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ON SOME HOMOMORPHISMS ASSOCIATED WITH FINITE BINARY RELATIONS

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Abstract

In the previous paper, we introduced a "pre-ring" and "pre-module" and also defined a pre-ring homomorphism and pre-module homomorphism. A power set with a binary operation "$\cup$" forms a pre-module over some two elements pre-ring together with an appropriate scalar multiplication. In this paper, considering non-empty finite sets A and B, we present a fundamental one-to-one correspondence between the homomorphisms from the pre-module $P(A)$ to the pre-module $P(B)$ and the relations from A to B. We also present some related consequences.

1. Preliminaries and Conventions

The study of this paper is based on the results of our previous study [1, 2]. We use the terminology and notation introduced in the previous study.

A binary relation (simply a relation) is a set of ordered pairs. A function f is the relation f such that if $(x, y) \in f$ and $(x, z) \in f$, this implies $y = z$. If f is a function, then the unique y such that $(x, y) \in f$ is the value of f at x.

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We adopt the following conventions. Let R be a relation (including functions). Then the notation xRy is equivalent to $(x, y) \in R$. If f is a function, then the notation $(x)f = y$ is used for the value y of f at x. Let R and S be relations. Then the composition of R and S, denoted by $R \circ S$, is the relation such that (x, y) belongs to $R \circ S$ if xRz, zSy, and $z \in (\mathrm{ran}(R) \cap \mathrm{dom}(S))$. Let f and g be functions such that $\mathrm{ran}(f) \subseteq \mathrm{dom}(g)$. Then the composition of f and g, denoted by $f \circ g$, is the function with $\mathrm{dom}(f \circ g) = \mathrm{dom}(f)$ such that $(x)(f \circ g) = ((x)f)g$ for every $x \in \mathrm{dom}(f)$.

Let A and B be sets. Then we adopt the following conventions. A relation R from A to B is a subset of $A \times B$. The empty relation $\varnothing$ is denoted by the symbol O. If R is a relation from A to B, then the inverse of R, denoted by R^{-1}, is the relation from B to A such that $R^{-1} = \{(y, x) \mid (x, y) \in R\}$. A function f from A to B is the relation f from A to B such that $\mathrm{dom}(f) = A$ and if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. A relation on A is a relation from A to A. A function on A is a function from A to A.

To reduce the parentheses in expressions with a sequence of symbols, we adopt the usual conventions. In this case the symbols "$\in$, $\notin$, $=$, $\neq$, $\subseteq$" are dominant. However, the two symbols "$\rightarrow$, $\leftrightarrow$" are more dominant symbols.

1.1 Some notation for finite relations. We briefly review some notation and related properties of finite relations [2].

Let A and B be non-empty finite sets and let $R \subseteq A \times B$, with $a \in A$ and $b \in B$. The symbols aR and Rb are defined as follows:

• aR is the set such that "$y \in aR$ if $(a, y) \in R$ for some $y \in B$" or "$aR = \varnothing$ if $(a, y) \notin R$ for every $y \in B$".

• Rb is the set such that "$x \in Rb$ if $(x, b) \in R$ for some $x \in A$" or "$Rb = \varnothing$ if $(x, b) \notin R$ for every $x \in A$".

In relation to the above notation, the following properties hold.

Let A and B be non-empty finite sets and let f be a function from A to

B. Then af is the singleton $\{(a)f\}$ for every $a \in A$. If $b \in B$ and $b \notin \operatorname{ran}(f)$, then fb is an empty set $\varnothing$.

Let A and B be non-empty finite sets and let $R \subseteq A \times B$, with $a \in A$ and $b \in B$. Then

1. $y \in aR$ if and only if aRy, and $x \in Rb$ if and only if xRb.

2. $aR = R^{-1}a$ and $Rb = bR^{-1}$.

Let A and B be non-empty finite sets and let $R, S \subseteq A \times B$, with $a \in A$ and $b \in B$. Then

1. $R = S$ if and only if "$aR = aS$ for every $a \in A$" or "$Rb = Sb$ for every $b \in B$".

2. $a(R \cup S) = aR \cup aS$ and $(R \cup S)b = Rb \cup Sb$.

Let A, B and C be non-empty finite sets, and let $R \subseteq A \times B$ and $S \subseteq B \times C$. Then $(a, c) \in R \circ S$ if and only if $aR \cap Sc \neq \varnothing$ for every $a \in A$ and $c \in C$.

