Far East J. Dynamical Systems **6**(1) (2004), 67-77

EIGEN PARAMETER PERTURBATION AND APPLICATION TO LINEAR DYNAMIC SYSTEM MODIFICATIONS

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Abstract

Rapid re-analysis of dynamic systems after any hardware modification is a problem of considerable practical importance. Re-analysis or structural dynamic modifications is the method of obtaining quick solution for modified systems by those of the unmodified system without going for time taking exact solution. The present paper addresses itself the above problem. A faster eigen parameter perturbation approach is applied in such cases. The method is illustrated by a numerical example of general symmetric matrix and a multi-degree of freedom spring mass system. Results are compared with the exact solution.

1. Introduction

A frequently encountered problem in dynamic mechanical systems is to obtain the modified behaviour of the system after the hardware changes have been incorporated. This modified behaviour may be desired to improve the performance in the existing systems. In case the

2000 Mathematics Subject Classification: 37-XX, 82-XX.

Key words and phrases: perturbation, linear dynamic system modification, multi-degree of freedom system, eigenvalue, eigenvectors.

Communicated by Ernesto Pérez Chavela

Received May 31, 2003

modification desired is drastically high, a completely new analysis and computation cycle is necessary. But if the new design varies only slightly from the old one, then the question is whether the information from the old design can be used to extract information concerning the new design. In particular, the question of interest here is whether the eigensolution already available can be used to derive the eigensolution corresponding to the new design, without extensive additional computation.

Eigenvalue re-analysis of linear dynamic systems have been studied by many investigators [1-15]. Several methods have been reported in the literature, which are reviewed in references [5, 12, 13]. In one of the earliest work by Fox and Kapoor [3] gave exact expressions for the derivatives of eigenvalues and eigenvectors with respect to any design variable. The expressions derived in this reference are valid for symmetric undamped systems. Later, many authors [2, 4, 6, 7, 8, 9, 10, 11, 15] used the approach given by Fox and Kapoor for determining eigensolution derivatives for more general cases. Very recently derivative of eigenvalues and eigenvectors of multi-degree-of-freedom damped linear dynamic systems with respect to arbitrary design changes is presented by Adhikari [1]. In the sequel, it is proposed to study the efficiency of the perturbation technique of modification process by first considering an arbitrary real matrix and then specializing the results to linear dynamic systems consisting of spring and mass elements.

2. Eigen Parameter Perturbation Theory

Let A be a characteristic matrix. The eigenvalues λ_{0i}, $i = 1, \ldots, n$, of A_0 are taken to be distinct. Such system matrix models are common in vehicle dynamics, rotor dynamics, structural vibrations analysis, and electrical systems analysis.

The right eigenvectors x_{0i}, $i = 1, \ldots, n$ of A_0 satisfy

$$A_0 x_{0i} = \lambda_{0i} x_{0i}, \tag{1a}$$

while transpose of A_0 satisfy

$$A_0^T y_{0j} = \lambda_{0j} y_{0j}, \quad j = 1, \ldots, n, \tag{1b}$$

where x_{0i} and y_{0j} are orthogonal and can be scaled such that $x_{0i}^T y_{0j} = y_{0j}^T x_{0i} = \delta_{ij}$, where δ_{ij} is the Kronecker delta, zero for $i \neq j$, and unity for $i = j$, in which case the eigenvectors are said to be orthogonalized. Next, considering the eigenvalue perturbation problem associated with the real $n \times n$ symmetric matrix such that

$$A = A_0 + A_1, \tag{2}$$

where A_0 is the original matrix and A_1 is $n \times n$ matrix representing small changes from A_0. The matrix A will be referred as modified or perturbed matrix. The modified eigenvalue problem can be written in the form

$$Ax_i = \lambda_i x_i, \quad i = 1, ..., n, \tag{3a}$$

$$y_i^T A = \lambda_i y_i^T, \quad i = 1, ..., n, \tag{3b}$$

where λ_i, x_i, y_i are the modified eigenvalues, the modified right eigenvectors and the modified left eigenvectors respectively. In this context, equations (1) represent the unmodified eigenvalue problem. Since the eigenvalues are assumed to be distinct, the eigenvectors are orthogonal, i.e.,

$$y_j^T x_i = \delta_{ij}, \quad y_j^T A x_i = \lambda_i \delta_{ij}, \quad i, j = 1, ..., n. \tag{4}$$

The perturbed eigenvalues and eigenvectors can be written as

$$\lambda_i = \lambda_{0i} + \lambda_{1i}, \quad x_i = x_{0i} + x_{1i}, \quad y_i = y_{0i} + y_{1i}, \quad i = 1, ..., n, \tag{5}$$

where λ_{1i}, x_{1i}, y_{1i} are first-order perturbations.

