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MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIALS AND BLOCKS

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Abstract

Let F be a field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \in F^{n \times n},$$

where the blocks $C_{i,j}$ are of type $p_i \times p_j$, $i, j \in \{1, 2, 3\}$ and $C_{1,1}$, $C_{2,2}$ and $C_{3,3}$ are square submatrices. In this paper we establish conditions for which it is possible to prescribe arbitrarily the characteristic polynomial of C, when $C_{1,1}$, $C_{2,2}$ and $C_{2,3}$ are fixed and the remaining blocks vary.

1. Introduction

The problem studied in this paper is inserted in the Matrix Completion Problems. A particular case of this type of problems was proposed by G. N. Oliveira in 1975 [6].

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Let F be a field. Then we denote by $F^{p \times q}$ the set of all matrices of type $p \times q$, with entries in F.

Problem [6]**.** Let n, p, q be positive integers such that $n = p + q$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in F^{n \times n}, \tag{1}$$

be a partitioned matrix, where $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$. Suppose that some of the blocks $A_{i,j}$, $i, j \in \{1, 2\}$, are known. Under which conditions does there exist a matrix of the form (1) with characteristic polynomial $f(x)$?

Many authors have studied this problem, see for example [5, 6, 7, 9, 10, 11, 14]. It is important to emphasize that the problem is completely solved for some prescription of blocks, however there are cases for which there are only partial solutions. Concerning the prescription of $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$, there is no answer.

Our aim is to generalize the previous problem to a matrix partitioned into 3×3 blocks. Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$ and let $C_{i,j} \in F^{p_i \times p_j}$, $i, j \in \{1, 2, 3\}$. We start by studying the possible characteristic polynomials of a matrix of the form

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \in F^{n \times n}, \tag{2}$$

when some of the blocks $C_{i,j}$ are prescribed and the others vary, over an algebraically closed field. This problem is equivalent to describe the possible eigenvalues of (2), for the same prescription of blocks. Obviously we obtain several problems, according to the prescription of some blocks of C.

Recently [2], we studied the possible eigenvalues of (2), when $C_{1,2}$, $C_{1,3}$ and $C_{2,1}$ are prescribed and the remaining blocks vary. In [1] we also described the list of the eigenvalues of (2), when $C_{1,1}$, $C_{1,2}$ and $C_{3,3}$ are prescribed and the other blocks vary.

In this paper we had the purpose to study the possible eigenvalues of (2), when $C_{1,1}$, $C_{2,2}$ and $C_{2,3}$ are prescribed and the remaining blocks vary. The approach used, allows us to solve the more general problem, of describing the possible characteristic polynomials of (2), for the same prescription of blocks (i.e., $C_{1,1}$, $C_{2,2}$ and $C_{2,3}$ are prescribed and the other blocks vary). Note that this situation is more general because still covers the case where the eigenvalues are outside of the field F.

2. Preliminaries

Let F be a field.

Let $D = F$ or $D = F[x]$ and let m, n be positive integers. We denote by $D^{m \times n}$ the set of all matrices in D of type $m \times n$.

The symbol $|$ is used in the following way: if $f(x)$, $g(x) \in F[x]$, then $f(x) | g(x)$ means "$f(x)$ divides $g(x)$".

Let R be the set of all monic polynomials in $F[x]$ and the zero polynomial.

Definition 1 [3]**.** Let $A(x) \in F[x]^{m \times n}$. Then the greatest common divisor chosen in R, of the determinants of the submatrices of size $k \times k$ of $A(x)$, $k \in \{1, ..., \min\{m, n\}\}$ is denoted by $d_k(x)$. If $k \leq \operatorname{rank} A(x)$, then we say that $d_k(x)$ is the k-th *determinantal divisor* of $A(x)$. Make convention that $d_0(x) = 1$.

It is known [3] that if $A(x) \in F[x]^{m \times n}$ and rank $A(x) = r$, then

(i) $d_k(x) \neq 0$ if and only if $k \leq r$;

(ii) $d_{k-1}(x) | d_k(x)$, $k \in \{1, ..., r\}$.

