Far East J. Appl. Math. **25**(2) (2006), 137-158

RECONSTRUCTION OF ELLIPTIC VOIDS IN THE ELASTIC HALF-SPACE: ANTI-PLANE PROBLEM

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Abstract

In the present paper, we study the reconstruction of geometry (position and size) of elliptic voids located in the elastic half-space, in frames of anti-plane two-dimensional problem. We assume that a known point force is applied to the boundary surface of the half-space, and we can measure the shape of the surface over a certain finite-length interval. Then, if the shape of the defect is unknown, we construct an algorithm to restore its position and size. Some numerical examples demonstrate a good stability of the proposed algorithm.

1. Introduction

In the engineering applications of strength theory the detection and recognition of defects in elastic materials is one of the most important problems of Non-destructive Evaluation. Various methods are used for

2000 Mathematics Subject Classification: 45Q05, 74S15.

Keywords and phrases: inverse problem, reconstruction, boundary integral equation, optimization.

Received September 10, 2005

this purpose, and one of them is founded on the theory of inverse problems. In order to detect and recognize the image of the void, one may apply over a boundary of the sample a certain type of load, so that to measure the boundary deformation caused by this load. Then one may suppose that the presence (or absence) of interior defects will influence the measured obtained data. It is also quite natural to suppose that if there is an interior void in the sample, then its position and geometry can influence significantly the shape of the deformed boundary. This idea creates a good basis for defects reconstruction from the measured deformation of the boundary of loaded samples.

A number of theoretical works were devoted to the inverse problems of this kind, with applications to recognition of cracks [1, 2, 7]. Some important papers concern uniqueness of the solution, some others develop explicit-form analytical results or numerical algorithms [3, 4]. Unfortunately, much less results are devoted to reconstruction of volumetric (non-thin) defects in elastic samples under the same conditions and with the same type of input data.

In the present work, we study the anti-plane (i.e., scalar) problem of linear isotropic elasticity in the half-space, with an outer load applied to its boundary surface. We show that so formulated direct problem can be reduced to the Laplace partial differential equation. Then we construct Green's function, which automatically satisfies the trivial boundary condition over the plane surface of the considered half-space. Such Green's function allows us to formulate the direct problem as a single integral equation holding over the boundary of the void, in the case when a volumetric defect is located in the elastic half-space. Solution of this integral equation permits to determine the shape of the boundary surface, if the form of the void is known. Further, we formulate the inverse problem, which is to restore the geometry of the void from the measured input data, which is the known deformation of the boundary line over some finite-length interval. A specially proposed numerical algorithm is suitable to solve this inverse problem. This is reduced to a sort of minimization of the discrepancy functional. Finally, we give some example of application of the proposed method, in the case of the reconstruction (location and geometry) of elliptic voids.

2. Mathematical Formulation of the Problem

Let us consider the (two-dimensional) *anti-plane* problem about a volumetric flaw with the boundary L located in the homogeneous and isotropic elastic half-space (see Figure). The anti-plane formulation implies that the components of the displacement vector $\mathbf{u}$ is of the following form:

$$\mathbf{u}(x, y, z) = \{0, 0, w(x, y)\}, \tag{2.1}$$

where w is the component of the displacement vector in the direction z. Then the equation of equilibrium can be simplified to the ordinary Laplace equation (see, for example, [10])

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} = 0, \;\; \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \tag{2.2}$$

where λ and μ are elastic constants. It is obvious that the only nontrivial components of the elastic stress tensor, among other six ones, are σ_{xz} and σ_{yz}, which can be represented in terms of function $w(x, y)$ as follows: $\sigma_{xz}(x, y) = \mu\, \partial w / \partial x$, $\sigma_{yz}(x, y) = \mu\, \partial w/\partial y$.

In the direct problem the position and the geometry of the void are known, but in the inverse problem they should be determined from some input data. In order to provide such input data, let us assume that a known tangential point force $\sigma_{yz} = \sigma_0 \delta(x)$ is applied to the boundary line $y = 0$ of the half-space, for example, at the origin (see Figure). Due to linearity of the problem, let us represent the full solution of this problem as a superposition of the two ones: (1) corresponding to the applied load for the fully continuous (i.e., without any void) half-space, and (2) corresponding to the problem with the free boundary line $y = 0$ and with void located in the half-space whose boundary L is subjected to some tangential stress:

$$w(x, y) = \varphi^0(x, y) + \varphi(x, y). \tag{2.3}$$

The solution to the problem in the perfect half-plane free of void can be constructed by applying the Fourier transform along the x-axis to the both parts of the Laplace equation (2.2):

$$\Phi^0(\alpha,\ y) = \int_{-\infty}^{\infty} \varphi^0(x,\ y) e^{i\alpha x} dx,\ \varphi^0(x,\ y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^0(\alpha,\ y) e^{-i\alpha x} d\alpha. \quad (2.4)$$

