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# REALIZING NUMERICAL SEMIGROUPS AS WEIERSTRASS SEMIGROUPS: A COMPUTATIONAL APPROACH

**FRANCISCO L. R. PIMENTEL**

*Universidade Federal do Ceará*

*Campus do Pici*, *Bloco* 914

*CEP* 60455-750, *Fortaleza*, *Ceará*, *Brasil*

e-mail: pimentel@mat.ufc.br

**GILVAN OLIVEIRA**

*Departamento de Matemática*

*CCE*, *UFES Campus de Goiabeiras*

*CEP* 29075-910, *Vitória*, *ES*, *Brasil*

e-mail: gilvan@cce.ufes.br



**Abstract**

We propose a method for realizing numerical semigroups as Weierstrass semigroups. Given a specific numerical semigroup as input, if it is a Weierstrass semigroup, we obtain as the output the explicit equations of a canonically embedded nonsingular curve whose Weierstrass semigroup at a distinct point is this numerical semigroup. To illustrate the method we realize the numerical semigroup $\langle 6, 8, 10, 11 \rangle$ as a Weierstrass semigroup.

## 1. Introduction

Let $C$ be a nonsingular, complete, irreducible, algebraic curve of

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genus $g$ defined over an algebraically closed field $k$ of characteristic zero. For each point $P$ of $C$ we can associate the semigroup $H_C(P)$ of the pole orders of the rational functions on $C$ that are regular outside $P$. The semigroup $H_C(P)$ is a numerical semigroup, that is, a subsemigroup of the natural numbers $\mathbb{N}$ whose complement is finite, and it is called the *Weierstrass semigroup* of $C$ at $P$. A classical question, posed by Hurwitz [7] in 1893, was whether any numerical semigroup was a Weierstrass semigroup. In 1980 Buchweitz [1] answered negatively this question by showing that the numerical semigroup $\langle 13, 14, 15, 16, 17, 18, 20, 22, 23\rangle$ cannot be a Weierstrass semigroup $H_C(P)$ of a curve $C$. The question of which numerical semigroups are Weierstrass semigroups is difficult and only partial results are known (see, for example, [5], [9], [10] and [17]).

In this note we propound the use of Stöhr's method (cf. [16], [13] and [11]) for constructing the moduli space of pointed curves with a given Weierstrass semigroup to handle the related problem that consists in giving a specific numerical semigroup and asking if it is a Weierstrass semigroup. In general, this kind of problem is solved by ad hoc techniques. Our method is systematic, the technique used is very flexible and in principle can be applied to any semigroup, the computational complexity being the only impediment. To illustrate the method we realize the numerical semigroup $\langle 6, 8, 10, 11\rangle$ as a Weierstrass semigroup and exhibit a pointed canonically embedded curve $(C, P)$ with $H_C(P) = \langle 6, 8, 10, 11\rangle$. To the best of our knowledge it was not known whether this semigroup was a Weierstrass semigroup, anyway other numerical semigroup could be considered.



## 2. The Method

Let $C$ be a nonsingular, complete, irreducible, nonhyperelliptic, nontrigonal curve of genus $g$ defined over an algebraically closed field $k$. Let $P$ be a point of $C$ with Weierstrass semigroup $H_C(P)$ and let $L = \mathbb{N} - H_C(P) = \{l_1, l_2, \ldots, l_g\}$ be the set of Weierstrass gaps at $P$. Let $\omega_C$ be the dualizing sheaf of $C$. By the Riemann-Roch theorem there exists a $P$-Hermitian basis for $\mathrm{Hom}^0(C, \omega_C)$, or equivalently, there exist regular differentials $\omega_{l_1}, \omega_{l_2}, \ldots, \omega_{l_g}$ on $C$ whose orders at $P$ are $l_1 - 1$, $l_2 - 1, \ldots, l_g - 1$, respectively. Since, by hypothesis, $C$ is nonhyperelliptic, we can identify it with its image under the canonical embedding

$$(\omega_{l_1} : \omega_{l_2} : \cdots : \omega_{l_g}) : C \hookrightarrow \mathbb{P}_k^{g-1}.$$

