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ALGEBRAS IN $MC_n(k)$

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Abstract

Let $(R, J(R), k)$ be a local commutative k-subalgebra of $M_n(k)$ with nilpotent maximal ideal $J(R)$ and residue class field k. In this paper, we classify maximal commutative k-subalgebras of $M_n(k)$ up to C_1-construction and C_2-construction according to $\dim_k(J(R)/\text{soc}(R))$.

1. Introduction

In this paper, k denotes an arbitrary field and $(R, J(R), k)$ denotes a local commutative k-subalgebra of $M_n(k)$ with nilpotent maximal ideal $J(R)$ and residue class field k. We denote the set of all local maximal commutative k-subalgebras of $M_n(k)$ by $MC_n(k)$.

Brown and Call introduced C_1-construction and Brown introduced C_2-construction [1, 2]. These constructions are useful to construct maximal commutative k-subalgebras of $M_n(k)$ having dimension less than the size n of matrices.

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In Section 3, we classify the k-algebra R in $MC_n(k)$ up to C_1-construction and C_2-construction according to dimension of $J(R)/\mathrm{soc}(R)$, where $\mathrm{soc}(R)$ is the socle of R.

2. Theorems Prerequisite to the Main Results

In this section, we will restate the definitions and some related properties of C_1-construction and C_2-construction.

Let $(B, J(B), k)$ be a finite dimensional commutative k-algebra with identity and N be a finitely generated faithful B-module. For a natural number ℓ, $R = B \oplus N^{\ell}$ is a commutative k-algebra and $M = B^{\ell} \oplus N$ is a faithful R-module via the following multiplications:

$$\alpha(b, n_1, \ldots, n_\ell) = (\alpha b, \alpha n_1, \ldots, \alpha n_\ell),$$

$$(b, n_1, \ldots, n_\ell)(b', n_1', \ldots, n_\ell) = (bb', n_1 b' + n_1' b, \ldots, n_\ell b' + n_\ell' b),$$

$$(b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell) = \left(b_1 b, \ldots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i\right),$$

where $\alpha \in k$, $b, b_i \in B$, and $n, n_i, n_i' \in N$ for $i = 1, 2, \ldots, \ell$.

Then $R \cong \mathrm{Hom}_R(M, M)$ via the regular representation. Thus R is in $MC_n(k)$, where $n = \dim_k(M)$.

Definition 2.1. The k-algebra R defined above is called a C_1*-construction*.

Let R be a commutative k-algebra. Then R is a C_1-construction if R has an ideal I satisfying the conditions in the following theorem. The proof can be found in [1].

Theorem 2.2. *Suppose* $(R, J(R), k)$ *is a commutative k-algebra. Then R is a C_1-construction if and only if there is an ideal I satisfying the following conditions*:

(1) $\mathrm{Ann}_R(I) = I$.

(2) $0 \to I \to R \to R/I \to 0$ *splits as k-algebras.*

Theorem 2.3. *Suppose* $(B, J(B), k)$ *is a finite dimensional commutative k-algebra with identity and N is a finitely generated faithful B-module. Suppose* $B \cong \text{Hom}_B(N, N)$ *via the regular representation. Then there exists an element* $w \in \text{soc}(B)$ *with* $\dim_k(Nw) = 1$.

Definition 2.4. Let $(B, J(B), k)$ be a finite dimensional commutative k-algebra with identity. If $R \cong B[X]/(J(B)X, X^p - w)$ for some $w \in \text{soc}(B) - \{0\}$ and a positive integer $p > 1$, then we say that the k-algebra R is a C_2*-construction*.

Theorem 2.5 is an equivalent condition for a k-algebra R to be a C_2-construction. The proof can be found in [3].

Theorem 2.5. *Suppose* $(R, J(R), k)$ *is a commutative k-algebra. Then R is a* C_2*-construction if and only if R contains a commutative k-subalgebra* $(B, J(B), k)$ *and an element* $x \in J(R)$ *satisfying the following conditions*:

(1) $0 \neq x^p \in \text{soc}(B)$ *for some positive integer* $p > 1$.

(2) $J(B)x = (0)$.

(3) $\dim_k(R) = \dim_k(B) + (p - 1)$.

