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THE SOLUTION OF BURGERS EQUATION BY THE ADOMIAN DECOMPOSITION METHOD

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Abstract

In this paper, we discuss the solution of Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

by the Adomian decomposition method.

1. Introduction

In this paper, we solve Burgers equation by the Adomian decomposition method. Burgers equation is a useful model for many physically interesting problems, particularly those of a fluid-flow nature, in which either shocks or viscous dissipation is significant in part of the region. For many combinations of initial and boundary conditions, an exact solution of Burgers equation is presented. Burgers equation is one of the simplest nonlinear partial differential equations for which it is possible to obtain exact solution, it behaves as an elliptic, parabolic or hyperbolic partial differential equation. Therefore, Burgers equation has been used widely as a model equation for testing and comparing computational techniques.

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2. The Solution by the Adomian Decomposition Method

We consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

hence we have

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}.$$

To get canonical form of above equation, on putting $L = \frac{\partial}{\partial t}$, we have

$$Lu = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}, \tag{3}$$

where L is an invertible operator and its inverse has the form $L^{-1} = \int_0^t (.)dt$. Putting L^{-1} in both sides of the relation (3), we get

$$u(x, t) = u(x, 0) + \int_0^t \left(-u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}\right)dt. \tag{4}$$

To solve equation (4) by the Adomian decomposition method, we consider as usual in this method

$$u(x, t) = \textstyle\sum_{n=0}^{\infty} u_n(x, t),$$

$$-u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}\partial t = \textstyle\sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n),$$

and substituting into (4), we have

$$\textstyle\sum_{n=0}^{\infty} u_n(x, t) = u(x, 0) + \int_0^t \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n)dt$$

$$= u(x, 0) + \textstyle\sum_{n=0}^{\infty} \int_0^t A_n(u_0, u_1, u_2, \ldots, u_n)dt. \tag{5}$$

Therefore

$$u_0(x, t) + u_1(x, t) + \cdots + u_n(x, t) + \cdots$$

$$= u(x, 0) + \int_0^t (A_0(u_0) + A_1(u_0, u_1) + \cdots + A_n(u_0, u_1, \ldots, u_n))dt.$$

Then

$$\begin{cases} u_0(x, t) = u(x, 0), \\ u_{n+1} = \int_0^t A_n(u_0, u_1, \ldots, u_n)dt, \quad n = 0, 1, 2, \ldots. \end{cases}$$

Based on methods presented for calculation of Adomian polynomials, we have

$$u(x, t) = \textstyle\sum_{n=0}^{\infty} u_n(x, t)$$

$$u_x(x, t) = \textstyle\sum_{n=0}^{\infty} (u_n(x, t))_x$$

$$u_{xx}(x, t) = \textstyle\sum_{n=0}^{\infty} (u_n(x, t))_{xx}.$$

Thus

$$\begin{aligned} -uu_x + \nu u_{xx} &= -(u_0 + u_1 + u_2 + \cdots)(u_{0x} + u_{1x} + u_{2x} + \cdots) \\ &\quad + \nu(u_{0xx} + u_{1xx} + u_{2xx} + \cdots) \\ &= -(u_0u_{0x} + u_0u_{1x} + u_0u_{2x} + \cdots + u_1u_{0x} \\ &\quad + u_1u_{1x} + \cdots + u_2u_{0x} + u_2u_{1x} + u_2u_{2x} + \cdots) \\ &\quad + \nu(u_{0xx} + u_{1xx} + u_{2xx} + \cdots) \\ &= (-u_0u_{0x} + \nu u_{0xx}) + (-u_0u_{1x} - u_1u_{0x} + \nu u_{1xx}) \\ &\quad + (-u_0u_{2x} - u_1u_{1x} - u_2u_{0x} + \nu u_{2xx}) + \cdots. \end{aligned}$$