Using equations (2) and (5), equation (3) can be written as

$$(A_0 + A_1)(x_{0i} + x_{1i}) = (\lambda_{0i} + \lambda_{1i})(x_{0i} + x_{1i}), \tag{6a}$$

$$(y_{0i}^T + y_{1i}^T)(A_0 + A_1) = (\lambda_{0i} + \lambda_{1i})(y_{0i}^T + y_{1i}^T), \tag{6b}$$

where $i = 1, ..., n$.

It is clear that if λ_{1i} and x_{1i} are known, then y_{1i} can be obtained from analogy. Therefore, the determination of λ_{1i} and x_{1i} are the main objectives from the known parameters A_0, A_1, λ_{0i}, x_{0i} and y_{0i}.

Since eigenvectors $x_{01}, x_{02}, ..., x_{0n}$ are linearly independent, the perturbed eigenvectors can be written as

$$x_i = x_{0i} + \sum_{k=1}^{n} \varepsilon_{ik} x_{0k}, \qquad \varepsilon_{ii} = 0 \text{ for } i = 1, ..., n,$$

where ε_{ik} $(i \neq k)$ are small coefficients. Hence, the eigenvector perturbation is simply

$$x_{1i} = \sum_{k=1}^{n} \varepsilon_{ik} x_{0k}, \qquad \varepsilon_{ii} = 0 \text{ for } i = 1, ..., n \tag{7}$$

therefore, λ_{1i} and ε_{ik} $(i \neq k)$ need to be obtained.

In reply to seeking a solution accurate to the first order, second order terms may be ignored. This reduces the equation (6a) to

$$A_0 x_{1i} + A_1 x_{0i} = \lambda_{0i} x_{1i} + \lambda_{1i} x_{0i}, \quad i = 1, ..., n. \tag{8}$$

Pre-multiplying equation (8) by y_{0j}^T, it can be rewritten as

$$y_{0j}^T A_0 x_{1i} + y_{0j}^T A_1 x_{0i} = \lambda_{0i} y_{0j}^T x_{1i} + \lambda_{1i} y_{0j}^T x_{0i} \text{ for } i = 1, ..., n. \tag{9}$$

Recalling equation (7); however, following expression can be alternately written as

$$\begin{aligned} y_{0j}^T A_0 x_{1i} &= y_{0j}^T A_0 \sum_{k=1}^{n} \varepsilon_{ik} x_{0k} = \sum_{k=1}^{n} \varepsilon_{ik} y_{0j}^T A_0 x_{0k} \\ &= \sum_{k=1}^{n} \varepsilon_{ik} \lambda_{0j} \delta_{jk} = \varepsilon_{ij} \lambda_{0j} \end{aligned} \tag{10}$$

$$\begin{aligned} y_{0j}^T x_{1i} &= y_{0j}^T \sum_{k=1}^{n} \varepsilon_{ik} x_{0k} = \sum_{k=1}^{n} \varepsilon_{ik} y_{0j}^T x_{0k} \\ &= \sum_{k=1}^{n} \varepsilon_{ik} \delta_{jk} = \varepsilon_{ij}. \end{aligned} \tag{11}$$

Using equations (10-11), equation (9) can be written as

$$\varepsilon_{ij}(\lambda_{0j} - \lambda_{0i}) + y_{0j}^T A_1 x_{0i} = \lambda_{1i}\delta_{ij}. \tag{12}$$

But when $i = j$, $\varepsilon_{ij} = 0$, therefore equation (12) yields the perturbed eigenvalue

$$\lambda_{1i} = y_{0i}^T A_1 x_{0i}, \quad i = 1, \dots, n. \tag{13}$$

On the other hand, when $i \neq j$, $\delta_{ij} = 0$, therefore equation (12) yields

$$\varepsilon_{ik} = \frac{y_{0k}^T A_1 x_{0i}}{(\lambda_{0i} - \lambda_{0k})}, \quad i, k = 1, \dots, n; \; i \neq k \tag{14}$$

substituting equation (14) into (7), the eigenvector perturbation vectors can be obtained as

$$x_{1i} = \sum_{k=1}^{n} \varepsilon_{ik} x_{0k}, \quad \varepsilon_{ik} = 0, \quad i, k = 1, \dots, n. \tag{15}$$