Definition 2 [3]**.** The *k-th invariant factor* of $A(x)$ is the element

$$i_k(x) = \frac{d_k(x)}{d_{k-1}(x)}, \quad k \in \{1, \ldots, \text{rank } A(x)\},$$

with the convention that $i_0(x) = 1$.

Note that according to the previous definitions, the determinantal divisors and the invariant factors of the matrix $A(x)$ are monic polynomials.

It is known [3] that $i_{k-1}(x) \mid i_k(x)$, $k \in \{1, \ldots, r\}$.

Since $F[x]$ is a unique factorization domain, if $i_1(x), \ldots, i_r(x)$ are the invariant factors of $A(x) \in F[x]^{m \times n}$, then every $i_k(x)$, $k \in \{1, \ldots, r\}$, can be factored as the following product:

$$i_k(x) = p_1^{n_{k,1}}(x) \cdots p_t^{n_{k,t}}(x),$$

where $p_j(x)$, $j \in \{1, \ldots, t\}$, are distinct irreducible monic polynomials over F, and $n_{k,j} \geq 0$, $k \in \{1, \ldots, r\}$, $j \in \{1, \ldots, t\}$.

The polynomials

$$p_j^{n_{k,j}}(x), \quad k \in \{1, \ldots, r\}, \quad j \in \{1, \ldots, t\},$$

for which $n_{k,j} > 0$, are called the *infinite elementary divisors* of $A(x)$ in F [3].

It is known [3] that if $A(x) = A_1 x + A_2 \in F[x]^{m \times n}$, then $A(x)$ has

$$\text{rank } A(x) - \text{rank } A_1$$

infinite elementary divisors.

Let $A \in F^{m \times m}$. Then the polynomial matrix $xI_m - A$ is called the *characteristic matrix* of A and its determinant is called the *characteristic polynomial* of A [3].

The invariant factors of $xI_m - A$ are called the *invariant polynomials* of A [3].

Note that the matrix $xI_m - A$ has rank m, since its determinant is different from zero. Consequently A has m invariant polynomials,

$$f_1(x)|\cdots|f_m(x).$$

It is also known [3] that the characteristic polynomial of a matrix $A \in F^{m\times m}$ it is equal to the product of its invariant polynomials.

It is known from systems theory [4] that a pair (A, B), where $A \in F^{p\times p}$, $B \in F^{p\times q}$ is *completely controllable* if and only if all the invariant factors of the matrix pencil

$$[xI_p - A \quad -B]$$

are equal to 1.

3. Main Result

Throughout this section, F denotes an arbitrary field.

Theorem 3. *Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1\times p_1}$, $C_{2,2} \in F^{p_2\times p_2}$ and $C_{2,3} \in F^{p_2\times p_3}$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Then apart from the exception listed below, there exist $C_{1,2} \in F^{p_1\times p_2}$, $C_{1,3} \in F^{p_1\times p_3}$, $C_{2,1} \in F^{p_2\times p_1}$, $C_{3,1} \in F^{p_3\times p_1}$, $C_{3,2} \in F^{p_3\times p_2}$ and $C_{3,3} \in F^{p_3\times p_3}$ such that the matrix of the form* (2), *has characteristic polynomial $f(x)$. The exception is the following*:

(E) *There exist prescribed submatrices of* (2) *of type $q_1 \times q_2$, with at least one prescribed principal block, such that $n < q_1 + q_2$.*

Theorem 4 [8, 13]**.** *Let $l_1(x), \ldots, l_s(x) \in F[x]$ be monic polynomials such that $l_1(x)|\cdots|l_s(x)$. Let $A(x) \in F[x]^{p\times q}$ and let $i_1(x), \ldots, i_r(x)$ be the invariant factors of $A(x)$. Then, there exist $B(x) \in F[x]^{p\times(n-q)}$, $C(x) \in$*

$F[x]^{(m-p)\times q}$, $D(x) \in F[x]^{(m-p)\times(n-q)}$ *such that*

$$\begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} \in F[x]^{m\times n}$$

has invariant factors $l_1(x), \ldots, l_s(x)$ *if and only if the following conditions are satisfied*:

(a) $r \leq s \leq r + (m-p) + (n-q)$;

(b) $s \leq \min\{m, n\}$;

(c) $l_k \mid i_k$, *for every* $k \in \{1, \ldots, r\}$;

(d) $i_k \mid l_{k+(m-p)+(n-q)}$, *for every* $k \in \{1, \ldots, r\}$ *such that* $k + (m-p) + (n-q) \leq s$.