Therefore

$$A_0(u_0) = -u_0u_{0x} + \nu u_{0xx}$$

$$A_1(u_0, u_1) = -u_0u_{1x} - u_1u_{0x} + \nu u_{1xx}$$

$$A_2(u_0, u_1, u_2) = -u_0u_{2x} - u_1u_{1x} - u_2u_{0x} + \nu u_{2xx}$$

$$\vdots$$

Based on values of A_n, we calculate terms u_n as follows:

$$u_0(x, t) = u(x, 0)$$

$$u_1(x, t) = \int_0^t A_0(u_0)dt = \int_0^t (-u_0u_{0x} + \nu u_{0xx})dt$$

$$u_2(x, t) = \int_0^t A_1(u_0, u_1)dt = \int_0^t (-u_0u_{1x} - u_1u_{1x} + \nu u_{1xx})dt$$

$$u_3(x, t) = \int_0^t A_2(u_0, u_1, u_2)dt = \int_0^t (-u_0u_{2x} - u_1u_{1x} - u_2u_{0x} + \nu u_{2xx})dt$$

$$\vdots$$

Therefore, we have

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots.$$

We can determine the components $u_n(x, t)$, so far as we like to enhance the accuracy of the approximation. Therefore, $m+1$-terms $u(x, t) = \sum_{n=0}^{m} u_n(x, t)$ can be used to approximate the solution.

3. Examples

Example 1. We consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{6}$$

with the boundary condition

$$u_0(x) = u(x, 0) = \begin{cases} 1 & \text{if } -1 \le x < 0, \\ 0 & \text{if } 0 < x \le 1, \end{cases}$$

where $\nu = \frac{1}{\mathrm{Re}}$ (where Re is the Reynolds number). By using the Adomian decomposition method, if $-1 \leq x \leq 0$, then by (5), we have

$$u_0(x, t) = u(x, 0) = 1$$

$$u_1(x, t) = \int_0^t A_0 dt = \int_0^t (-u_0 u_{0x} + \nu u_{0xx}) dt = \int_0^t (0) dt = 0$$

$$u_2(x, t) = 0$$

$$\vdots$$

Thus

$$u_n(x, t) = 0, \quad n = 1, 2, \ldots .$$

The exact solution is

$$u(x, t) = u_0(x, t) = 1.$$

If $0 < x \leq 1$, then by (5), we have

$$u_0(x, t) = u(x, 0) = 0$$

$$u_1(x, t) = \int_0^t (-u_0 u_{0x} + \nu u_{0xx}) dt = \int_0^t (0) dt = 0$$

$$u_2(x, t) = 0$$

$$\vdots$$

Thus

$$u_n(x, t) = 0, \quad n = 0, 1, 2, 3, \ldots .$$

The exact solution is

$$u(x, t) = u_0(x, t) = 0.$$

Example 2. We consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

with the boundary condition

$$u(x,\ 0) = -\sin(\pi x),$$

where $\nu = \frac{10^{-2}}{\pi}$. By using the Adomian decomposition method, we have

$$A_0(u_0) = (\pi \sin(\pi x)\cos(\pi x) + 10^{-2}\pi \sin(\pi x))$$

$$A_1(u_0,\ u_1) = (-2\pi^2 \sin(\pi x)\cos^2(\pi x) + 6 * 10^{-2}\pi^2 \sin(\pi x)\cos(\pi x)$$
$$+ \pi^2 \sin^3(\pi x) - 10^{-4}\pi^2 \sin(\pi x))t$$
$$\vdots$$

Therefore by (5), we have

$$u_0(x,\ t) = -\sin(\pi x)$$

$$u_1(x,\ t) = (-\pi \sin(\pi x)\cos(\pi x) + 10^{-2}\pi \sin(\pi x))t$$

$$u_2(x,\ t) = (-2\pi^2 \sin(\pi x)\cos^2(\pi x) + 6 * 10^{-2}\pi^2 \sin(\pi x)\cos(\pi x)$$
$$+ \pi^2 \sin^3(\pi x) - 10^{-4}\pi^2 \sin(\pi x))\frac{t^2}{2}.$$