By analogy, the adjoint perturbation vectors can be written in the form

$$y_{ij} = \sum_{k=1}^{n} \gamma_{jk} y_{0k}, \quad \gamma_{jk} = 0, \quad j = 1, \dots, n, \tag{16}$$

where the coefficients γ_{jk} can be given as

$$\gamma_{jk} = \frac{x_{0k}^T A_1 y_{0j}}{(\lambda_{0j} - \lambda_{0k})}, \quad j, k = 1, \dots, n; \quad j \neq k. \tag{17}$$

For real and symmetric matrix, right and left eigenvectors coincide. Therefore, the eigenvalue perturbation becomes

$$\lambda_{1i} = x_{0i}^T A_1 x_{0i}, \quad i = 1, \dots, n \tag{18}$$

and coefficient of eigenvector perturbation expansion is written as

$$\varepsilon_{ik} = \frac{x_{0k}^T A_1 x_{0i}}{(\lambda_{0i} - \lambda_{0k})}, \quad i, k = 1, \dots, n, \quad i \neq k. \tag{19}$$

The eigenvector perturbations are real, i.e.,

$$x_{1i} = \sum_{k=1}^{n} \frac{x_{0k}^T A_1 x_{0i}}{(\lambda_{0i} - \lambda_{0k})} x_{0k}, \quad \varepsilon_{ii} = 0, \quad i = 1, 2, \ldots, n. \tag{20}$$

Equations (18) and (20) can be used as a first order eigenvalue perturbation respectively. Therefore perturbed eigenvalue and eigenvectors can be obtained from (5). The above mathematical formulation is applied first to a typical general symmetric matrix and further verified on a linear spring mass dynamic system, which follows.

3. Application of Perturbation Theory

In practice it is useful to have a method of calculating new minor changes in the components of a dynamic system affect the eigen parameters of the whole dynamic system without being obliged to resolve the new eigen parameters. The perturbation theory derived in the previous section can provide such a method and thereby allows economics in human and computer resource usage. To depict this economy following two examples are demonstrated.

3.1. Example 1

Let A_0 be a general symmetric matrix and given as

$$A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & -3 & 10 \end{bmatrix}.$$

The eigen parameters are obtained after eigensolution as

$$\lambda_0 = \begin{bmatrix} 1.3700 & 0 & 0 \\ 0 & 3.3714 & 0 \\ 0 & 0 & 11.2586 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0.8319 & 0.5533 & 0.0417 \\ 0.5241 & -0.7589 & -0.3865 \\ 0.1822 & -0.3434 & -0.9213 \end{bmatrix}.$$

The small change from A_0 can be represented by A_1 which can be typically assumed as

$$A_1 = \begin{bmatrix} 0.1 & -0.2 & 0 \\ -0.2 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The perturbed A_0 matrix can be represented as A and can be given by using equation (2) as

$$A = A_0 + A_1 = \begin{bmatrix} 2.1 & -1.2 & 0 \\ -1.2 & 4.2 & -3 \\ 0 & -3 & 10 \end{bmatrix}.$$

The eigen parameters are obtained for matrix A by two methods as discussed in the previous section. First, it is resolved for eigen parameters which are given as

$$\lambda_{\text{resolved}} = \begin{bmatrix} 1.3191 & 0 & 0 \\ 0 & 3.6841 & 0 \\ 0 & 0 & 11.2968 \end{bmatrix}$$

$$x_{\text{resolved}} = \begin{bmatrix} 0.8237 & 0.5647 & 0.0517 \\ -0.5360 & -0.7455 & -0.3962 \\ 0.1852 & -0.3541 & 0.9167 \end{bmatrix}.$$

Secondly, using the perturbation method as described in the previous section. Thus the perturbed eigen parameters are obtained using equations (18) and (20) respectively as

$$\lambda_{\text{perturbed}} = \begin{bmatrix} 1.2689 & 0 & 0 \\ 0 & 3.9978 & 0 \\ 0 & 0 & 11.3333 \end{bmatrix}$$

$$x_{\text{perturbed}} = \begin{bmatrix} 0.8237 & 0.5650 & 0.0517 \\ 0.5362 & -0.7456 & -0.3962 \\ 0.1851 & -0.3541 & 0.9162 \end{bmatrix}.$$

It is observed from the above numerical example that the perturbation method can be utilized for obtaining the eigen parameters for the modified system matrices with good accuracy. The advantage of this method would be appreciated when the system matrix is of large in size and resolution may no longer be economical. This method is further applied to a linear dynamic spring mass system, which is follows.