Theorem 5 [14]. *Let n, p, q be positive integers such that* $n = p + q$. *Let* $A_{1,1} \in F^{p\times p}$, $A_{1,2} \in F^{p\times q}$ *and let* $f(x) \in F[x]$ *be a monic polynomial of degree n. Let* $f_1(x) | \cdots | f_p(x)$ *be the invariant factors of*

$$[xI_p - A_{1,1} \quad -A_{1,2}].$$

There exist $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ *such that the matrix of the form* (1) *has characteristic polynomial* $f(x)$ *if and only if*

$$f_1(x) \cdots f_p(x) \mid f(x). \tag{3}$$

Lemma 6 [12]. *Suppose that* $C_{i,j} \in F^{p_i \times p_j}$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. *The following conditions are equivalent*:

(a) *There exists a* $C_{2,1} \in F^{p_2 \times p_1}$ *such that*

$$\left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right) \tag{4}$$

is completely controllable.

(b) *The following conditions are satisfied*:

(i) *The pair*

$$(C_{1,1}, [C_{1,2} \quad C_{1,3}]) \tag{5}$$

is completely controllable.

(ii) *The matrix pencil*

$$A(x) = \begin{bmatrix} -C_{1,2} & -C_{1,3} \\ xI_{p_2} - C_{2,2} & -C_{2,3} \end{bmatrix} \tag{6}$$

has, *at least*, *one infinite elementary divisor and the number of its infinite elementary divisors is greater than or equal to the number of its invariant factors different from* (1).

Lemma 7. *If the exception condition* (E) *holds*, *then there exists no matrix of the form* (2) *with arbitrary characteristic polynomial* $f(x)$.

Proof. Suppose that the condition (E) holds and assume that there exist $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$ and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (2) has characteristic polynomial $f(x)$. Since the prescribed submatrices of (2) with principal prescribed blocks are $C_{1,1}$, $C_{2,2}$ and $[C_{2,2} \quad C_{2,3}]$, at least one of the following conditions must occur: $2p_1 > n$, $2p_2 > n$ or $2p_2 + p_3 > n$. Clearly, if $2p_2 > n$, then necessarily $2p_2 + p_3 > n$.

Let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant polynomials of $C_{1,1}$, $\beta_1 | \cdots | \beta_{p_2}$ be the invariant polynomials of $C_{2,2}$, $\gamma_1 | \cdots | \gamma_{p_2}$ be the invariant factors of

$$[xI_{p_2} - C_{2,2} \quad -C_{2,3}]$$

and $\mu_1 | \cdots | \mu_n$ be the invariant polynomials of (2).

Case 1. Suppose that $2p_1 > n$. Then, $p_1 > p_2 + p_3$. According to Theorem 4, it follows that

$$\alpha_i \,|\, \mu_{i+2p_2+2p_3}, \quad i \leq p_1 - p_2 - p_3.$$

Hence,

$$\alpha_1 \cdots \alpha_{p_1 - p_2 - p_3} \mid \mu_{1+2p_2+2p_3} \cdots \mu_n \mid \mu_1 \cdots \mu_n = f(x).$$

Consequently, the roots of $\alpha_1, \ldots, \alpha_{p_1 - p_2 - p_3}$ must be roots of $f(x)$, so the prescription of $f(x)$ is not arbitrary.