This results in the ordinary differential equation of the second order, with a certain parameter s:

$$\frac{d^2\Phi^0(\alpha,\ y)}{dy^2} - \alpha^2 \Phi^0(\alpha,\ y) = 0, \quad (2.5)$$

whose bounded solution is

$$\Phi^0(\alpha,\ y) = A(\alpha) e^{-|\alpha| y}, \quad y \geq 0. \quad (2.6)$$

Here the unknown quantity $A(\alpha)$ should be determined from the boundary condition:

$$\frac{\partial \varphi^0}{\partial y}(y = 0) = \frac{\sigma_0}{\mu} \delta(x), \quad \Rightarrow \frac{d\Phi^0(\alpha,\ 0)}{dy} = \frac{\sigma_0}{\mu}, \quad \Rightarrow A = \frac{\sigma_0}{\mu |\alpha|} \quad (2.7)$$

hence

$$\varphi^0(x,\ y) = \frac{\sigma_0}{\mu} \int_{-\infty}^{\infty} \frac{e^{-|\alpha| y - i\alpha x}}{|\alpha|} d\alpha. \quad (2.8)$$

It should be noted that this integral diverges due to a non-integrable singularity of the integrand as $\alpha \to 0$. This conforms to the well known classical paradox of the potential theory for unbounded domains: under any non-self-balanced system of forces the displacement field in the far zone increases infinitely as a logarithmic function (see, for example, [9]). However, the stress field is determined correctly, since the components of the stress tensor are expressed as some derivatives applied to function $\varphi^0(x,\ y)$. Obviously, any derivative with respect to x or y in (2.8) cancels the non-integrable singularity in the denominator.

Another way to arrange a regular stress-strain field in the half-plane is to remove the singularity in Eq. (2.8). In fact, if we add any constant to the right-hand side of (2.8), then the so-written function provides again all the required conditions, i.e., this satisfies the Laplace partial differential equation and the boundary condition (2.7). For this reason we can rewrite representation (2.8) for φ^0 as follows:

$$\varphi^0(x,\ y) = \frac{\sigma_0}{\mu}\int_{-\infty}^{\infty}\frac{e^{-|\alpha|y - i\alpha x} - e^{-|\alpha|}}{|\alpha|}\,d\alpha$$

$$= \frac{2\sigma_0}{\mu}\int_0^{\infty}\frac{e^{-\alpha y}\cos(\alpha x) - e^{-\alpha}}{\alpha}\,d\alpha, \qquad (2.9)$$

which is regular for all $|x| < \infty$ and all $y \geq 0$. By such a manner we avoid the irregular behavior of the displacement, because any motion of the medium as an absolutely rigid body is out of any interest when studying the strength properties of the material.

3. The Basic Boundary Integral Equation and Green's Function for the Half-space

Let a void with the boundary L free of load be located in the linear isotropic elastic half-space (see Figure). Under conditions of the anti-plane problem, among all three components of the stress vector on the boundary curve L there is the only one, which reads as

$$T_z = \sigma_{xz}n_x + \sigma_{yz}n_y = \mu\left(n_x\frac{\partial w}{\partial x} + n_y\frac{\partial w}{\partial y}\right) = \mu\frac{\partial w}{\partial n}, \qquad (3.1)$$

where $\mathbf{n}$ is the normal to L. If the upper plane surface $y = 0$ is loaded at the origin by a point force σ_0, then the mathematical formulation of the direct problem is to solve the Laplace equation (2.2), with the boundary conditions

$$\frac{\partial w}{\partial y}(y = 0) = \frac{\sigma_0}{\mu}\delta(x), \qquad \left.\frac{\partial w}{\partial n}\right|_L = 0. \qquad (3.2)$$

Then, taking into account representation (2.3) and boundary condition (2.7) for function $\varphi^0(x,\ y)$, we can conclude that the boundary value problem for function $\varphi(x,\ y)$ is formulated as follows:

$$\frac{\partial^2\varphi(x,\ y)}{\partial x^2} + \frac{\partial^2\varphi(x,\ y)}{\partial y^2} = 0, \quad \frac{\partial\varphi}{\partial y}(y = 0) = 0, \quad \left.\frac{\partial\varphi}{\partial n}\right|_L = -\left.\frac{\partial\varphi^0}{\partial n}\right|_L. \qquad (3.3)$$

Let $\Phi(\xi, \eta, x, y)$ denote Green's function for the Laplacian in the considered half-space, i.e., the function which satisfies the equation

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = -\delta(\xi - x)\delta(\eta - y), \quad y > 0, \; \eta \geq 0, \tag{3.4}$$

in the half space $\eta \geq 0$ with the trivial boundary condition:

$$\frac{\partial \Phi}{\partial \eta} = 0, \quad \eta = 0. \tag{3.5}$$

Let us consider the pair of two functions, $\varphi(\xi, \eta)$ and $\Phi(\xi, \eta, x, y)$, both being the functions of variables (ξ, η) on the domain Ω which is our half-plane without the void's interior S: $\Omega = \{(\xi, \eta) \in R^2 : \eta > 0, (\xi, \eta) \notin S\}$, and write out the following combination:

$$\int_{\Omega} (\varphi \Delta \Phi - \Phi \Delta \varphi) d\xi d\eta. \tag{3.6}$$

By using Green's integral formula and taking into account relations (3.3), (3.4), this can be rewritten as

$$-\varphi(x, y) = \int_{\Omega} (\varphi \Delta \Phi - \Phi \Delta \varphi) d\xi d\eta = \int_{\partial\Omega} \left(\varphi \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \varphi}{\partial n} \right) dl. \tag{3.7}$$

Let us recall that the full boundary $\partial\Omega$ of the considered domain Ω consists of the two contours: L and $\{\eta = 0\}$. Since the normal derivatives of the both functions satisfy the trivial boundary condition (see Eqs. (3.3) and (3.5)), the right integral over the line $\{\eta = 0\}$ is null in Eq. (3.7). If we change in addition the direction of the unit normal $\mathbf{n}$, so that this be directed outwards the contour L, we can conclude from the last relation that

$$\varphi(x, y) = \int_L \left(\varphi \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \varphi}{\partial n} \right) dl = \int_L \left(\varphi \frac{\partial \Phi}{\partial n} + \Phi \frac{\partial \varphi^0}{\partial n} \right) dl, \tag{3.8}$$

where we have taken into account again boundary condition (3.3).

