Far East J. Dynamical Systems **8**(2) (2006), 175-184
This paper is available online at http://www.pphmj.com

STOCHASTIC PROPERTIES IN LORENZ MAPS

YAGANG ZHANG

Department of Applied Mathematics
North China Electric Power University
Baoding, Hebei 071003, P. R. China
and
Center for Nonlinear Complex Systems
School of Mathematics and Physics
Yunnan University, Kunming, Yunnan 650091, P. R. China
e-mail: ygzhg@163.com

Abstract

The inherent stochastic properties in Lorenz type maps are clarified in this paper. The distributions of frequency and inter-occurrence times are discussed carefully. Unimodal surjective map and Lorenz map have same distributions, which are also explained in theory. By constructing 4-letters Lorenz map, comparing similarities with 4-letters surjective map, one elicits catholicity of inherent stochastic properties. In the field of application, we hope to afford a symbolic platform which satisfies these stochastic properties and study some properties of DNA sequences, 20 amino acids symbolic sequences of proteid structure, and the time series that can be symbolic in finance market etc.

1. Introduction

As early as in 1947, Ulam and von Neumann studied the densities distribution [8] of orbital points $\{x_i\}_{i=0}^{\infty}$ about the surjective parabolic maps, and got the famous Chebyshev distribution

---

2000 Mathematics Subject Classification: 37H99, 37N25.

Keywords and phrases: Lorenz maps, multinomial distribution, stochastic properties.

Communicated by Florin N. Diacu

Received April 11, 2006

$$\rho(x) = \frac{1}{\pi\sqrt{1 - x^2}}.$$

The study of chaotic symbolic sequences is gradually developing in theory. However, applied research of stochastic chaotic sequences has not been fully carried out, for most of studies focus on controlling or avoiding chaos. Chaos, nevertheless, affords inherent stochastic properties that can be calculated, which is an important applied domain. The stochastic symbolic sequences bear the following three features. First, computer can generate them iteratively. Second, like false stochastic numbers, they can set up a stochastic sequence simulation (in contradiction, they are based on corresponding symbolic spaces). Third, they can produce numerous symbolic spaces, which is not characteristic of common stochastic numbers. Therefore, the symbolic dynamics [3] developed by this means is supposed to be very useful. This paper aims at inquiring preliminarily into various properties of inherent stochastic properties in Lorenz maps. We have been familiar with two symbolic sequences and have clarified the inherent randomicity in 4-symbolic dynamics in paper [10], the multiformity of inherent randomicity and visitation density in n symbolic dynamics in paper [9], while the generic properties of Lorenz type maps are still unknown.

These researches reveal the probability distributions of symbolic appearance and inter-occurrence times in Lorenz maps, Unimodal map, 4-letters Lorenz map, 4-letters surjective map. And this paper is organized as follows. In Section 2, introduced the symbolic dynamics of Lorenz maps. In Section 3, the stochastic properties in Lorenz map are amplified in detail. In Section 4, compared the similar stochastic properties in Lorenz map and Unimodal map. In Section 5, one discussed the stochastic properties in 4-letters Lorenz maps.

2. Symbolic Dynamics of Lorenz Maps

Lorenz equation:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = (r - z)x - y, \\ \dot{z} = xy - bz. \end{cases}$$

On the Poincaré section, some geometrical structure of Lorenz flow may be reduced to a one-dimensional Lorenz map $f : [-\mu, \nu] \to [-\mu, \nu]$,

$$f(x) = \begin{cases} f_L(x) = \nu - \alpha |x|^\lambda + \text{h.o.t}, & x \le 0, \\ f_R(x) = -\mu + \beta x^\lambda + \text{h.o.t}, & x > 0, \end{cases} \quad (\mu, \nu > 0, \xi > 1),$$

where λ is a constant greater than 1, "h.o.t" represents high-level term. Both of the branches f_L and f_R are monotone increasing. In order to get iterative sequences in the part of chaos, the Lorenz map used in this research is:

$$f(x) = \begin{cases} f_L(x) = 1 - 2|x|^2, & x \le 0, \\ f_R(x) = -1 + 2x^2, & x > 0. \end{cases} \tag{1}$$