3.2. Example 2

To further strengthen the above economical and efficient method of obtaining modified behaviour of the perturbed system matrices, a linear multi degree of freedom spring mass system shown in the Figure 1 is considered. The design parameters are typically chosen as $K_1 = K_2 = K_3 = K_4 = K_5 = K_6 = 1000$ N/m and $m_1 = 0.5$ kg, $m_2 = 1$ kg and $m_3 = 1.5$ kg. The system matrices can be obtained [9] and given as

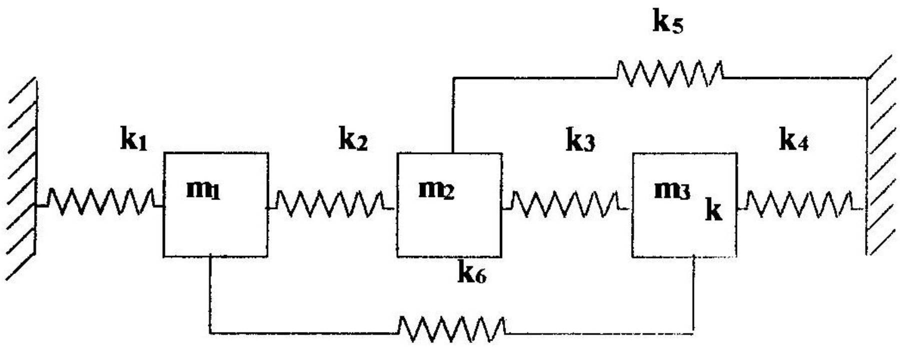

Figure 1. Multi degree of freedom spring mass system

$$K = 10^3 * \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}.$$

The equation of motion for free vibration is written and a significant part of the dynamic analysis is to obtain the eigensolution of the equation in terms of K and M as $KX = \lambda MX$ or $M^{-1}KX = \lambda X$, where the scalar quantities λ (eigenvalues) and the corresponding non-trivial vectors x (eigenvectors) are to be obtained. Typically $M^{-1}K$ may be treated similar to A_0 as shown in the Example 1. Therefore A_0 in this case can be written as

$$A_0 = 10^3 * \begin{bmatrix} 6 & -2 & -2 \\ -1 & 3 & -1 \\ -2/3 & -2/3 & 2 \end{bmatrix}.$$

If a change in system parameters shown in Figure 1, is desired such that k_2 is modified from 1000 to 1500 N/m, a perturbation matrix can be written for this case as

$$A_1 = 10^3 * \begin{bmatrix} 1 & -1 & 1 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore perturbed matrix A can be written as

$$A = 10^3 * \begin{bmatrix} 7 & -3 & -2 \\ -1.5 & 3.5 & -1 \\ -2/3 & -2/3 & 2 \end{bmatrix}.$$

Now the new eigen parameters may be obtained by two methods as depicted in the Example 1 as

$$\lambda_{\text{resolved}} = \begin{bmatrix} 8083.4 & 0 & 0 \\ 0 & 952.3 & 0 \\ 0 & 0 & 3464.3 \end{bmatrix}$$

$$x_{\text{resolved}} = \begin{bmatrix} -0.9525 & 0.4960 & -0.3638 \\ 0.2960 & 0.5544 & -0.7741 \\ 0.0719 & 0.6683 & 0.5181 \end{bmatrix}$$

$$\lambda_{\text{perturbed}} = \begin{bmatrix} 7998.2 & 0 & 0 \\ 0 & 934.6 & 0 \\ 0 & 0 & 3459.9 \end{bmatrix}$$

$$x_{\text{perturbed}} = \begin{bmatrix} -0.9728 & 0.4445 & -0.2955 \\ 0.2628 & 0.5686 & -0.8112 \\ 0.0415 & 0.6747 & 0.5222 \end{bmatrix}.$$

The above results further support the application of perturbation method. In general resolution of the system are always time consuming and costly affairs. Therefore perturbation method would be always preferred to modify the system, if the changes required is less. The broad conclusion may be made as follows.

4. Conclusions

The method presented allows one to solve the problem of modification of linear dynamic system. The method produces analytical formulas defining the perturbed eigenvalues and eigenvectors, which considerably facilitates and accelerates in evaluating the modified behaviour of the dynamic system due to small change in system parameters. The resolution for small design changes from an initial configuration is computationally expensive, inelegant, and inefficient in terms of analyst's time and effort, and in many instances the perturbation method should be preferable.

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