Now, in order to derive a boundary integral equation of the direct problem, we should use some results of the classical potential theory.

Namely, if any point $(X, Y) \in L$, then

$$\lim_{(x, y)\to(X, Y)} \int_L \varphi \frac{\partial\Phi}{\partial n}(\xi, \eta; x, y) dl = \frac{\varphi(X, Y)}{2} + \int_L \varphi \frac{\partial\Phi}{\partial n}(\xi, \eta; X, Y) dl. \tag{3.9}$$

This allows us to derive from Eq. (3.8) the following boundary integral equation:

$$\frac{1}{2}\varphi(X, Y) - \int_L \varphi(\xi, \eta) \frac{\partial\Phi}{\partial n}(\xi, \eta, X, Y) dl = f(X, Y), \tag{3.10a}$$

$$f(X, Y) = \int_L \frac{\partial\varphi^0}{\partial n}(\xi, \eta)\Phi(\xi, \eta, X, Y) dl, \tag{3.10b}$$

where again, function φ^0 is given by Eq. (2.8) but function Φ should be constructed as a solution of Eq. (3.4) with boundary condition (3.5).

Integral equation (3.10) is the basic boundary integral equation of the direct problem. The only remaining technical detail is to construct Green's function as a solution to Eq. (3.4) satisfying trivial boundary condition (3.5) over the boundary surface of the half-space. In order to develop explicit representation for such Green's function, let us apply again the Fourier transform to Eq. (3.4) with respect to the variable ξ. Then we obtain

$$\frac{d^2\widetilde{\Phi}}{d\eta^2} - \alpha^2\widetilde{\Phi} = -e^{i\alpha x}\delta(\eta - y), \tag{3.11}$$

since

$$\int_{-\infty}^{\infty} \delta(\xi - x) e^{i\alpha\xi} d\xi = e^{i\alpha x}. \tag{3.12}$$

Here $\widetilde{\Phi}(\alpha, \eta, x, y)$ designates the Fourier transform of the function $\Phi(\xi, \eta, x, y)$.

The solution to Eq. (3.12) can be constructed as a sum of a general solution of respective homogeneous equation and a particular solution of the full (inhomogeneous) equation:

$$\widetilde{\Phi}(\alpha, \eta, x, y) = A(\alpha, x, y)e^{-|\alpha|\eta} + B(\alpha, x, y)e^{|\alpha|\eta} + \widetilde{\Phi}_{part}(\alpha, \eta, x, y). \tag{3.13}$$

In order to define $\widetilde{\Phi}_{part}$ from inhomogeneous equation (3.11), let us note that the particular solution may not satisfy any boundary condition on the upper surface $\eta = 0$. For this reason let us apply again the Fourier transform with respect to the variable η:

$$(\alpha^2 + \beta^2)\widetilde{\widetilde{\Phi}}(\alpha, \beta, x, y) = e^{i(\alpha x + \beta y)} \Rightarrow \widetilde{\widetilde{\Phi}}(\alpha, \beta, x, y) = \frac{e^{i(\alpha x + \beta y)}}{\alpha^2 + \beta^2}, \tag{3.14}$$

where $\widetilde{\widetilde{\Phi}}$ denotes the double Fourier transform of the function Φ, with respect to both ξ and η variables. The inverse Fourier transform (along the second variable) of expression (3.14) is

$$\widetilde{\Phi}(\alpha, \eta, x, y) = \frac{e^{i\alpha x}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(\beta y - \beta\eta)}}{\alpha^2 + \beta^2} d\beta$$

$$= \frac{e^{i\alpha x}}{\pi}\int_{0}^{\infty} \frac{\cos[\beta(y - \eta)]}{\alpha^2 + \beta^2} d\beta = \frac{e^{i\alpha x}e^{-|\alpha||\eta - y|}}{2|\alpha|}, \tag{3.15}$$

hence

$$\widetilde{\Phi}(\alpha, \eta, x, y) = A(\alpha, x, y)e^{-|\alpha|\eta} + B(\alpha, x, y)e^{|\alpha|\eta} + \frac{e^{i\alpha x}e^{-|\alpha||\eta - y|}}{2|\alpha|}. \tag{3.16}$$

The correct solution should be bounded at infinity, i.e., as $n \to +\infty$, consequently the second term here must be rejected. We thus come to the representation

$$\widetilde{\Phi}(\alpha, \eta, x, y) = A(\alpha, x, y)e^{-|\alpha|\eta} + \frac{e^{i\alpha x}e^{-|\alpha||\eta - y|}}{2|\alpha|}. \tag{3.17}$$

Boundary condition (3.5) implies $\partial\widetilde{\Phi}/\partial\eta\,(\eta = 0) = 0$ that allows us to define the quantity $A(\alpha, x, y)$ in the following form (notice that if $y > 0$

and $\eta = 0$, then $|\eta - y| = y - \eta$):

$$A(\alpha) = \frac{e^{i\alpha x} e^{-|\alpha| y}}{2|\alpha|}. \tag{3.18}$$

Now substitution of Eq. (3.18) into Eq. (3.17) and application of the inverse Fourier transform leads to the representation of the sought Green's function

$$\Phi(\xi, \eta, x, y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|\alpha|(\eta + y)} + e^{-|\alpha||\eta - y|}}{|\alpha|} e^{i\alpha(x - \xi)} d\alpha. \tag{3.19}$$

It should be noted that again, like in formula (2.8), we have arrived at a divergent integral. However, as we will see below, the boundary equation with this Green's function can be formulated quite correctly.