Then

$$u_0 = -\sin(\pi x)$$

$$u_1 = (-\pi \sin(\pi x) + 10^{-2}\pi \sin(\pi x))t$$

$$u_2 = (-2\pi^2 \sin(\pi x)\cos^2(\pi x) + 6 * 10^{-2}\pi^2 \sin(\pi x)\cos(\pi x)$$
$$+ \pi^2 \sin^3(\pi x) - 10^{-4}\pi^2 \sin(\pi x))\frac{t^2}{2}.$$

Therefore, we have

$$u(x,\ t) = \sum_{n=0}^{\infty} u_n(x,\ t) = u_0(x,\ t) + u_1(x,\ t) + u_2(x,\ t) + \cdots.$$

Now we solve it by using Maple and list the results in the following table and term Adomian is calculated for 3-terms. We present in the following table exact solution for $t = 0$ and approximate solution for $t = 0.13,\ 0.25$.

x	t = 0.00	t = 0.13	t = 0.25
– 1.000	0.0000	0.0000	0.0000
– 0.900	0.3090	0.2334	0.2445
– 0.800	0.5878	0.4408	0.3932
– 0.700	0.8090	0.6164	0.4471
– 0.600	0.9511	0.7720	0.5089
– 0.500	1.000	0.9125	0.6838
– 0.400	0.9511	1.009	0.9597
– 0.300	0.8090	1.0000	1.1765
– 0.200	0.5878	0.8245	1.1225
– 0.100	0.3090	0.4704	0.6953
0.0000	0.0000	0.0000	0.0000
0.100	– 0.3090	– 0.4705	– 0.6953
0.200	– 0.5878	– 0.8245	– 1.1225
0.300	– 0.8090	– 1.0000	– 1.1765
0.400	– 0.9511	– 1.0091	– 0.9597
0.500	– 1.0000	– 0.9125	– 0.6838
0.600	– 0.9511	– 0.7720	– 0.5089
0.700	– 0.8090	– 0.6164	– 0.4471
0.800	– 0.5878	– 0.4408	– 0.3932
0.900	– 0.3090	– 0.2334	– 0.2445
1.000	0.0000	0.0000	0.0000

Example 3. We consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

with the boundary condition

$$u(x, 0) = e^{-8(1-x)},$$

where $\nu = 2 * 10^{-3}$, we have

$$A_0(u_0) = -8e^{-16(1-x)} + 128 * 10^{-3} e^{-8(1-x)}$$

$$A_1(u_0, u_1) = (-64e^{-6(1-x)} - 4.096e^{-16(1-x)} - 0.016384e^{-8(1-x)}).$$

Therefore by (5), we have

$$u_0(x, t) = e^{-8(1-x)}$$

$$u_1(x, t) = (-8e^{-16(1-x)} + 0.128e^{-8(1-x)})t$$

$$u_2(x, t) = (64e^{-26(1-x)} - 4.096e^{-16(1-x)} - 0.016384e^{-8(1-x)})\frac{t^2}{2}$$

$$\vdots$$

Then

$$u_0 = e^{-8(1-x)}$$

$$u_1 = (-8e^{-16(1-x)} + 0.128e^{-8(1-x)})t$$

$$u_2 = (64e^{-26(1-x)} - 4.096e^{-16(1-x)} - 0.016384e^{-8(1-x)})\frac{t^2}{2}$$

$$\vdots$$

Therefore, we have

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots.$$

Now we solve it by using Maple and list the results in the following table and term Adomian is calculated for 3-terms. We present in the following table exact solution for $t = 0$ and approximate solution for $t = 0.3, 0.18$.

x	$t = 0.000$	$t = 0.3$	$t = 0.18$
0.000	0.0003	0.0003	0.0003
0.100	0.0007	0.0008	0.0008
0.200	0.0017	0.0017	0.0017
0.300	0.0037	0.0038	0.0038
0.400	0.0082	0.0084	0.0083
0.500	0.0183	0.0182	0.0183
0.600	0.0408	0.0388	0.0393
0.700	0.0907	0.0756	0.0814
0.800	0.2019	0.1275	0.1535
0.900	0.4493	0.1956	0.2458
1.000	1	1.5177	0.6196

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