Let us study the symbolic dynamics of Lorenz maps [7]. Following the kneading theory [5], the address $A(x)$ of any point x on the interval $[-1, 1]$ is

$$A(x) = \begin{cases} R, & x \in [-1, 0), \\ L, & x \in [0, 1). \end{cases}$$

$x = 0$ is the turning (discontinuous) point, and one can define C and D as

$$C = \lim_{x \to 0^-} f_L(x),$$

$$D = \lim_{x \to 0^+} f_R(x).$$

Two infinite or finite symbolic sequences starting from C and D are kneading sequences which can be ordered lexicographically by $L \prec C$, $D \prec R$. For two kneading sequences, $\gamma_1\gamma_2 \cdots \gamma_i\gamma_{i+1} \cdots$ and $\eta_1\eta_2 \cdots \eta_i\eta_{i+1} \cdots$, with maximal common leading part $\gamma_1\gamma_2 \cdots \gamma_i = \eta_1\eta_2 \cdots \eta_i$, one has $\gamma_1\gamma_2 \cdots \gamma_i\gamma_{i+1} \cdots \prec \eta_1\eta_2 \cdots \eta_i\eta_{i+1} \cdots$ if and only if $\gamma_{i+1} \prec \eta_{i+1}$. The shift operator φ is defined as $\varphi^k(\xi) = \xi_{k+1}\xi_{k+2} \cdots$ for the sequence $\xi = \xi_1\xi_2 \cdots \xi_k\xi_{k+1} \cdots$. For any two sequences $\xi = \xi_1\xi_2 \cdots \xi_i\xi_{i+1} \cdots$ and $\zeta = \zeta_1\zeta_2 \cdots \zeta_j\zeta_{j+1} \cdots$, $\xi_i, \zeta_j \in \{R, L\}$, if $\varphi^k(\xi) \preceq \xi$ and $\zeta \preceq \varphi^k(\zeta)$, for all $K \in \mathbb{Z}_+$, then ξ and ζ are called *maximal* and *minimal*, respectively, and

$S = (\xi, \zeta)$ is an *extremal pair*. Let the integers k_L and k_R be the order coordinates of a letter in the sequence such that $\varphi^{k_L - 1}(\xi) = L\ldots,$ and $\varphi^{k_R - 1}(\xi) = R\ldots,$ the set k_L or k_R describe successive sequences of L or R. Then, if the pair S further satisfies the following condition:

$$\varphi^{k_L}(\xi) \preceq K^1, \quad \varphi^{k_R}(\xi) \succeq K^2, \quad \{k_L\} \cup \{k_R\} = \{k\} \in \mathbb{Z}_+,$$

$$\varphi^{k'_L}(\zeta) \preceq K^1, \quad \varphi^{k'_R}(\zeta) \succeq K^2, \quad \{k'_L\} \cup \{k'_R\} = \{k'\} \in \mathbb{Z}_+,$$

S is *admissible* with respect to the kneading sequences K^1 and K^2. All the admissible pairs form an admissible set K and fill up the whole kneading parameter plane of dynamical systems of two letters.

3. The Stochastic Properties in Lorenz Map

3.1. The distribution of frequency

Let us define an alphabet of 2 numbers, which is corresponding to the likely states of a random discrete dynamical system, or all the likely outcomes of a random experiment:

$$\Omega = \{0, 1\} = \{\text{“failure”}, \text{“success”}\}.$$

The forward sequence constitutes a space (or a set) composed of the generated outcomes:

$$\Omega^N = \{(\xi_0, \xi_1, \xi_2, \ldots) : \xi_i \in \Omega, \forall i \in \{0, 1, 2, \ldots\}\}.$$