First of all, let us show that boundary integral equation (BIE) (3.10) can be re-written in a more advanced form, when its right-hand side $f(X, Y)$ is free of any integration. For this aim, we prove that

$$\int_L \left(\varphi^0 \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \varphi^0}{\partial n} \right) dl = 0. \tag{3.20}$$

This statement can be proved directly, if one considers again Green's integral formula applied to the pair of functions: $\varphi^0(\xi, \eta)$ and $\Phi(\xi, \eta, x, y)$ inside the contour L. Really, both functions are regular in this domain, if the point (x, y) is outside L, and satisfy there the Laplace equation. Consequently, the construction like presented in Eq. (3.6) immediately results in (3.20). Now the summation of (3.20) with the left equality in (3.8) gives

$$\varphi(x, y) = \int_L \left(w \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial w}{\partial n} \right) dl = \int_L w(\xi, \eta) \frac{\partial \Phi}{\partial n_{\xi\eta}} (\xi, \eta, x, y) dl_{\xi\eta}, \tag{3.21}$$

due to boundary condition (3.2). Here we have explicitly marked that the outer unit normal to the boundary contour L is drawn at the point (ξ, η), not (x, y). So, with $(x, y) \to (X, Y)$ we come to the representation following from (3.21):

$$\varphi(X, Y) = \frac{w(X, Y)}{2} + \int_L w(\xi, \eta) \frac{\partial \Phi}{\partial n_{\xi\eta}}(\xi, \eta, X, Y) dl_{\xi\eta}, \quad (X, Y) \in L. \quad (3.22)$$

Since $\varphi(x, y) = w(x, y) - \varphi^0(x, y)$, this results in a considerably simpler integral equation:

$$\frac{w(X, Y)}{2} - \int_L w(\xi, \eta) \frac{\partial \Phi}{\partial n_{\xi\eta}}(\xi, \eta, x, y) dl_{\xi\eta} = \varphi^0(X, Y), \quad (X, Y) \in L, \quad (3.23a)$$

than the one written in (3.10). Note that the right-hand side $\varphi^0(X, Y)$ here is given by Eq. (2.9) that can be expressed in a finite form as follows:

$$\varphi^0(x, y) = \frac{2\sigma_0}{\mu} \int_0^\infty \frac{e^{-\alpha Y} \cos(\alpha X) - e^{-\alpha}}{\alpha} d\alpha = -\frac{\sigma_0}{\mu} \ln(X^2 + Y^2). \quad (3.23b)$$

A more detailed representation for the kernel Φ in (3.23a) can also be given in a finite form. Let us notice that

$$\begin{aligned}
\frac{\partial \Phi}{\partial n_{\xi\eta}} &= \frac{\partial \Phi}{\partial \xi} n_\xi + \frac{\partial \Phi}{\partial \eta} n_\eta \\
&= \frac{1}{2\pi} \int_0^\infty \left(n_\xi \frac{\partial}{\partial \xi} + n_\eta \frac{\partial}{\partial \eta} \right) \frac{e^{-\alpha(\eta + y)} + e^{-\alpha|\eta - y|}}{\alpha} \times \cos[\alpha(x - \xi)] d\alpha \\
&= \frac{1}{2\pi} \int_0^\infty \{ n_\xi \sin[\alpha(x - \xi)] [e^{-\alpha(\eta + y)} + e^{-\alpha|\eta - y|}] \\
&\qquad - n_\eta \cos[\alpha(x - \xi)] [e^{-\alpha(\eta + y)} + e^{-\alpha|\eta - y|} \operatorname{sign}(\eta - y)] \} d\alpha \\
&= \frac{1}{2\pi} \left\{ n_\xi \left[\frac{x - \xi}{(x - \xi)^2 + (\eta + y)^2} + \frac{x - \xi}{(x - \xi)^2 + (\eta - y)^2} \right] \right. \\
&\qquad \left. - n_\eta \left[\frac{\eta + y}{(x - \xi)^2 + (\eta + y)^2} + \frac{\eta - y}{(x - \xi)^2 + (\eta - y)^2} \right] \right\}. \quad (3.24)
\end{aligned}$$