Let A, B, C and D be sets. Let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$. The following properties are well known. If $R = O$ or $S = O$, then $R \circ S = O$. $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$ and $(R \circ S) \circ T = R \circ (S \circ T)$.

We consider an indexed family of sets with a finite index set for the later study. Let $\{A_i \mid i \in I\}$ be an indexed family of sets with a finite index set I. Let $I = \{1, 2, \ldots, n\}$, where n is a positive integer. Then $\bigcup \{A_i \mid i \in I\} = A_1 \cup A_2 \cup \cdots \cup A_n$. If $I = \varnothing$, then $\bigcup \{A_i \mid i \in I\} = \varnothing$.

Proposition 1.1. *Let $\{A_i \mid i \in I\}$ and $\{A_j \mid j \in J\}$ be two indexed families of sets such that if $i = j$, then $A_i = A_j$, where I and J are finite index sets. Then $\bigcup \{A_k \mid k \in I \cup J\} = \bigcup \{A_i \mid i \in I\} \cup \bigcup \{A_j \mid j \in J\}$.*

Proof. Clearly, by the hypothesis, $\{A_k \mid k \in I \cup J\}$ is an indexed family of sets with the finite index set $I \cup J$. Note that if $i = j$, then $A_i = A_j = A_i \cup A_j$. Then by considering the associativity and commutativity of $\bigcup$, together with the fact above, we can prove the equality. □

Lemma 1.2. *Let* $\{A_i \mid i \in I\}$ *be an indexed family of sets with a finite index set I. Define* $N(x) = \{i \mid x \in A_i$ *and* $i \in I\}$. *Then* $x \in \bigcup\{A_i \mid i \in I\}$ *if and only if* $N(x) \neq \varnothing$.

Proof. Let $x \in \bigcup\{A_i \mid i \in I\}$. Then there is some $k \in I$ such that $x \in A_k$. By the definition of $N(x)$, we have $k \in N(x)$ and hence $N(x) \neq \varnothing$. Conversely, let $N(x) \neq \varnothing$. Then by the definition of $N(x)$, there is some $i \in I$ such that $x \in A_i$. This implies that $x \in \bigcup\{A_i \mid i \in I\}$. Thus, we have $x \in \bigcup\{A_i \mid i \in I\} \leftrightarrow N(x) \neq \varnothing$. □

Proposition 1.3. *Let A, B and C be non-empty finite sets. Let* $R \subseteq A \times B$ *and* $S \subseteq B \times C$, *with* $a \in A$ *and* $c \in C$. *Let* $\{bS \mid b \in aR\}$ *be an indexed family of sets with an index set aR. Let* $\{Rb \mid b \in Sc\}$ *be an indexed family of sets with an index set Sc. Then* $a(R \circ S) = \bigcup\{bS \mid b \in aR\}$ *and* $(R \circ S)c = \bigcup\{Rb \mid b \in Sc\}$.

Proof. To prove $a(R \circ S) = \bigcup\{bS \mid b \in aR\}$, let $C_a = \bigcup\{bS \mid b \in aR\}$. We show that $a(R \circ S) = C_a$.

Let $N(x) = \{b \mid x \in bS$ and $b \in aR\}$. This is equivalent to $N(x) = \{b \mid b \in Sx$ and $b \in aR\}$. This means $N(x) = aR \cap Sx$. By Lemma 1.2, we have $x \in C_a \leftrightarrow aR \cap Sx \neq \varnothing \leftrightarrow a(R \circ S)x \leftrightarrow x \in a(R \circ S)$. This implies $a(R \circ S) = C_a$.

On the other hand, $(R \circ S)c = c(R \circ S)^{-1} = c(S^{-1} \circ R^{-1})$. From the result above, it follows that $c(S^{-1} \circ R^{-1}) = \bigcup\{bR^{-1} \mid b \in cS^{-1}\} = \bigcup\{Rb \mid b \in Sc\}$. Thus, we have $(R \circ S)c = \bigcup\{Rb \mid b \in Sc\}$. □

1.2 A pre-module of a power set. The following is a brief review for a pre-ring and pre-module, and related terms [1].

Let A be a non-empty set. The addition + and the multiplication $\cdot$ are binary operations on A. The ordered triplet $\langle A, +, \cdot \rangle$ is a pre-ring if it satisfies the following conditions.