Thus $C$ becomes a projective, nondegenerate curve of genus $g$ and degree $2g - 2$ in $\mathbb{P}_k^{g-1}$, and the integers $l_1 - 1, l_2 - 1, \ldots, l_g - 1$ are precisely the intersection multiplicities of $C$ with the hyperplanes at the point $P = (1 : 0 : \cdots : 0)$. Let $I(C)$ be the ideal of $C$. Then we have that $I(C)$ is the set of polynomials $f$ in the indeterminates $W_{l_1}, \ldots, W_{l_g}$ satisfying $f(\omega_{l_1}, \ldots, \omega_{l_g}) = 0$. So $I(C)$ is the homogeneous ideal $\oplus_{i=2}^{\infty} I_n(C)$, where $I_n(C)$ is the vector space of $n$-forms that vanish identically on $C$. By a theorem of Noether (cf. [15], Theorem 1.2) the homomorphism

$$k[W_{l_1}, \ldots, W_{l_g}]_n \to \operatorname{Hom}^0(C, \omega_C^{\otimes n})$$

induced by the liftings $W_{l_i} \to \omega_{l_i}$, $i = 1, \ldots, g$, is onto for each $n$ and the canonical curve $C \subset \mathbb{P}_k^{g-1}$ is arithmetically Cohen-Macaulay. Then we have that

$$\dim_k \frac{k[W_{l_1}, \ldots, W_{l_g}]_n}{I_n(C)} = (2n-1)(g-1)$$

or equivalently, $\dim_k I_n(C) = \binom{n+g-1}{n} - (2n-1)(g-1)$, for each $n \geq 2$. In particular, the vector space $I_2(C)$ of quadratic relations has dimension $\frac{(g-2)(g-3)}{2}$. Let $L_2 = \{l_i + l_j \mid l_i, l_j \in L\}$. Then for each $s \in L_2$ we consider all the partitions of $s$ as sums of two gaps

$$s = a_{si} + b_{si} \quad (i = 0, \ldots, \nu_s)$$

with $a_{si} \leq b_{si}$ and $a_{si}, b_{si} \in L$. We put $a_s = a_{s0}$ and $b_s = b_{s0}$, where $b_{s0}$ is the largest among the $b_{si}$'s. Observing that the $\omega_{a_s}\omega_{b_s}$'s have different orders at $P$, there exists a monomial basis $\mathcal{B}$ of $\operatorname{Hom}^0(C, \omega_C^{\otimes 2})$

containing all the $\omega_{a_s}\omega_{b_s}$ (see [12], Theorem 2.1). After normalization we can obtain $\frac{(g-2)(g-3)}{2}$ equations, one for each $\omega_{a_{si}}\omega_{b_{si}}$ not in $\mathcal{B}$,

$$\omega_{a_{si}}\omega_{b_{si}} = \omega_{a_s}\omega_{b_s} + \sum_{\substack{l+l'>s \\ \omega_l\omega_{l'}\in\mathcal{B}}} c_{sill'}\omega_l\omega_{l'},$$

where $c_{sill'} \in k$ and $l + l' \in L_2$. All the $\frac{(g-2)(g-3)}{2}$ forms

$$F_{si} = W_{a_{si}}W_{b_{si}} - W_{a_s}W_{b_s} - \sum_{\substack{l+l'>s \\ \omega_l\omega_{l'}\in\mathcal{B}}} c_{sill'}W_lW_{l'} \tag{1}$$

vanish identically on $C$ and are linearly independent. Thus, they form a $k$-basis for the vector space of the quadratic relations $I_2(C)$. Indeed, since $C$ is nontrigonal, by a classical theorem of Petry (cf. [15]) it follows that the forms $F_{si}$ generate the ideal $I(C)$. It is worth observing that we know exactly the Weierstrass semigroups on trigonal curves (cf. [2], [3] and [8]). Moreover, our method can be adapted to the trigonal case by adding $g-3$ cubic forms to the generators of the ideal $I(C)$ (see [15]).



Stöhr's idea consists in reversing the above considerations. Initially, we define for each form $F_{si}$ the homogeneous and isobaric form

$$F_{si}^0 = W_{a_{si}}W_{b_{si}} - W_{a_s}W_{b_s} \tag{2}$$

and write $I^0$ for the ideal defined by the forms $F_{si}^0$'s. Now let $C^0$ be the subscheme of $\mathbb{P}_k^{g-1}$ defined by the homogeneous ideal $I^0$. Typically $C^0$ will be a highly singular canonical curve that realizes the Weierstrass gap sequence in the sense that the intersection multiplicities of $C^0$ with the hyperplanes at the point $P = (1 : 0 : \cdots : 0)$ are precisely the integers $l_1 - 1, l_2 - 1, \ldots, l_g - 1$. We have just seen that any canonical curve $C$ having a point with the gap sequence $l_1 - 1, l_2 - 1, \ldots, l_g - 1$ and $\mathcal{B}$ as a basis of $\mathrm{Hom}^0(C, \omega_C^{\otimes 2})$ is given by $\frac{(g-2)(g-3)}{2}$ quadratic forms $F_{si}$, similar to those in (1), thus it can be seen as a deformation of the curve $C^0$. We impose conditions on the coefficients $c_{sill'}$ in order to the intersections of the quadrics $F_{si}$ in $\mathbb{P}_k^{g-1}$ be a canonical curve $C$ having a nonsingular point $P = (1 : 0 : \cdots : 0)$, where the intersection multiplicities of $C$ with the osculating spaces of $C$ at $P$ are $l_1 - 1, l_2 - 1, \ldots, l_g - 1.$ The conditions we have to impose are given by Stöhr's deformation theorem.