Case 2. Suppose that $2p_2 > n$. Then, $p_2 > p_1 + p_3$. By Theorem 4, it follows that

$$\beta_i \mid \mu_{i+2p_1+2p_3}, \quad i \leq p_2 - p_1 - p_3.$$

Therefore,

$$\beta_1 \cdots \beta_{p_2 - p_1 - p_3} \mid \mu_{1+2p_1+2p_3} \cdots \mu_n \mid \mu_1 \cdots \mu_n = f(x).$$

Consequently, the roots of $\beta_1, \ldots, \beta_{p_2 - p_1 - p_3}$ must be roots of $f(x)$. Again the prescription of $f(x)$ is not arbitrary.

Case 3. Suppose that $2p_2 + p_3 > n$. Then, $p_2 > p_1$. According to Theorem 4, it follows that

$$\gamma_i \mid \mu_{i+2p_1+p_3}, \quad i \leq p_2 - p_1.$$

Then,

$$\gamma_1 \cdots \gamma_{p_2 - p_1} \mid \mu_{1+2p_1+p_3} \cdots \mu_n \mid \mu_1 \cdots \mu_n = f(x).$$

Consequently, the roots of $\gamma_1, \ldots, \gamma_{p_2 - p_1}$ must be roots of $f(x)$. Once again the prescription of $f(x)$ is not arbitrary.

Proof of Theorem 3. Suppose that the exception (E) is not satisfied. According to the hypothesis, it follows that $2p_1 \leq n$, $2p_2 \leq n$ and $2p_2 + p_3 \leq n$. Consequently, $p_1 \leq p_2 + p_3$, $p_2 \leq p_1 + p_3$ and $p_2 \leq p_1$.

Case 1. Suppose that $p_2 < p_1$. Let

$$C_{1,2} = \begin{bmatrix} I_{p_2} \\ 0 \end{bmatrix} \in F^{p_1 \times p_2} \text{ and } C_{1,3} = \begin{bmatrix} 0 & 0 \\ I_{p_1 - p_2} & 0 \end{bmatrix} \in F^{p_1 \times p_3}.$$

The pair of the form (5) is completely controllable. Let $A(x)$ be the matrix pencil of the form (6). Clearly, $A(x)$ has rank greater than or equal to p_1. Hence $A(x)$ has, at least, $p_1 - p_2 > 0$ infinite elementary divisors. It is clear that the matrix pencil $A(x)$ has all its invariant factors equal to 1. Then according to Lemma 6, there exists a $C_{2,1} \in F^{p_2 \times p_1}$ such that the pair of the form (4) is completely controllable. By Theorem 5, we can state that there exist $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$ and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (2) has characteristic polynomial $f(x)$. Clearly, the matrix C has the prescribed form.

Case 2. Suppose that $p_1 = p_2$. Let

$$C_{1,2} = \begin{bmatrix} I_{p_1-1} & 0 \\ 0 & 0 \end{bmatrix} \in F^{p_1 \times p_1} \text{ and } C_{1,3} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in F^{p_1 \times p_3}.$$

Then the pair of the form (5) is completely controllable. Consider again the matrix pencil $A(x)$ of the form (6). Clearly, $A(x)$ has rank greater than or equal to $p_1 + 1$. Then $A(x)$ has, at least, one infinite elementary divisor. On the other hand, the number of nonconstant invariant factors of $A(x)$ is less than or equal to 1. Hence, according to Lemma 6 there exists a $C_{2,1} \in F^{p_2 \times p_1}$ such that the pair of the form (4) is completely controllable. Now applying Theorem 5, there exist $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$ and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (2) has characteristic polynomial $f(x)$. Clearly, the matrix C has the prescribed form.

Corollary 8. *Let* n, p_1, p_2, p_3 *be positive integers such that* $n = p_1 + p_2 + p_3$. *Let* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$ *and* $C_{2,3} \in F^{p_2 \times p_3}$. *Let* $c_1, \ldots, c_n \in F$. *Then apart from the exception* (E), *there exist* $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$ *and* $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (2), *has eigenvalues* $c_1, \ldots, c_n$.

Proof. Straightforward.

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