PR 1. $\langle A, + \rangle$ is a commutative monoid.

PR 2. $0 \cdot x = x \cdot 0 = 0$ for every $x \in A$, where 0 is unity for +.

PR 3. Multiplication $\cdot$ is associative, and has unity.

PR 4. For each $x, y, z \in A$:

1. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
2. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.

Let $\langle A, +, \cdot \rangle$ be a pre-ring (or simply a pre-ring A), and let U be the set of units of the pre-ring A. Then $\langle U, \cdot \rangle$ is the unit group of the pre-ring A.

Let $\langle A, +, \cdot \rangle$ and $\langle B, +, \cdot \rangle$ be pre-rings. If f is a monoid homomorphism from $\langle A, + \rangle$ to $\langle B, + \rangle$, and also a monoid homomorphism from $\langle A, \cdot \rangle$ to $\langle B, \cdot \rangle$, then the function f is a pre-ring homomorphism from the pre-ring A to pre-ring B. And f is a pre-ring isomorphism if f is a pre-ring homomorphism and bijective.

A pre-ring is commutative if its multiplication is commutative. Let $\langle Q, +, \cdot \rangle$ be a commutative pre-ring with additive unity 0 and multiplicative unity 1. Let $\langle M, + \rangle$ be a commutative monoid with unity 0_M. Let $\mu : Q \times M \to M$ be a scalar multiplication, with $(r, x)\mu$ denoted by rx for every $r \in Q$ and $x \in M$. Then $\langle M, + \rangle$ (or simply M) is a pre-module over Q, if it satisfies the following conditions for every r, $s \in Q$ and x, $y \in M$, where $r \cdot s$ is denoted simply by rs.

PM 1. $r(x + y) = rx + ry$.

PM 2. $(r + s)x = rx + sx$.

PM 3. $(rs)x = r(sx)$.

PM 4. $1x = x$.

PM 5. $0x = 0_M$.

The element 0_M is the zero element of the pre-module M. In relation to a scalar multiplication, the following holds generally: $r0_M = 0_M$ for every $r \in Q$.

Let M and N be pre-modules over the same commutative pre-ring Q. Then the function $f : M \to N$ is a pre-module homomorphism (or simply a homomorphism) from M to N if it satisfies the following conditions. For each $x, y \in M$ and $r \in Q$,

1. $(x + y)f = (x)f + (y)f$.

2. $(rx)f = r(x)f$.

If f is a homomorphism from M to N, then $(0_M)f = 0_N$, where 0_M and 0_N are the zero elements of M and N, respectively.

A pre-module isomorphism (or simply an isomorphism) f from M to N means that f is a homomorphism and bijective. If f is an isomorphism from M to N, then f^{-1} is an isomorphism from N to M.

By K_2 we mean two elements pre-ring with additive unity 0 and multiplicative unity 1 having the following operations:

1. $0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 1;$

2. $0 \cdot 0 = 0,\ 0 \cdot 1 = 0,\ 1 \cdot 0 = 0,\ 1 \cdot 1 = 1.$

Let A be a non-empty set. Then $\langle P(A), \cup \rangle$ is a commutative monoid with unity $\varnothing$, where $P(A)$ is the power set of A. The following gives a pre-module of a power set.

Proposition 1.4. *Let A be a non-empty set. Define a scalar multiplication by K_2 on $P(A)$ such that $0x = \varnothing$ and $1x = x$ for every $x \in P(A)$. Then $\langle P(A), \cup \rangle$ forms a pre-module over K_2.*

Proof. Let $K_2 = \langle \{0, 1\}, +, \cdot \rangle$. We show that $P(A)$ satisfies all conditions PM 1-PM 5 for a pre-module under the given scalar multiplication. For every $x, y \in P(A)$:

PM 1. $0(x \cup y) = \varnothing = \varnothing \cup \varnothing = 0x \cup 0y,\ 1(x \cup y) = x \cup y = 1x \cup 1y.$

PM 2. $(1+1)x = 1x = x = x \cup x = 1x \cup 1x$, $(1+0)x = 1x = x = x \cup \varnothing = 1x \cup 0x$, $(0+1)x = 1x = x = \varnothing \cup x = 0x \cup 1x$, $(0+0)x = 0x = \varnothing = \varnothing \cup \varnothing = 0x \cup 0x$.