3. Classifications

In this section, we will classify the algebra R in $MC_n(k)$ up to C_1-construction and C_2-construction according to $\dim_k(J(R)/\text{soc}(R))$. If $i(J(R))$, the index of nilpotency of $J(R)$, is two, then obviously R is a C_1-construction, but not a C_2-construction. Thus we will assume $i(J(R)) \geq 3$ in this section.

The following theorem can be found in [3].

Theorem 3.1. *Suppose* $(R, J(R), k) \in MC_n(k)$ *and* $\dim_k(J(R)/\text{soc}(R)) = 1$. *Then R is a* C_2*-construction but not a* C_1*-construction.*

Example 3.2. Let $R = k[E_{21} + E_{32}, E_{31}, E_{41}, E_{51}, E_{61}]$. Then the algebra R is in $MC_6(k)$ and $\mathrm{soc}(R) = (E_{31}, E_{41}, E_{51}, E_{61})$, the ideal generated by elements $E_{31}, E_{41}, E_{51}, E_{61}$. Thus $\dim_k(J(R)/\mathrm{soc}(R)) = 1$ and by Theorem 3.1, the algebra R is a C_2-construction but not a C_1-construction. In fact, if we let $B = k[E_{31}, E_{41}, E_{51}, E_{61}]$. Then $m_B = (E_{31}, E_{41}, E_{51}, E_{61})$. Since $J(B) = \mathrm{soc}(R)$, by letting $x = E_{21} + E_{32}$ and $p = 2$, the conditions in Theorem 2.4 are obviously satisfied.

The conditions for an algebra R to be a C_1-construction or C_2-construction is now naturally asked in the case of $\dim_k(J(R)/\mathrm{soc}(R)) > 1$.

Theorem 3.3. *Suppose* $(R, J(R), k) \in MC_n(k)$ *and* $\dim_k(J(R)/\mathrm{soc}(R)) = 2$. *Then* $r^2 = 0$ *for all* $r \in J(R)$ *if and only if R is a* C_1*-construction.*

Proof. Let $\dim_k(J(R)) = m$ and let $\mathrm{soc}(R)$ be generated by the elements $s_1, s_2, ..., s_{m-2}$, for some $s_i \in \mathrm{soc}(R)$, $i = 1, ..., m-2$. Then there exist elements $r_1, r_2 \in J(R) - \mathrm{soc}(R)$ such that the m vectors $r_1, r_2, s_1, \ldots, s_{m-2}$ generate $J(R)$. Let x and y be in k with $(x, y) \neq (0, 0)$ and let I be an ideal generated by $xr_1 + yr_2, s_1, \ldots, s_{m-2}$. Then by the hypothesis, we have $I^2 = (0)$ and so $I \subseteq \mathrm{Ann}_R(I)$. Note that $\mathrm{soc}(R) \subseteq I$. Now let $r \in \mathrm{Ann}_R(I) - \mathrm{soc}(R)$. Then for some $x_i \in k$, the element r is in the following form:

$$r = x_1r_1 + x_2r_2 + \sum_{i=3}^{m} x_is_{i-2},\ (x_1, x_2) \neq (0, 0).$$

Moreover

$$0 = r(xr_1 + yr_2) = (x_1r_1 + x_2r_2)(xr_1 + yr_2) = (x_1y + x_2x)r_1r_2.$$

If $r_1r_2 = 0$, then $J(R)^2 = (0)$ which is impossible since $i(J(R)) \geq 3$. Thus $r_1r_2 \neq 0$ and so $x_1y + x_2x = 0$. Since $(x, y) \neq (0, 0)$, we have either $x \neq 0$ or $y \neq 0$. If we assume $x \neq 0$, then

$$x_1 = (x_1x^{-1})x, \quad x_2 = -(x_1x^{-1})y.$$

If we assume $y \neq 0$, then

$$x_1 = (-x_2 y^{-1})x, \quad x_2 = -(-x_2 y^{-1})y.$$

This implies $x_1 r_1 + x_2 r_2 = t(xr_1 - yr_2)$ for some $t \in k$. Since $(r_1 + yr_2)^2 = 0$, we have $2yr_1 r_2 = 0$. But $r_1 r_2 \neq 0$ and hence $2y = 0$ which implies $y = -y$. Thus

$$x_1 r_1 + x_2 r_2 = t(xr_1 + yr_2)$$

which implies $r \in I$. Therefore I is an ideal of R satisfying $\operatorname{Ann}_R(I) = I$. Now, consider the following exact sequence:

