplus IL.$$

Thus β is an isomorphism. Therefore α is an epimorphism if and only if γ is a monomorphism, (see [7]). It is easily to see that $\alpha(I \otimes (N \cap L)) = I(N \cap L)$. Hence, α is onto if and only if M has the PIP by Theorem 4.1. Moreover, γ is a monomorphism if and only if $I \otimes (N + L) \cong \gamma(I \otimes (N + L)) = I(N + L)$. Thus γ is monomorphism if and only if $N + L$ is a flat R-module, by Theorem 2.1(3). Thus M has the PIP if and only if $N + L$ is a flat R-module for any pure submodules N and L of M.

The converse follows from Proposition 4.10.

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References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer- Verlag, New York, 1992.

[2] P. M. Cohn, On the free product of associative rings, Math. Z. 71 (1959), 380-398.

[3] F. Kasch, Modules and Rings, Academic Press, London, 1982.

[4] T. Y. Lam, Lectures on Modules and Rings, Springer, 1999.

[5] P. Ribenboim, Algebraic Numbers, Wiley, 1972.

[6] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[7] R. Wisbauer, Foundations of Module and Ring Theory, Gordon & Breach, Reading, 1991.

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