PM 3. $(1 \cdot 1)x = 1x = 1(1x)$, $(1 \cdot 0)x = 0x = 1(0x)$, $(0 \cdot 1)x = 0x = 0(1x)$, $(0 \cdot 0)x = 0x = \varnothing = 0\varnothing = 0(0x)$.

PM 4. $1x = x$.

PM 5. $0x = \varnothing$.

Thus, $\langle P(A), \cup \rangle$ is a pre-module over K_2. □

For a simplicity of a notation, the pre-module $\langle P(A), \cup \rangle$ over K_2 is denoted by $P(A, \cup)$.

2. Pre-module Homomorphisms Related to Finite Relations

Suppose that A and B are non-empty finite sets, and $X = P(A, \cup)$ and $Y = P(B, \cup)$. In this section, we are mainly concerned with a homomorphism between X and Y, and a relation between A and B. The symbol $\hat{a}$ denotes the singleton $\hat{a} = \{a\}$ for every $a \in A$.

Proposition 2.1. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let f be a function from X to Y. Then f is a homomorphism from X to Y if and only if $(x)f = \bigcup\{(\hat{a})f \mid a \in x\}$ for every $x \in X$.*

Proof. Let f be a homomorphism from X to Y. Then by the definition of a homomorphism, clearly, we have $(x)f = \bigcup\{(\hat{a})f \mid a \in x\}$ for every $x \in X$.

Conversely, let f be a function from X to Y such that $(x)f = \bigcup\{(\hat{a})f \mid a \in x\}$ for every $x \in X$.

Let $x_1, x_2 \in X$. Then by the hypothesis and Proposition 1.1, we have

$$(x_1 \cup x_2)f = \bigcup\{(\hat{a})f \mid a \in x_1 \cup x_2\}$$

$$= \bigcup\{(\hat{a})f \mid a \in x_1\} \cup \bigcup\{(\hat{a})f \mid a \in x_2\}$$

$$= (x_1)f \cup (x_2)f.$$

Hence $(x_1 \cup x_2)f = (x_1)f \cup (x_2)f$.

Let $x \in X$ and $r \in K_2$. We show that $(1x)f = 1(x)f$ and $(0x)f = 0(x)f$. By the scalar multiplications on X and Y, we have $(1x)f = 1(x)f$. On the other hand, by the hypothesis, $(\varnothing)f = \varnothing$. Then by the scalar multiplications on X and Y, the equality $(\varnothing)f = \varnothing$ implies that $(0x)f = 0(x)f$. Hence $(rx)f = r(x)f$.

The two facts above imply that f is a homomorphism from X to Y.

Thus the proof is completed. □

Lemma 2.2. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let R be a relation from A to B. If f_R is a function from X to Y such that $(x)f_R = \bigcup\{aR \mid a \in x\}$ for every $x \in X$, then f_R is a homomorphism from X to Y.*

Proof. Clearly, $\bigcup\{aR \mid a \in x\}$ is a subset of B for every $x \in X$. This implies that f_R is a well defined function from X to Y. By the hypothesis, we have $(\hat{a})f_R = aR$ for every $a \in A$. Then $(x)f_R = \bigcup\{(\hat{a})f_R \mid a \in x\}$ for every $x \in X$. By Proposition 2.1, the function f_R is a homomorphism from X to Y. □

By the notation $H(X, Y)$, we mean the set of all homomorphisms from a pre-module X to pre-module Y.

Proposition 2.3. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Then there is a bijective function $\theta : H(X, Y) \to P(A \times B)$ such that $(f)\theta = R$ for every $f \in H(X, Y)$, satisfying the condition $(\hat{a})f = aR$ for every $a \in A$, where $R \in P(A \times B)$.*

Proof. Let f be an arbitrary element of $H(X, Y)$. Define R to be a relation from A to B such that $aR = (\hat{a})f$ for every $a \in A$. Then the relation R is unique. This fact implies that there exists a function $\theta : H(X, Y) \to P(A \times B)$ such that $(f)\theta = R$ for every $f \in H(X, Y)$, satisfying the condition $(\hat{a})f = aR$ for every $a \in A$, where $R \in P(A \times B)$.

Now we show that the function θ is bijective.