4. Calculation of Physical Quantities and Formulation of the Reconstruction Problem

As soon as integral equation (3.23) is solved, i.e., the function $w(x, y)$ is determined, the displacement field at arbitrary point of the elastic medium can be calculated by using Eq. (3.21):

$$w(x, y) = \varphi^0(x, y) + \int_L w(\xi, \eta) \frac{\partial \Phi}{\partial n_{\xi\eta}} dl$$

$$= -\frac{\sigma_0}{\mu} \ln(x^2 + y^2) + \int_L w(\xi, \eta) \frac{\partial \Phi}{\partial n_{\xi\eta}} dl, \tag{4.1}$$

where the quantity $\partial\Phi/\partial n$ is given again by Eq. (3.24). After that the components of the stress tensor can be calculated as $\sigma_{xz} = \mu\, \partial w/\partial x$, $\sigma_{yz} = \mu\, \partial w/\partial y$ (see Section 2). We thus can calculate all physical quantities at the arbitrary point (x, y) inside the medium. In particular, the shape of the upper boundary surface $(y = 0)$ is

$$F(x) = -\frac{2\sigma_0}{\mu} \ln|x| + \frac{1}{\pi} \int_L \frac{(x - \xi) n_\xi - \eta n_\eta}{(x - \xi)^2 + \eta^2} w(\xi, \eta) dl. \tag{4.2}$$

From this formula we can easily extract the contribution given by two physically different components: (1) The deformation of the boundary in the perfect (i.e., free of any void) half-space under the applied force σ_0, that is, given by the first logarithmic term in (4.2), and (2) The contribution given by the influence of presence of the flaw, the second integral term in (4.2). The latter can be calculated as:

$$F_0(x) = \frac{1}{\pi} \int_L \frac{(x - \xi) n_\xi - \eta n_\eta}{(x - \xi)^2 + \eta^2} w(\xi, \eta) dl, \tag{4.3}$$

and gives, as has been said above, the contribution of the defect presence to the deformation of the boundary surface.

Concrete realization of the proposed ideas is founded on the *collocation* technique (see, for example, [5]). Let us arrange a dense set of nodes (x_i, y_i), $i = 1, \ldots, N$ on the contour L: $(x_i, y_i) \in L, \forall i$, which subdivides the contour to N small intervals of the length ℓ_i. As follows from the general theory of integral equations of the second kind (see, for example, [11, Chapter 1]), if $\max_i(\ell_i) \to 0$, then the approximate numerical solution to Eq. (3.23) can be obtained by solving the linear algebraic system:

$$\sum_{j=1}^{N} a_{ij} w_j = \varphi_i^0, \quad i = 1, \ldots, N, \tag{4.4a}$$

with

$$w_i = w(x_i, y_i), \quad \varphi_i^0 = -\frac{\sigma_0}{\mu} \ln(x_i^2 + y_i^2), \quad a_{ii} = \frac{1}{2},$$

$$a_{ij} = \frac{1}{2\pi} \left\{ n_x(x_j, y_j) \left[\frac{x_i - x_j}{(x_i - x_j)^2 + (y_j + y_i)^2} + \frac{x_i - x_j}{(x_i - x_j)^2 + (y_j - y_i)^2} \right] \right.$$

$$\left. - n_y(x_j, y_j) \left[\frac{y_j + y_i}{(x_i - x_j)^2 + (y_j + y_i)^2} + \frac{y_j - y_i}{(x_i - x_j)^2 + (y_j - y_i)^2} \right] \right\} \ell_j, \quad i \neq j. \tag{4.4b}$$

This system is constructed so that the set of the "inner" discrete integration point $\{(\xi_j, \eta_j)\}$, over which the integration is being performed, coincide with the set of the "outer" nodes $\{(x_i, y_i)\}$, which are used to provide the equality between the left- and the right-hand sides in (3.23a). In this case $(\xi_j, \eta_j) = (x_i, y_i)$, and this justifies why the method is called the *collocation* technique.

It should also be noted that the long elements a_{ij}, as expressed by the last equality in Eqs. (4.4), are excluded from the diagonal elements (the case $i = j$). These elements correspond to the case when $(\xi, \eta) = (X, Y)$ in the kernel of integral (3.23), and they have a singular behavior as $(\xi, \eta) \to (X, Y)$. However, the contribution of such elements to the full sum (4.4) is small (as $\max(\ell_j) \to 0$) when compared with the contribution of the "outer" term in (3.23a) staying outside the integral, which in the discrete form results in the diagonal element $a_{ii} = 1/2$ in (4.4). Such an approach allows us to avoid unnecessary troubles connected with a potentially singular behavior of the kernel in (3.23a) as $(\xi, \eta) \to (X, Y)$.

The problem of void reconstruction from boundary measurements can be formulated in the following way. Let the shape of the deformed boundary surface $(y = 0)$, which is caused by the applied point force $\sigma_0 \delta(x)$, be known on a certain finite-length interval $x \in (x^-, x^+)$ of this boundary line $y = 0$. In this case we know the function $F_0(x)$ in (4.3) for $x \in (x^-, x^+)$, but we do not know the form of the void, contour L. The task is just to reconstruct this contour from the given input data for function $F_0(x)$.

Mathematically, the problem can be presented as a (nonlinear) system of two Eqs. (3.23) and (4.3) for the two unknown functions: (1) function $w(\xi, \eta)$, $(\xi, \eta) \in L$, and (2) defining equation of the contour L. The considered system seems to be linear with respect to function $w(\xi, \eta)$ but this is so only at first sight, because this function is defined over the contour L, which is unknown *a priori*.

In practice, the measurements on the deformation of the boundary surface cannot be carried out absolutely precisely. This predetermines the input data to be known with a certain error. Therefore, the proposed algorithm should provide a stability with respect to small perturbations of the input data.