These sequences themselves are iteratively generated, in fact it is a shift map $\sigma : \Omega^N \to \Omega^N$, which acting on the sequences by $\sigma(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \ldots)$. Another definition is μ, which is the product measure [2] on Ω^N generated by the measure $(1 - p, p)$ on $\{0, 1\}$, and will be denoted by $(1 - p, p)^N$. Defining $f : \Omega^N \to \{0, 1\}$ by $f\{\xi_0, \xi_1, \xi_2, \ldots\} = \xi_0$, it is coarse graining in theory, one can get: $X_i(\xi) = \varphi(\sigma^i \xi)$, (for $i = 0, 1, 2, \ldots$), which are sequences of independent identically distributed (i.i.d.) random variables defined on the probability space (Ω^N, μ) (all the following discussions are based on the random variables), that is, the random variables represented by $\xi_0, \xi_1, \xi_2, \ldots$ are i.i.d., and ξ_i $(i = 0, 1, 2, \ldots)$ is based on Ω,

$$Y_n = X_0 + X_1 + \cdots + X_{n-1} = \sum_{i=0}^{n-1} \varphi(\sigma^i \xi) = \xi_0 + \xi_1 + \cdots + \xi_{n-1}.$$

The stochastic symbolic sequences in 2-letters map satisfy Binomial distribution:

$$\mu\{\xi \in \Omega^N : Y_n(x) = k\} = C_n^k p^k (1-p)^{n-k}.$$

3.2. The inter-occurrence times of Lorenz map

Now let us make a further study a given word's occurrence times in an independent repeated experiment, such as "*success*" in the alphabet of 2 numbers. Given outcomes of a random sequence

$$\xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \Omega^N,$$

we are mainly interested in n such that $\xi_n = 1$, let

$$\tau(\xi) = \tau_1(\xi) = \inf\{n \geq 0 : \xi_n = 1\},$$

and accordingly, for $j \geq 2$,

$$\tau_j(\xi) = \inf\{n > \tau_{j-1}(\xi) : \xi_n = 1\}.$$

Then for all $k \geq 0$, the result is, for fixed $k > 0$ and all $k_1 < \cdots < k_{j-1}$,

$$P(\tau_j - \tau_{j-1} = k \mid \tau_{j-1} = k_{j-1}, \ldots, \tau_1 = k_1) = P(\tau_j - \tau_{j-1} = k) = p(1-p)^{k-1}.$$

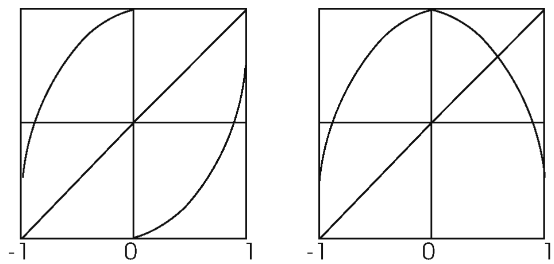
the inter-occurrence times $1 + \tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots.$ are i.i.d. with parameters p.

Using this method similar to study the distribution of first passage time T_y of one-dimensional simple random wander in stochastic processes, one has proved τ_j satisfies Negative Binomial distribution $BN(j, p)$:

$$P\{\tau_j = k\} = C_{k-1}^{r-1} p^r q^{k-r}, \quad k = r, r+1, r+2, \ldots, \quad 0 < p < 1, \quad q = 1 - p.$$

4. The Comparability of Stochastic Properties in Lorenz Map and Unimodal Map

In fact, the former stochastic properties in Lorenz map are similar entirely to that in Unimodal map. This kind of comparability is determined by the relationship of Lorenz map and Unimodal map. (See Fig. 1)



(a) (b)

Figure 1. Lorenz map and Unimodal map.