To prove that θ is surjective, let R be an element of $P(A \times B)$. Then, by Lemma 2.2, there is a homomorphism $f \in H(X, Y)$ such that $(x)f = \bigcup \{aR \mid a \in x\}$ for every $x \in X$, which satisfies the condition $(\hat{a})f = aR$ for every $a \in A$. This means that if a relation $R \in P(A \times B)$ is given, then there is some homomorphism $f \in H(X, Y)$ such that $(f)\theta = R$ satisfying the condition $(\hat{a})f = aR$ for every $a \in A$. This implies that θ is surjective.

Let $f_1, f_2 \in H(X, Y)$, and let $(f_1)\theta = R_1$ and $(f_2)\theta = R_2$, where R_1, $R_2 \in P(A \times B)$. To prove the injectivity of θ, we show that if $f_1 \neq f_2$, then $R_1 \neq R_2$. Let $f_1 \neq f_2$. Then there is some element $x' \in X$ such that $(x')f_1 \neq (x')f_2$. By Proposition 2.1, $\bigcup \{(\hat{a})f_1 \mid a \in x'\} \neq \bigcup \{(\hat{a})f_2 \mid a \in x'\}$. By the condition for θ, we have $\bigcup \{aR_1 \mid a \in x'\} \neq \bigcup \{aR_2 \mid a \in x'\}$. This means that there is some element $a' \in x'$ such that $a'R_1 \neq a'R_2$ and hence $R_1 \neq R_2$. Hence θ is injective.

Thus the function θ is bijective. □

In Proposition 2.3, the function θ gives one to one correspondence between the pre-module homomorphisms from X to Y and the relations from A to B. The bijective function θ is called the θ-*function* for $H(X, Y)$. If $(f)\theta = R$, then R is called the relation *corresponding to* the pre-module homomorphism f, and f is called the pre-module homomorphism *corresponding to* the relation R.

Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let f and g be homomorphisms from X to Y. By $f * g$ we mean the homomorphism from X to Y such that $(x)(f * g) = (x)f \cup (x)g$ for every $x \in X$.

Proposition 2.4. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let f and g be homomorphisms from X to Y, and let R and S be the relations corresponding to f and g, respectively. Then $R \cup S$ is the relation corresponding to the homomorphism $f * g$.*

Proof. Let θ be the θ-function for $H(X, Y)$. We show that $(f * g)\theta = R \cup S$.

Let $h = f * g$ and $(h)\theta = T$. By the definition of $*$, we have $(\hat{a})h = (\hat{a})f \cup (\hat{a})g$ for every $a \in A$. Note that $(f)\theta = R$ and $(g)\theta = S$. Then by the properties of a θ-function, we have $aT = aR \cup aS$ for every $a \in A$. Then $aT = a(R \cup S)$ for every $a \in A$ and hence $T = R \cup S$. Thus, we have $(f * g)\theta = R \cup S$. □

Let X and Y be pre-modules. The zero homomorphism H_0 is the homomorphism such that $(x)H_0 = 0_N$ for every $x \in X$, where 0_N is the zero element of Y.

Corollary 2.5. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Then the empty relation O is the relation corresponding to the zero homomorphism H_0 from X to Y.*

Proof. Let θ be the θ-function for $H(X, Y)$. We show that $(H_0)\theta = O$.

Let $(H_0)\theta = R$. Then by the properties of a θ-function, $(\hat{a})H_0 = aR$ for every $a \in A$. By the properties of H_0, we have $aR = \varnothing$ for every $a \in A$ and hence $R = O$. Thus $(H_0)\theta = O$. □

Corollary 2.6. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let θ be the θ-function for $H(X, Y)$. Then θ is a monoid isomorphism from $\langle H(X, Y), *\rangle$ to $\langle P(A \times B), \cup\rangle$.*

Proof. $\langle H(X, Y), *\rangle$ and $\langle P(A \times B), \cup\rangle$ are commutative monoids with unities H_0 and O, respectively.