5. Numerical Algorithm and Examples of Reconstruction in the Case of Elliptic Flaw

The proposed numerical method is founded on the collocation technique described above. For concrete implementation of algebraic system (4.4) we put in the case of smooth contour L:

$$\ell_i = \frac{\sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} + \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}}{2},$$

$$n_x(x_i, y_i) = n_i^x = \frac{y_{i+1} - y_{i-1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}},$$

$$n_y(x_i,\, y_i) = n_i^y = \frac{x_{i-1} - x_{i+1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}}, \quad (1 < i < N),$$

$$\ell_1 = \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_1 - x_N)^2 + (y_1 - y_N)^2}}{2},$$

$$\ell_N = \frac{\sqrt{(x_N - x_1)^2 + (y_N - y_1)^2} + \sqrt{(x_N - x_{N-1})^2 + (y_N - y_{N-1})^2}}{2}$$

$$n_1^x = \frac{y_2 - y_N}{\sqrt{(x_2 - x_N)^2 + (y_2 - y_N)^2}}, \quad n_1^y = \frac{x_N - x_2}{\sqrt{(x_2 - x_N)^2 + (y_2 - y_N)^2}},$$

$$n_N^x = \frac{y_1 - y_{N-1}}{\sqrt{(x_1 - x_{N-1})^2 + (y_1 - y_{N-1})^2}},$$

$$n_N^y = \frac{x_{N-1} - x_1}{\sqrt{(x_1 - x_{N-1})^2 + (y_1 - y_{N-1})^2}}. \tag{5.1}$$

Such a construction automatically provides symmetric solution for symmetric contours.

These formulas are valid for arbitrary smooth contour L. However, if the flaw is an elliptic cylinder with the semi-axes a and b, with its center being located at the point $(c,\, h)$ and with the angle of inclination θ with respect to the x-axis, then the above formulas can be written in a more concrete form since

$$\begin{cases} x_i = c + \dfrac{ab\cos(\beta_i)}{\sqrt{a^2\sin^2(\beta_i + \theta) + b^2\cos^2(\beta_i + \theta)}}, \\ y_i = h + \dfrac{ab\sin(\beta_i)}{\sqrt{a^2\sin^2(\beta_i + \theta) + b^2\cos^2(\beta_i + \theta)}}, \end{cases}$$

$$\beta_i = \varepsilon(i - 0.5), \qquad \varepsilon = \frac{2\pi}{N}. \tag{5.2}$$

Under such conditions the reconstruction problem becomes five-dimensional, in the sense that this consists of a search of the five parameters h, c, a, b, θ.

Two different algorithms can be proposed to resolve this inverse problem.

The first one operates with the set of $N + 5$ unknown quantities h, c, a, b, θ, and w_i, $(i = 1, \ldots, N)$ from the system (4.4) (N equations), which has to be considered after substitution of (5.2) into (4.4). The additional set of required equations may be constructed by choosing a number of nodes $x_m \in (x^-, x^+)$, $(m = 1, \ldots, M)$ in the equality (4.3) over the interval where the input data of the measured values $F_0(x_m)$ is collected. If we take $M > 5$, then we come to an overdetermined system of $N + M$ relations for $N + 5$ unknown quantities.

In order to resolve this system of equations, we can pose a problem on minimization of the discrepancy functional:

$$\min[\Omega_1(h, c, a, b, \theta, \{w_i\})],$$

$$\Omega_1 = \left\| \sum_{j=1}^{N} a_{ij} w_j - \varphi_i^0 \right\|^2 + \left\| \frac{1}{\pi} \sum_{j=1}^{N} \frac{(x_m - \xi_j) n_j^x - \eta_j n_j^y}{(x_m - \xi_j)^2 + \eta_j^2} w_j \ell_j - F_0(x_m) \right\|^2$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij} w_j - \varphi_i^0 \right)^2 + \frac{1}{\pi^2} \sum_{m=1}^{M} \left[\sum_{j=1}^{N} \frac{(x_m - \xi_j) n_j^x - \eta_j n_j^y}{(x_m - \xi_j)^2 + \eta_j^2} w_j \ell_j - F_0(x_m) \right]^2. \tag{5.3}$$

In the case when the input data is given exactly this functional attains the minimum: $\min(\Omega_1) = 0$ just on the exact contour. The question whether there is another combination of the sought quantities, which provides the zero value of the functional Ω_1, is related to uniqueness of the solution to the formulated reconstruction problem. This question is out of the present study, whose principal goal is to propose an efficient concrete numerical algorithm of the reconstruction and to test it in the case of elliptic flaws, rather than to investigate (also important in some aspects) questions about solvability and uniqueness.

Another approach can be founded on an explicit (numerical) resolution of system (4.4) considered as a linear algebraic system only for the values of w_i. If this system is represented in the operator form as

$$Aw = \varphi^0, \ \ A = (a_{ij}), \ \ w = (w_i), \ \ \varphi^0 = (\varphi_i^0), \ \ (i, \ j = 1, \dots, N), \tag{5.4}$$

then its inversion is

$$w = A^{-1}\varphi^0, \Rightarrow w_i = (A^{-1}\varphi^0)_i. \tag{5.5}$$

Obviously, operator A^{-1} depends on the five parameters: $A^{-1} = A^{-1}(h, c, a, b, \theta)$, so the substitution of (5.5) into (4.3) results, in the discrete form, in the overdetermined system of nonlinear equations for parameters h, c, a, b, θ:

$$\frac{1}{\pi}\sum_{j=1}^{N}\frac{(x_m - \xi_j)n_j^x - \eta_j n_j^y}{(x_m - \xi_j)^2 + \eta_j^2}[A^{-1}(h, c, a, b, \theta)\varphi^0]_j \ell_j = F_0(x_m),$$