**Theorem 2.1.** *Suppose that the scheme* $C^0$ *defined as above is a canonically embedded curve and let* $I$ *be the ideal generated by the* $\frac{(g-2)(g-3)}{2}$ *quadratic forms* $F_{si}$, $s \in L_2$ *and* $\omega_{a_{si}}\omega_{b_{si}} \notin \mathcal{B}$. *Then the following statements are equivalent*:

(i) *The quadratic forms* $F_{si}$ *define a non-degenerated Gorenstein curve* $C$ *of arithmetic genus* $g$ *and degree* $2g - 2$ *in* $\mathbb{P}_k^{g-1}$;

(ii) *Each homogeneous syzygy of degree* $n$ *between the forms* $F_{si}$ *is induced by a homogeneous syzygy of degree* $n$ *between the forms* $F_{si}^{(0)}$.

*In addition, the curve* $C$ *in item* (i) *has a unique component passing through the nonsingular point* $P = (1 : 0 : \cdots : 0)$ *whose contact orders with the hyperplanes defined by the equations* $W_l = 0$ *are the integers* $l - 1$ *with* $l \in L = \{l_1, l_2, \ldots, l_g\}$.

**Proof.** See the proof of Theorem 4.1 in [13].

After dividing by the action of the subgroup of $GL_k(g)$ that preserves $P$-Hermitian basis, or more precisely, the $\frac{g(g-1)}{2}$-dimensional group of upper triangular matrices whose diagonal elements are of the form $z^{l_i}$ for each $i = 1, \ldots, g,$ for a non-zero constant $z$ (see [16], Section 3 for details), Theorem 2.1 gives defining equations for the parameter space of the pointed Gorenstein curves with gap sequence $l_1, l_2, \ldots, l_g$ and because the nonsingular curves correspond to an open subset in the moduli variety (cf. [14]), if we choose an "aleatory point" in the moduli it should correspond to a nonsingular curve for "almost every" choice. Thus we can obtain the desired nonsingular curve that realizes the given numerical semigroup (if it does exist).

Our method for representing a given numerical semigroup $H$ can be summarized by the following steps:

(1) Study of a special singular canonical curve that realizes the numerical semigroup $H$ at a distinct nonsingular point in the sense that the sequence of intersection multiplicities of the curve with the hyperplanes at this point has the correct values, that is, it is equal to $l_1 - 1,\ l_2 - 1, \ldots, l_g - 1$, where $l_1, l_2, \ldots, l_g$ is the sequence of gaps $\mathbb{N} - H$ of the semigroup. (If a curve is nonsingular, it is well known that this is equivalent to saying that the Weierstrass semigroup of the curve at this point is $H$.)

(2) Deform the special curve in (1), by using Stöhr's deformation theorem (see Theorem 2.1), in order to obtain the parameter space of isomorphism classes of pointed canonical curves with the prescribed Weierstrass gap sequence $l_1, l_2, \ldots, l_g$.

(3) Obtain the examples by choosing arbitrary curves among those constructed in (2) and proving that they are nonsingular by the Jacobian criterion.



## 3. An Example: The Semigroup $\langle 6, 8, 10, 11\rangle$

To illustrate the method we take the numerical semigroup $\langle 6, 8, 10, 11\rangle$. We used the software SINGULAR [6] for the algebraic computations.

**Step 1.** For this semigroup we have that $L = \{1, 2, 3, 4, 5, 7, 9, 13, 15\}$ is the set of gaps. We obtain the ideal $I^0$ by observing that

$$4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 14,\ 16,\ 17,\ 18,\ 20,\ 22 \in L_2$$

can be written as a sum of two gaps in more than one way. (The $\frac{(g-2)(g-3)}{2} = 21$ quadratic equations $F_{si}^0$'s that generate $I^0$ are the isobaric binomials in the forms $F_{si}$ given in Step 3.) Let $R = k[W_1, W_2, W_3, W_4, W_5, W_7, W_9, W_{13}, W_{15}]/I^0$ be the homogeneous ideal of the scheme $C^0$. The Hilbert polynomial of $C^0$ is $16n - 8$, so $C^0$ is a curve of genus 9 and degree 16 in $\mathbb{P}_k^8$. To conclude that it is a canonically embedded curve it is sufficient to prove that the ring $R$ is Gorenstein. Since $W_{15}$, $W_1 - W_{13}$ is an $R$-sequence, it follows that $R$ is Gorenstein if and only if the Artinian ring $A = \dfrac{R}{(W_1 - W_{13}, W_{15})}$ is a Gorenstein ring. Now, observe that $A$ can be written as the sum of vector spaces $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$, where $\dim_k A_3 = 1$ and $A_3 = kW_1^2W_2$ is the annihilator of the maximal ideal of $A$, so $A$ is a Gorenstein ring (cf. [4], Proposition 21.5).