Let $f, g \in H(X, Y)$, and let $(f)\theta = R$ and $(g)\theta = S$. Then, by

Proposition 2.4, we have $(f * g)\theta = R \cup S = (f)\theta \cup (g)\theta$. By Corollary 2.5, $(H_0)\theta = O$. A θ-function is bijective. Hence θ is a monoid isomorphism from $\langle H(X, Y), *\rangle$ to $\langle P(A \times B), \cup\rangle$. □

Corollary 2.7. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let $\langle P(A), \subseteq\rangle$ and $\langle P(B), \subseteq\rangle$ be partially ordered sets by set inclusion $\subseteq$. If f is a homomorphism from X and Y, then f is an order-preserving function from $\langle P(A), \subseteq\rangle$ to $\langle P(B), \subseteq\rangle$.*

Proof. Let $x_1, x_2 \in P(A)$ and $x_1 \subseteq x_2$. Then there is some $x_d \in P(A)$ such that $x_2 = x_1 \cup x_d$. Since f is a homomorphism, $(x_2)f = (x_1 \cup x_d)f = (x_1)f \cup (x_d)f$. This implies that $(x_1)f \subseteq (x_2)f$. Hence f is an order-preserving function from $\langle P(A), \subseteq\rangle$ to $\langle P(B), \subseteq\rangle$. □

Suppose that X, Y and Z are pre-modules. If f and g are homomorphisms from X to Y and Y to Z, respectively, then $f \circ g$ is a homomorphism from X to Z.

Proposition 2.8. *Let A, B and C be non-empty finite sets, and let $X = P(A, \cup)$, $Y = P(B, \cup)$ and $Z = P(C, \cup)$. Let f and g be homomorphisms from X to Y and Y to Z, respectively. Let R and S be the relations corresponding to f and g, respectively. Then $R \circ S$ is the relation corresponding to the homomorphism $f \circ g$ from X to Z.*

Proof. Let θ_1, θ_2 and θ be the θ-functions for $H(X, Y)$, $H(Y, Z)$ and $H(X, Z)$, respectively. We show that $(f \circ g)\theta = R \circ S$.

By the hypothesis, $(f)\theta_1 = R$ and $(g)\theta_2 = S$. Then by the properties of a θ-function, $(\hat{a})f = aR$ for every $a \in A$ and $(\hat{b})g = bS$ for every $b \in B$. On the other hand, let $h = f \circ g$ and $(h)\theta = T$. Then by the properties of a θ-function, $(\hat{a})h = aT$ for every $a \in A$.

From Proposition 2.1, it follows that $(\hat{a})h = (\hat{a})(f \circ g) = ((\hat{a})f)g = (aR)g = \bigcup\{(\hat{b})g \mid b \in aR\} = \bigcup\{bS \mid b \in aR\}$ for every $a \in A$. Then, by Proposition 1.3, we have $(\hat{a})h = a(R \circ S)$ for every $a \in A$. This implies that $aT = a(R \circ S)$ for every $a \in A$ and hence $T = R \circ S$. Thus we have $(f \circ g)\theta = R \circ S$. □

Let A be a non-empty finite set and let $X = P(A, \cup)$. Then i_A denotes the identity relation on A and i_X is the identity function on X.

Corollary 2.9. *Let A be a non-empty finite set and let $X = P(A, \cup)$. Then identity relation i_A on A is the relation corresponding to the isomorphism i_X on X.*

Proof. Let θ be the θ-function for $H(X, X)$. The identity function i_X is an isomorphisms on X. We show that $(i_X)\theta = i_A$.

Let $(i_X)\theta = R$, where $R \in P(A \times A)$. Then by the properties of a θ-function, $(\hat{a})i_X = aR$ for every $a \in A$. This implies that $\{a\} = aR$ for every $a \in A$ and hence $R = i_A$. Thus we have $(i_X)\theta = i_A$. □

Lemma 2.10. *Let A, B and C be non-empty finite sets, and let $X = P(A, \cup)$, $Y = P(B, \cup)$ and $Z = P(C, \cup)$. Let θ_1, θ_2 and θ be the θ-functions for $H(X, Y)$, $H(Y, Z)$ and $H(X, Z)$, respectively. Let $f \in H(X, Y)$ and $(f)\theta_1 = R$, and let $g \in H(Y, Z)$ and $(g)\theta_2 = S$, where $R \in P(A \times B)$ and $S \in P(B \times C)$. Then $(f \circ g)\theta = (f)\theta_1 \circ (g)\theta_2$ and $(R \circ S)\theta^{-1} = (R)\theta_1^{-1} \circ (S)\theta_2^{-1}$.*