$$0 \to I \to R \overset{\nu}{\to} R/I \to 0.$$

Here $\nu : R \to R/I$ is the natural homomorphism. In the element $xr_1 + yr_2 \in I$, we may assume $x \neq 0$. Since $xr_1 + yr_2 \in I$, we have

$$0 = \nu(xr_1 + yr_2) = x\nu(r_1) + y\nu(r_2).$$

Thus $\nu(r_1) = (-x^{-1}y)\nu(r_2)$. This implies $R/I = k[\nu(r_2)]$. Now define a map $\mu : R/I \to R$ by $\mu(\nu(r_2)) = r_2$, $\mu(\alpha) = \alpha$ for all $\alpha \in k$. Obviously, μ is a k-algebra homomorphism and

$$\nu\mu(\alpha\nu(r_2)) = \nu(\alpha r_2) = \alpha\nu(r_2).$$

Thus $\nu\mu$ is the identity homomorphism on R/I and hence the exact sequence splits as k-algebras. Therefore R is a C_1-construction.

Conversely, suppose R is a C_1-construction. Then there exists an ideal I satisfying $\operatorname{Ann}_R(I) = I$. If $s \in \operatorname{soc}(R)$, then $sI = (0)$ and hence $s \in \operatorname{Ann}_R(I) = I$. This implies $\operatorname{soc}(R) \subseteq I \subseteq J(R)$. Since $\dim_k(J(R)/\operatorname{soc}(R)) = 2$, there are following three cases:

Case 1. $\dim_k(I/\operatorname{soc}(R)) = 2$ and $\dim_k(J(R)/I) = 0$.

Case 2. $\dim_k(I/\operatorname{soc}(R)) = 0$ and $\dim_k(J(R)/I) = 2$.

Case 3. $\dim_k(I/\operatorname{soc}(R)) = 1$ and $\dim_k(J(R)/I) = 1$.

First of all, case 1 is impossible since $i(J(R)) \geq 3$. In case 2, $r \in J(R)$ implies $rI = r\operatorname{soc}(R) = (0)$ and hence $r \in \operatorname{Ann}_R(I) = I$. Thus $J(R) = I$ which is also impossible. Thus we have only the case 3. Let $r \in J(R) - I$. Since $\dim_k(J(R)/I) = 1$, we have $J(R) = I \oplus kr$ as k-vector spaces. Thus $r^2 = s + \alpha r$ for some $s \in I$ and $\alpha \in k$. If $\alpha \neq 0$, then $r - \alpha I_n$ is a unit and so $r = s(r - \alpha I_n)^{-1}$. But then $r \in I$ which is impossible. Thus $\alpha = 0$ and $r^2 = s \in I$. From the hypothesis, the following exact sequence splits as k-algebras via the k-algebra homomorphism $\mu : R/I \to R$,

$$0 \to I \to R \overset{\nu}{\to} R/I \to 0.$$

Here $\nu : R \to R/I$ is the natural homomorphism. Note $\mu(r + I) = r + r_1$ for some $r_1 \in I$. Moreover, we have

$$0 = \mu(r + I)^2 = r^2 + r_1^2 + 2rr_1 = r(r + 2r_1).$$

Here $r + 2r_1 \in J(R)$ but $r \in 2r_1 \notin I$. Since $\dim_k(J(R)/I) = 1$, we have $r + 2r_1 = \beta r$ for some nonzero β in k. Thus

$$\beta r^2 = r(r + 2r_1) = 0.$$

Since $\beta \neq 0$, we have $r^2 = 0$. Moreover, $I^2 = (0)$ implies $r^2 = 0$ for all $r \in J(R)$.

Example 3.4. Suppose k is a field of characteristic two. Let $R = k[E_{21} + E_{43}, E_{31} + E_{42}, E_{41}]$. Then the algebra R is in $MC_4(k)$. Moreover, we have $\dim_k(J(R)/\operatorname{soc}(R)) = 2$, $i(J(R)) = 3$, and $r^2 = 0$ for all $r \in J(R)$. Thus R is a C_1-construction by Theorem 3.3.

Example 3.5. Let $R = k[E_{21} + E_{32} + E_{43}, E_{31} + E_{42}, E_{41}, E_{51}]$. Then R is in $MC_5(k)$, $\dim_k(J(R)/\operatorname{soc}(R)) = 2$, and $i(J(R)) = 4$. If we let $r = a(E_{21} + E_{32} + E_{43})$ for some $a \neq 0 \in k$, then $r^2 \neq 0$. Thus the algebra R is not a C_1-construction by Theorem 3.3.