$$m = 1, \dots, M. \tag{5.6}$$

This can also be resolved by a minimization of the discrepancy functional:

$$\min[\Omega_2(h, c, a, b, \theta)],$$

$$\Omega_2(h, c, a, b, \theta)$$

$$= \left\| \frac{1}{\pi}\sum_{j=1}^{N}\frac{(x_m - \xi_j)n_j^x - \eta_j n_j^y}{(x_m - \xi_j)^2 + \eta_j^2}[A^{-1}(h, c, a, b, \theta)\varphi^0]_j \ell_j - F_0(x_m) \right\|^2$$

$$= \frac{1}{\pi^2}\sum_{m=1}^{M}\left\{\sum_{j=1}^{N}\frac{(x_m - \xi_j)n_j^x - \eta_j n_j^y}{(x_m - \xi_j)^2 + \eta_j^2}[A^{-1}(h, c, a, b, \theta)\varphi^0]_j \ell_j - F_0(x_m)\right\}^2. \tag{5.7}$$

By the same reason as in the minimization of functional Ω_1, Eq. (5.3), in the case of exact input data a zero minimum of Ω_2 corresponds to the exact solution of the inverse problem under consideration and we are not able to predict *a priori* whether this solution is unique.

By their basic ideas and in some technical aspects these approaches are different. The principal difference is that the first algorithm does not require any solution of the direct problem, as it is formulated in the form of integral equation (3.23), or in the equivalent discrete form of linear algebraic system (4.4).

Another difference between the minimization of the two functionals, Ω_1 and Ω_2, is that the latter is really a usual function of five variables. From this point of view the minimization of Ω_2 seems to be a simpler problem.

However, any regular method to solve the both minimization problems is based on a calculation of the gradient operator of the corresponding functional. For Ω_1 this can be found analytically in an explicit form since the dependence of the functional upon w_i is quadratic and explicit, and the dependence on the parameters (h, c, a, b, θ) is more complex but still explicit (in fact, a_{ij} is explicitly written in (4.4b) in terms of (x_i, y_i), and the latter clearly contains the five parameters from (5.2)). From this point of view, the determination of the gradient is a more simple task just in the case of Ω_1 since for this functional the gradient can be easily calculated analytically. For the functional Ω_2 this gradient can be calculated numerically.

As soon as the method to calculate the gradient of Ω_1 and Ω_2 has been arranged, the minimization of these functionals can be achieved by some iterative gradient process, like Steepest Descent Method, or Conjugate Gradient Method (see, for example, [8]). Both iterative methods converge to exact solution in the case of linear operator equation, which corresponds to a quadratic functional. Our inverse problem is nonlinear, so we cannot prove strictly that the iterative scheme converge. However, since any smooth functional is locally quadratic, there is a good chance that this iterative method is convergent.

The more important point is that any iterative method provides a convergence to a local minimum only. In the case of nonlinear equations such values of local minima may be too far from the desired value $\Omega_{1,2} = 0$.

For this reason we used in our numerical experiments a version of the method of random search contiguous to the one described in detail in [6]. This provides a search of the global minima by moving step by step among the most promising iterations. When performing the numerical implementation, we could clearly observe that the minimization of Ω_1 and Ω_2 gives the results very close to each other, but operation with Ω_2 permits reconstruction within a shorter time of computations.

Some examples of the reconstruction are demonstrated in Table 1. For all examples demonstrated below we used $M = 200$ points of measurements over the interval $x \in (-5, 5)$, to form the array of the input data, so that $x_m = 0.05m$, $x_{m+100} = -0.05m$, $m = 1, \ldots, M/2$. It is clear that with such a choice of the trial points they represent a relatively uniform set around the applied force.

Table 1

Input data error	h	c	a	b	θ	Type of result
0%	1.500	0.000	1.000	0.500	0.000	exact
	1.520	0.025	0.995	0.527	0.020	restored
0%	2.500	1.000	1.300	0.500	$\pi/2 = 1.571$	exact
	2.491	0.999	1.299	0.497	1.569	restored
	2.470	1.001	0.493	1.293	3.140	restored
0%	5.000	4.000	2.000	0.700	$\pi/4 = 0.785$	exact
	4.987	3.980	0.639	2.039	2.344	restored
0%	2.000	– 3.000	0.400	0.600	$\pi/3 = 1.047$	exact
	2.022	– 3.008	0.605	0.399	2.621	restored

These results are related to the case of exact input data. The latter was obtained by a solution of the direct boundary value problem by Boundary Element Technique.

Then we studied the stability of the proposed algorithm if the input data is given with an error. In order to model the input data with a certain error, we first did construct the solution of respective direct problem, and then arranged some stochastic perturbations of the so obtained data. Some results of such a numerical simulation are shown in Table 2.