Proof. By Proposition 2.8, $(f \circ g)\theta = R \circ S$. Then $(f \circ g)\theta = (f)\theta_1 \circ (g)\theta_2$ because by the hypothesis, $(f)\theta_1 = R$ and $(g)\theta_2 = S$. On the other hand, since a θ-function is bijective, $f = (R)\theta_1^{-1}$, $g = (S)\theta_2^{-1}$ and $f \circ g = (R \circ S)\theta^{-1}$. This implies that $(R \circ S)\theta^{-1} = (R)\theta_1^{-1} \circ (S)\theta_2^{-1}$. □

Lemma 2.11. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let $J \circ K = i_A$ and $K \circ J = i_B$, where $J \in P(A \times B)$ and $K \in P(B \times A)$. Let j and k be the homomorphisms corresponding to J and K, respectively. Then j is an isomorphism from X to Y and $k = j^{-1}$.*

Proof. Let θ_1 and θ_2 be the θ-functions for $H(X, Y)$ and $H(Y, X)$, respectively. Let θ_x be the θ-function for $H(X, X)$.

From the condition $J \circ K = i_A$ and θ_x, it follows that $(J \circ K)\theta_x^{-1} = (i_A)\theta_x^{-1}$. Then, by Corollary 2.9 and Lemma 2.10, we have $(J)\theta_1^{-1} \circ (K)\theta_2^{-1} = i_X$. This implies that $j \circ k = i_X$ because by the hypothesis, $j = (J)\theta_1^{-1}$ and $k = (K)\theta_2^{-1}$. Similarly, by the condition $K \circ J = i_B$, we have $k \circ j = i_Y$. Then from the properties of a function [3], it follows that j is a bijective function from X to Y and $k = j^{-1}$.

Thus j is an isomorphism from X to Y and $k = j^{-1}$. □

Lemma 2.12. *Let A and B be non-empty finite sets, and let $X = P(A, \cup)$ and $Y = P(B, \cup)$. Let t be an isomorphism from X to Y and let T be the relation corresponding to t. Then T is a bijective function from A to B and T^{-1} is the relation corresponding to the isomorphism t^{-1}.*

Proof. Let θ_1 and θ_2 be the θ-functions for $H(X, Y)$ and $H(Y, X)$, respectively. Let θ_x and θ_y be the θ-functions for $H(X, X)$ and $H(Y, Y)$, respectively.

First we show that the relation T is a function from A to B.

By the hypothesis, $(t)\theta_1 = T$. Then by the properties of a θ-function, $(\hat{a})t = aT$ for every $a \in A$. Clearly, $(\varnothing)t = \varnothing$. Then we have $(\hat{a})t \neq \varnothing$ for every $a \in A$ because t is bijective. This implies that $aT \neq \varnothing$ for every $a \in A$.

To prove that T is a function from A to B, we show that aT is a singleton for every $a \in A$.

Assume that $a'T$ is not a singleton for some $a' \in A$. Then there is some $y \in Y$ such that $y \subseteq a'T$ and $|y| = 1$. On the other hand, t^{-1} is an isomorphism from Y to X. Then by Corollary 2.7, we have $(y)t^{-1} \subseteq (a'T)t^{-1}$. This implies that $(y)t^{-1} \subseteq \{a'\}$ because $(\{a'\})t = a'T$. Then $(y)t^{-1} = \varnothing$ or $(y)t^{-1} = \{a'\}$.

If $(y)t^{-1} = \varnothing$, then $y = \varnothing$. This is a contradiction because by the assumption $|y| = 1$. If $(y)t^{-1} = \{a'\}$, then $y = (\{a'\})t = a'T$. This is also a contradiction because, by the assumption, $a'T$ is not a singleton and $|y| = 1$. Thus, we can say that aT is a singleton for every $a \in A$ and hence T is a function from A to B.

Next we show that the function T is bijective. Let K be the relation corresponding to the isomorphism t^{-1} from Y to X. Then by the result above, we can say that K is a function from B to A.

By the properties of a function, $t \circ t^{-1} = i_X$ and $t^{-1} \circ t = i_Y$. By considering a θ-function, $(t \circ t^{-1})\theta_x = (i_X)\theta_x$ and $(t^{-1} \circ t)\theta_y = (i_Y)\theta_y$. By Corollary 2.9 and Lemma 2.10, we have $(t)\theta_1 \circ (t^{-1})\theta_2 = i_A$ and $(t^{-1})\theta_2 \circ (t)\theta_1 = i_B$. Since $(t)\theta_1 = T$ and $(t^{-1})\theta_2 = K$, we have $T \circ K = i_A$ and $K \circ T = i_B$. Then by the properties of a function [3], T is a bijective function from A to B and $T^{-1} = K$.