Theorem 3.6. *Suppose* $(R, J(R), k) \in MC_n(k)$ *and* $\dim_k(J(R)/\mathrm{soc}(R)) = t$ *for some positive integer t. If there exists an element* $r \in J(R) - \mathrm{soc}(R)$ *such that* $r^{t+1} = 0$ *and* $r^t \neq 0$, *then R is a* C_2*-construction.*

Proof. Since $\dim_k(J(R)/\mathrm{soc}(R)) = t$, the maximal ideal $J(R)$ can be expressed as follows:

$$J(R) = \mathrm{soc}(R) \oplus kr \oplus ks_1 \oplus \cdots \oplus ks_{t-1}$$

for some $s_i \in J(R)$ as k-vector spaces. Since $r^{t+1} = 0$ and $r^t \neq 0$, the following elements are all distinct:

$$r, \alpha_1 r^2, \alpha_2 r^3, \ldots, \alpha_{t-1} r^t \quad (\alpha_i \neq 0 \in k).$$

Now, let $B = k[\mathrm{soc}(R) \oplus kr^t]$. Then

$$J(B) = \mathrm{soc}(R) \oplus kr^t.$$

Since $r^t \mathrm{soc}(R) = (0)$ and $r^t r^t = 0$, we have $r^t J(B) = (0)$ which implies

$$r^t \in \mathrm{soc}(B) - \{0\}.$$

Moreover

$$J(B) r = (0).$$

By the definition of B,

$$\dim_k(R) = \dim_k(B) + (t - 1).$$

Thus by letting $x = r$ and $p = t$, the algebra R satisfies the three conditions in Theorem 2.5 and hence R is a C_2-construction.

Example 3.7. Let $R = k[E_{21} + E_{32}]$. Then the algebra R is a k-subalgebra in $MC_3(k)$. Moreover, $\dim_k(J(R)/\mathrm{soc}(R)) = 1$. If we let $r = E_{31}$, then $r^2 = 0$ and $r \neq 0$. Therefore, if we let $t = 1$, then by Theorem 3.6, the algebra R is a C_2-construction.

Theorem 3.8. *Suppose* $(R, J(R), k) \in MC_n(k)$ *and* $\dim_k(J(R)/\mathrm{soc}(R)) = t$ *for some positive integer t. If there exists an element* $r \in J(R) - \mathrm{soc}(R)$ *such that* $r^{t+2} = 0$ *and* $r^{t+1} \neq 0$, *then R is a* C_2*-construction.*

Proof. Since $\dim_k(J(R)/\text{soc}(R)) = t,$ the maximal ideal $J(R)$ can be expressed as follows:

$$J(R) = \text{soc}(R) \oplus kr \oplus ks_1 \oplus \cdots \oplus ks_{t-1}$$

for some $s_i \in J(R)$ as k-vector spaces. Let $B = k[\text{soc}(R)].$ Then

$$J(B) = \text{soc}(R) = \text{soc}(B).$$

Since $r^{t+1} \in J(R),$ $r^{t+1}\text{soc}(R) = (0)$ and hence $r^{t+1}J(B) = (0).$ Thus $r^{t+1} \in \text{soc}(B) - \{0\}.$ Obviously, we have

$$J(B)r = \text{soc}(R)r = (0).$$

Since $B = k[\text{soc}(R)],$ we have

$$\dim_k(R) = \dim_k(soc(R)) + 1 + (t-1) + 1 = \dim_k(B) + t.$$

Thus, by letting $x = r$ and $p = t + 1,$ the algebra R satisfies the three conditions in Theorem 2.5 and we conclude R is a C_2-construction.

Example 3.9. Let $R = k[E_{21} + E_{32}, E_{31}, E_{41}].$ Then the algebra R is a k-subalgebra in $MC_4(k).$ Moreover, $\dim_k(J(R)/\text{soc}(R)) = 1.$ If we let $r = E_{21} + E_{32},$ then $r^2 = E_{31} \neq 0$ and $r^3 = 0.$ Therefore, if we let $t = 1,$ then by Theorem 3.8, the algebra R is a C_2-construction.

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