**Step 2.** Observing that

$$\omega_1^2, \omega_1\omega_2, \omega_1\omega_3, \omega_1\omega_4, \omega_1\omega_5, \omega_2\omega_5, \omega_1\omega_7, \omega_2\omega_7, \omega_1\omega_9,$$

$$\omega_2\omega_9, \omega_3\omega_9, \omega_4\omega_9, \omega_1\omega_{13}, \omega_2\omega_{13}, \omega_1\omega_{15}, \omega_2\omega_{15}, \omega_3\omega_{15},$$

$$\omega_4\omega_{15}, \omega_5\omega_{15}, \omega_7\omega_{15}, \omega_9\omega_{15}, \omega_{13}^2, \omega_{13}\omega_{15}, \omega_{15}^2$$

have different orders at $P$ we conclude that they form a basis $\mathcal{B}$ for $\mathrm{Hom}^0(C, \omega_C^{\otimes 2})$. Thus we can obtain the forms $F_{si}$ that generate the ideal $I(C)$ as in (1) and then apply Theorem 2.1.

**Step 3.** We search for a pointed curve $(C, P)$ with $H_C(P) = \langle 6, 8, 10, 11 \rangle$ among those obtained by Step 2. Since the nonsingular curves correspond to an open set in that variety, we take an arbitrary curve and analyze it. By Theorem 2.1 we know that the curve $C$ defined by the following set of 21 equations:

$$F_{22,1} = W_9W_{13} - W_7W_{15},$$

$$F_{20,1} = W_7W_{13} - W_5W_{15} - W_{13}W_{15} - W_{15}^2,$$

$$F_{18,2} = W_9^2 - W_3W_{15} - W_4W_{15} - W_5W_{15} - W_7W_{15} - W_{15}^2,$$

$$F_{18,1} = W_5W_{13} - W_3W_{15},$$

$$F_{17,1} = W_4W_{13} - W_2W_{15},$$

$$F_{16,2} = W_7W_9 - W_1W_{15} - W_2W_{15} - W_3W_{15} - W_5W_{15} - 2W_{13}W_{15} - W_{15}^2,$$

$$F_{16,1} = W_3W_{13} - W_1W_{15},$$

$$F_{14,2} = W_7^2 - W_1W_{13} - W_2W_{13} - 2W_{13}^2 - W_1W_{15} - W_3W_{15} - W_{13}W_{15},$$

$$F_{14,1} = W_5W_9 - W_1W_{13} - W_2W_{13} - 2W_{13}^2 - W_3W_{15} + W_7W_{15}$$

$$+ W_9W_{15} - W_{13}W_{15},$$

$$F_{12,1} = W_5W_7 - W_3W_9,$$

$$F_{11,1} = W_4W_7 - W_2W_9,$$

$$F_{10,2} = W_5^2 - W_1W_9 + W_3W_{15} + W_5W_{15},$$

$$F_{10,1} = W_3W_7 - W_1W_9,$$

$$F_{9,1} = W_4W_5 - W_2W_7 + W_2W_{15} + W_4W_{15},$$

$$F_{8,2} = W_4^2 - W_1W_7 - W_3W_9 + W_1W_{15} - W_3W_{15} - W_5W_{15}$$

$$- W_7W_{15} - W_9W_{15} - W_{15}^2,$$

$$F_{8,1} = W_3W_5 - W_1W_7 + W_1W_{15} + W_3W_{15},$$

$$F_{7,1} = W_3W_4 - W_2W_5,$$

$$F_{6,2} = W_3^2 - W_1W_5,$$

$$F_{6,1} = W_2W_4 - W_1W_5 - W_1W_9 - 2W_1W_{15} - W_3W_{15}$$

$$- W_5W_{15} - W_7W_{15} - 2W_{13}W_{15} - W_{15}^2,$$

$$F_{5,1} = W_2W_3 - W_1W_4,$$

$$F_{4,1} = W_2^2 - W_1W_3 - 2W_1W_{13} - 2W_{13}^2 - W_1W_{15} - W_3W_{15}$$

$$- W_5W_{15} - 2W_{13}W_{15} - W_{15}^2,$$

is canonically embedded in $\mathbb{P}_k^8$ and has the correct sequence of intersection multiplicities

$$0, 1, 2, 3, 4, 6, 8, 12, 14$$

with the hyperplanes at the point $P = (1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$. If the curve is nonsingular, it is well known that this is equivalent to saying that the Weierstrass semigroup of the curve at the point $P$ is $\langle 6, 8, 10, 11 \rangle$. We can verify by means of the Jacobian criterion that in fact this curve is nonsingular.


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