The iterative form of Lorenz map is (1), and Unimodal map is: $y = 1 - 2x^2$. One can find this characteristic by Fig. 1,

$$\begin{cases} f_a = -f_b, & x \geq 0, \\ f_a = f_b, & x < 0. \end{cases}$$

An n-periods orbit of f_b corresponds to a couple of n-periods orbits of f_a. Both of them have the same topological entropy [4] and marker behavior. The fixed point of f_b exhibits two-periods behavior of f_a, which can be found clearly by contrasting their bifurcation diagrams. (See Fig. 2)

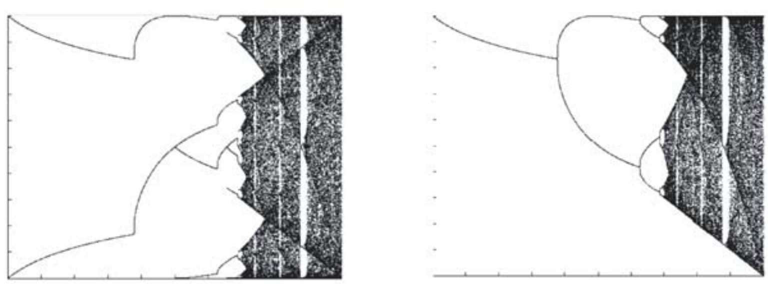


Figure 2. The bifurcation diagrams of Lorenz map and Unimodal surjective map.

Compared the right branch of Lorenz map and Unimodal map, the Lorenz map is only overturned by x coordinate axis. As these results reveal that this kind of overturn does not influence statistical properties of random sequences. We can also make out this character by comparing 4-letters Lorenz map and 4-letters surjective map. Compared with Unimodal map, Lorenz map belongs to a more complex category, which presents more abundant dynamics actions. But as above study, these statistical results present regulation as a whole.

These are stochastic properties in deterministic systems.

5. The Stochastic Properties in 4-letters Lorenz Maps

The iterative form of 4-letters Lorenz map is,

$$x_{n+1} = F(A, x_n) = \begin{cases} -8x_n^2 - 8x_n - 1, & -1 \le x < -0.5, \\ 8x_n^2 + 8x_n + 1, & -0.5 \le x < 0, \\ -8x_n^2 + 8x_n - 1, & 0 \le x < 0.5, \\ 8x_n^2 - 8x_n + 1, & 0.5 \le x \le -0.5. \end{cases}$$

5.1. The distribution of frequency

Define an alphabet of 4 numbers,

$\Omega = \{0, 1, 2, 3\} = \{$“L”, “M”, “N”, “R”$\}$

= {“Adenine”, “Guanine”, “Cytosine”, “Thymine”} = {“A”, “G”, “C”, “T”}.

The forward sequence of outcomes is this space,

$$\Omega^N = \{(\xi_0, \xi_1, \xi_2, \ldots) : \xi_i \in \Omega, \forall i \in \{0, 1, 2, \ldots\}\}.$$

These sequences are iterative themselves. Introduced a shift map $\sigma : \Omega^N \to \Omega^N$, which acting on the sequences by $\sigma(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \ldots)$. Defining $\varphi : \Omega^N \to \{0, 1, 2, 3\}$ by $\varphi\{\xi_0, \xi_1, \xi_2, \ldots\} = \xi_0$, one has:

$$X_i(\xi) = \varphi(\sigma^i \xi), \quad (\text{for } i = 0, 1, 2, \ldots).$$

The stochastic symbolic sequences in 4-letters Lorenz map satisfy Multinomial distribution:

$$T(N_1 = n_1, N_2 = n_2, N_3 = n_3, N_4 = n_4) = \frac{n!}{n_1!n_2!n_3!n_4!} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}, \quad (2)$$

$$\left(\sum_{i=1}^{4} p_i = 1, \sum_{i=1}^{4} n_i = n \right). \quad (3)$$

As we know, the random sequences of one-hump surjective map in 2-letters maps satisfy Binomial distribution. The base in theory of these results is that the topological entropy in n letters surjective maps is $\ln(n)$, which is a deduction of chaotic symbolic sequences' Bernoulli property.