Table 2

Input data error	h	c	a	b	θ	Type of result
10%	7.000	0.000	4.000	1.000	0.000	exact
	7.015	– 0.027	1.385	3.703	1.562	restored
	7.017	0.065	3.739	1.342	0.011	restored
10%	3.000	– 2.500	1.200	0.300	$\pi/4$ = 0.785	exact
	3.028	– 2.470	1.140	0.433	0.836	restored
	3.135	– 2.420	0.342	1.165	2.369	restored
10%	1.000	3.500	0.200	0.200	$\pi/2$ = 1.571	exact
	1.030	3.539	0.201	0.204	1.909	restored
10%	4.500	0.000	0.500	2.500	0.000	exact
	4.490	0.025	2.510	0.481	1.568	restored

Further increase in the error of the input data results in the following table:

Table 3

Input data error	h	c	a	b	θ	Type of result
20%	0.500	4.500	0.400	0.200	$\pi/2$ = 1.571	exact
	0.605	4.551	0.431	0.274	1.527	restored
20%	2.500	3.000	1.000	1.000	0.000	exact
	2.480	3.023	0.995	0.988	0.384	restored
20%	5.000	– 2.000	3.000	1.000	$\pi/4$ = 0.785	exact
	5.317	– 1.870	3.139	0.729	0.787	restored
	5.451	– 1.969	0.913	3.023	2.352	restored
20%	0.600	0.000	0.100	0.100	$\pi/3$ = 1.047	exact
	0.603	– 0.001	0.102	0.099	2.600	restored

From the presented results of the numerical simulation, we come to some important conclusions:

1. It is interesting to notice that Table 3 in some cases shows better results than Table 2, despite more rough input data. This indicates an important property that the precision of the reconstruction is less dependent on the error of the input data than on the geometry of the void.

2. The highest precision is attained in the reconstruction of ellipses with low aspect ratio a/b (see the last example in Table 1, the third example in Table 2, and the second and the last examples in Table 3.

3. Generally, precision of the reconstruction is high. In some cases almost the same results are obtained with formally different reconstructed geometries. For example, two different results of reconstruction in the first example, Table 1, seem to give absolutely different defects. However, more detailed view shows that they are almost coinciding ellipses, whose second ratio is the inverse value of the first one, and the principal axis is inclined approximately at the angle $\pi/2$. Consequently, they represent a well reconstructed images for the same defect. Similar examples are given by all boxes represented by pairs of (seemingly) different reconstructions.

4. The results do not depend significantly on the precision of the input data. This may be connected with the overdetermined character of the input data for the inverse problem. It also indicates implicitly that the input data randomly perturbed has the same mean value as a precise input data. In other words, the proposed algorithm arranges automatically something like a filtering of a randomly distributed error of the input data.

Acknowledgements

The present work has been supported in part by the Program on Russian leading scientific schools in mechanics (Grant 2113.2003.1). The paper has also been supported by Italian Ministry of University (M.U.R. S.T.) through its national and local (60%) projects.

References

[1] A. B. Abda et al., Line segment crack recovery from incomplete boundary data, Inverse Problems 18 (2002), 1057-1077.

[2] G. Alessandrini, E. Beretta and S. Vessella, Determining linear cracks by boundary measurements: Lipschitz stability, SIAM J. Math. Anal. 27 (1996), 361-375.

[3] S. Andrieux and A. B. Abda, Identification of planar cracks by complete overdetermined data: inversion formulae, Inverse Problems 12 (1996), 553-563.

[4] T. Bannour, A. B. Abda and M. Jaoua, A semi-explicit algorithm for the reconstruction of 3D planar cracks, Inverse Problems 13 (1997), 899-917.

[5] M. Bonnet, Boundary Integral Equations Methods for Solids and Fluids, John Wiley, New York, 1999.

[6] M. Corana et al., Minimizing multimodal functions of continuous variables with the simulated annealing algorithm, ACM Trans. Math. Software 13 (1987), 262-280.

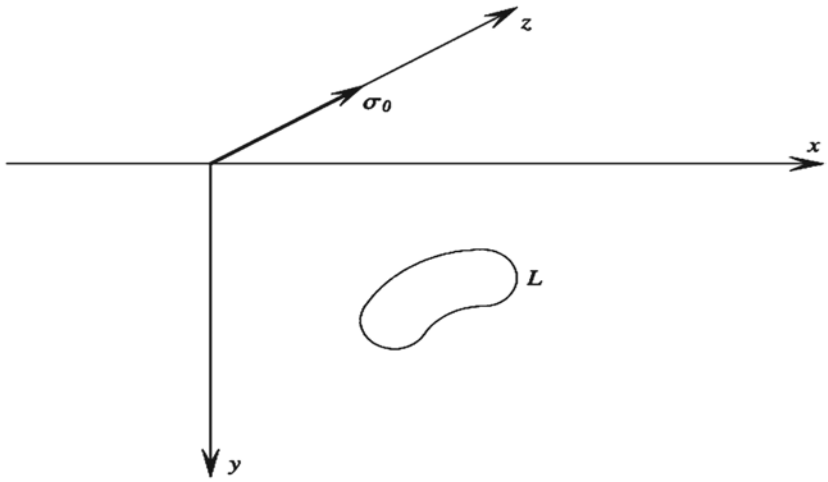
[7] A. Friedman and M. Vogelius, Determining cracks by boundary measurements, Indiana Univ. Math. J. 38 (1989), 527-556.

[8] P. E. Gill, W. Murray and M. H. Wright, Practical Optimization, Academic Press, London, 1981.

[9] N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, Kluwer, Dordrecht, 1975.

[10] A. S. Saada, Elasticity, Theory and Applications, 2nd ed., Krieger, Malabar, Florida, 1993.

[11] M. A. Sumbatyan and A. Scalia, Equations of Mathematical Diffraction Theory, CRC Press, Boca Raton, Florida, 2004.



Figure

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