Thus the proof is completed. □

Proposition 2.13. *Let A and B be non-empty finite sets. Let R be a relation from A to B and let S be a relation from B to A. Then "$R \circ S = i_A$ and $S \circ R = i_B$" if and only if "R is a bijective function from A to B and $S = R^{-1}$".*

Proof. Let p be the statement "$R \circ S = i_A$ and $S \circ R = i_B$". Let q be the statement "R is a bijective function from A to B and $S = R^{-1}$". Then, by considering Lemmas 2.11 and 2.12, we have $p \to q$. On the other hand, by the properties of a function, $q \to p$. Thus we have $p \leftrightarrow q$. □

In the following description, the notation $H(X, X)$ is abbreviated to $H(X)$.

Proposition 2.14. *Let A be a non-empty finite set and let $X = P(A, \bigcup)$. Let R_A be the set of all relations on A and let S_A be the set of all bijections on A. Let θ be the θ-function for $H(X)$. Then*

1. θ *is a pre-ring isomorphism from* $\langle H(X), *, \circ\rangle$ *to* $\langle R_A, \cup, \circ\rangle$.

2. *The symmetric group* $\langle S_A, \circ\rangle$ *is the unit group of the pre-ring* $\langle R_A, \cup, \circ\rangle$.

Proof. (1) The ordered triplet $\langle H(X), *, \circ\rangle$ is a pre-ring with additive unity H_0 and multiplicative unity i_X and $\langle R_A, \cup, \circ\rangle$ is also a pre-ring with additive unity O and multiplicative unity i_A [1], where H_0 is the zero homomorphism on X and O is empty relation.

Let $f, g \in \langle H(X), *, \circ\rangle$. Then, by Proposition 2.4 and Corollary 2.5, we have $(f * g)\theta = (f)\theta \cup (g)\theta$ and $(H_0)\theta = O$. By Corollary 2.9 and Lemma 2.10, we have $(f \circ g)\theta = (f)\theta \circ (g)\theta$ and $(i_X)\theta = i_A$. And θ is a bijective function from $H(X)$ to R_A.

Thus θ is a pre-ring isomorphism from $\langle H(X), *, \circ\rangle$ to $\langle R_A, \cup, \circ\rangle$.

(2) Let $\langle U, \circ\rangle$ be the unit group of the pre-ring $\langle R_A, \cup, \circ\rangle$. To prove $\langle U, \circ\rangle = \langle S_A, \circ\rangle$, we show that $U = S_A$.

Let $R \in U$ and let $\widetilde{R}$ be the multiplicative inverse of R. Then by the definition of U, we have $R \circ \widetilde{R} = \widetilde{R} \circ R = i_A$. By Proposition 2.13, the relation R is a bijection on A and hence $U \subseteq S_A$.

Conversely, let $f \in S_A$. Then $f^{-1} \in S_A$ because f is a bijective function on A. By the properties of a function, $f \circ f^{-1} = f^{-1} \circ f = i_A$. By the definition of U, $f \in U$ and hence $S_A \subseteq U$.

Thus we have $U = S_A$. □

Remarks. Suppose that A and B are non-empty finite sets with $|A| = m$ and $|B| = n$, where m and n are positive integers. Let $X = P(A, \cup)$ and $Y = P(B, \cup)$, and let $\hat{A} = \{\{a\} | a \in A\}$ and $\hat{B} = \{\{b\} | b \in B\}$ be ordered sets.

The sets $\hat{A}$ and $\hat{B}$ may be considered as ordered "bases" for the pre-modules X and Y, respectively. Then a homomorphism f from X to Y can be represented by some $m \times n$ $(0, 1)$-matrix over K_2. This suggests that the relation R from A to B corresponding to the homomorphism f can be identified with the same $m \times n$ $(0, 1)$-matrix over K_2.

On the other hand, in view of the converse, it can be said that any finite relation or $(0, 1)$-matrix induces a homomorphism between two appropriate pre-modules, such as $P(A, \cup)$ and $P(B, \cup)$.

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