5.2. The inter-occurrence times of 4-letters Lorenz map

Now let us consider a given word's occurrence times in an independent repetition experiment, such as "R", "T" or "Thymine" in the alphabet of 4 numbers. Given a random sequence of outcomes

$$\xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \Omega^N,$$

we are mainly interested in n such that $\xi_n = T$, let

$$\tau(\xi) = \tau_1(\xi) = \inf\{n \geq 0 : \xi_n = T\},$$

and define, for $j \geq 2$,

$$\tau_j(\xi) = \inf\{n > \tau_{j-1}(\xi) : \xi_n = T\}.$$

Then for all $k \geq 0$, the result is, for fixed $k > 0$ and all $k_1 < \cdots < k_{j-1}$,

$$P(\tau_j - \tau_{j-1} = k | \tau_{j-1} = k_{j-1}, \ldots, \tau_1 = k_1) = P(\tau_j - \tau_{j-1} = k) = p_1^a p_2^b p_3^c p_4, \quad (4)$$

$$\left(\sum_{i=1}^{4} p_i = 1; \, a, b, c \in \{0, 1, 2, \ldots\}, a + b + c = k - 1 \right). \quad (5)$$

Correspondingly, for $\xi_n = A, \xi_n = G, \xi_n = C,$ the results can be educed in this way. The inter-occurrence times $1 + \tau_1^\lambda, \tau_2^\lambda - \tau_1^\lambda, \tau_3^\lambda - \tau_2^\lambda, \ldots$ are i.i.d. with parameters p_1, p_2, p_3, p_4.

Similarity with 4-letters Lorenz map, 4-letters surjective map also has these distributions, namely (2) and (4). This kind of stochastic properties have catholicity.

6. Conclusion and Discussion

The inherent stochastic properties in Lorenz maps are clarified in this paper. These stochastic symbolic sequences bear three characters. These researches reveal the probability distributions of symbolic appearance and the inter-occurrence times in Lorenz maps (include 4-letters Lorenz map). Compared with Unimodal surjective map and 4-letters surjective map, the catholicity is clear.

We hope to afford a symbolic platform which satisfies these stochastic properties and study some properties of DNA [1, 6] sequences, 20 amino acids symbolic sequences of proteid structure, and the time series that can be symbolic in finance market etc., which are part of our future work. The symbolic platform provides a set of effective methods to approach problems of this kind. The establishment of this symbolic platform opens up a vast vista.

Acknowledgements

The author would like to thank Professor Shou-Li Peng for numerous discussions and valuable comments. This research was supported partly by the Natural Science Foundation of Hebei Province and the Youthful Teacher Research Foundation of North China Electric Power University (200511018).

References

[1] A. Bershadskii, Multifractal and probabilistic properties of DNA sequences, Phys. Lett. A 284 (2001), 136-140.

[2] Z. Coelho and P. Collet, Asymptotic limit law for the close approach of two trajectories in expanding maps of the circle, Probab. Theory Related Fields 99 (1994), 237-250.

[3] B. L. Hao, Elementary Symbolic Dynamics and Chaos in Dissipative Systems, World Scientific, Singapore, 1989.

[4] X. Liang and J. F. Jiang, On the topological entropy, nonwandering set and chaos of monotone and competitive dynamical systems, Chaos Solitons Fractals 14 (2002), 689-696.

[5] J. Milnor and W. Thurston, On iterated maps of the interval I and II, Preprints, Princeton, 1977; J. C. Alexander, ed., Published, in: Dynamical Systems, Proceedings, University of Maryland 1986-87, Lecture Notes in Mathematics, Vol. 1342, Springer-Verlag, Berlin, 1988, p.465.

[6] L. Natalia and S. Avy, Nonlinear waves in double-stranded DNA, Bulletin of Mathematical Biology 67 (2005), 701-718.

[7] S. L. Peng and L. M. Du, Dual star products and symbolic dynamics of Lorenz maps with the same entropy, Phys. Lett. A 261 (1999), 63-73.

[8] S. M. Ulam and J. V. Neumann, On combination of stochastic and deterministic processes, Bull. Amer. Math. Soc. 53 (1947), 1120.

[9] Y. G. Zhang and C. J. Wang, Multiformity of inherent randomicity and visitation density in n symbolic dynamics, Chaos Solitons Fractals (to appear).

[10] Y. G. Zhang, C. J. Wang and Z. Zhou, Inherent randomicity in 4-symbolic dynamics, Chaos Solitons Fractals 28(1) (2006), 236-243.

■