Adv. & Appl. in Stat. **6**(1) (2006), 111-120

RANDOM VOLATILITY AND OPTION PRICES WITH THE GENERALIZED STUDENT-*t* DISTRIBUTION

MARK FIELDING, FIMA C. KLEBANER* **and ZINOVIY LANDSMAN**

Department of Statistical Science
LaTrobe University, Australia

School of Mathematical Sciences
Bld 28M, Monash University
Clayton, Victoria 3800, Australia

Department of Statistics
University of Haifa, Haifa, Israel, 31905

Abstract

In finance the stock volatility is used for pricing of options and risk management. In this paper we propose the model of random volatility, and study its distribution by using volatility implied by options on stock. For three companies we examine on the Australian Stock Exchange, it is shown to follow the generalized Student-*t* distribution. Option prices, along with a smile curve, according to this model are shown to be close to the observed market values.

2000 Mathematics Subject Classification: 62P05, 62P20, 91B30, 91B70.

Keywords and phrases: implied volatility, Bayesian statistics, Black-Scholes, smile effect.

*Research supported by the Australian Research Council.

Received November 3, 2005

Introduction

In finance the concept of volatility is important for managing risk. In the classical Black-Scholes model the volatility of a stock is assumed to be constant, and their formula (1) gives the option price on this stock. For pricing options all parameters are readily available, except the stock volatility σ, which is defined as the standard deviation of returns.

When the market option prices are equated to their Black-Scholes values, the volatility parameter can be solved for, giving rise to the notion of implied volatility. This quantity is important for financial applications and modeling. It is seen and documented in the literature that the volatility parameter varies across option's maturities and strikes. This effect is known as the "smile-effect", discussed by many authors (see e.g., Hull and White [4], Hull [3, Ch. 17]). Here we concentrate on the smile curve as a graph of the volatility implied by the Black-Scholes formula against the strike price. To explain the smile-effect, volatility is often taken to be stochastic and is modelled by a stochastic process, such as Heston's stochastic volatility model. In this paper we examine empirically a much simpler model, in which the volatility is a random variable with the some unknown to us distribution. To obtain an idea of its distribution we consider the multitude of implied volatilities and fit a distribution.

Taylor and Xu [7] considered σ^2 as a random variable and used an approximation to the Black-Scholes formula by a quadratic polynomial with respect to σ^2. They obtained a theoretical equation for the smile curve. They point out that the empirical smiles differ from the theoretical ones. Taylor and Xu's approach takes into consideration only the first two moments (mean and the variance) of σ^2, which often give insufficient information about the underlying distribution. Moreover, the expansion they use is a local one, whereas the prices of options are used by integration over the whole line, which may be a source for the inconsistency. In our approach we model the distribution of the implied volatility directly.

In the next section we give a mathematical result showing that the implied volatility is properly defined by inverting the Black-Scholes formula for any observed market price of an option.

We then present empirical findings by analyzing data on three selected companies. After trying a number of standard families we find that the distribution of implied volatility may follow the so-called generalized Student-t. We then calculate option prices implied by this distribution of σ by averaging Black-Scholes prices with respect to the generalized Student-t density. Finally we give the modeling smile curve, which is compared with the market smile curve. Option prices, along with a smile curve, according to this model are shown to be close to the observed market values.

Distribution of Volatility

Implied volatility

The implied volatility σ is the solution of the equation

$$c = S\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2), \tag{1}$$

where t is the current time, $T-t$ is the time to expiration, r is interest rate, $S(t)$ is market price of the stock at time t, K is the strike price of the option, c is the option's market price, and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

First we show that a solution for σ always exists and is nonnegative.

Theorem 1. *For a given t, T, S and K and any market price of option, the implied volatility, being the solution of equation* (1), *exists, is unique and nonnegative.*

Proof. Equation (1) can be rewritten as follows:

$$c = S\Phi\left(\frac{A}{\sigma} + B\sigma\right) - K\exp(-r(T-t))\Phi\left(\frac{A}{\sigma} - B\sigma\right),$$

where

$$A = \frac{\ln(S/K)}{\sqrt{T-t}} + r\sqrt{T-t},$$

$$B = 1/2\sqrt{T-t}. \tag{2}$$

One can get that

$$\frac{\partial c}{\partial \sigma} = S\sqrt{T-t}\,\Phi'(d_1) > 0, \quad \sigma \in (-\infty, 0) \cup (0, \infty)$$

(see, e.g., Cox and Rubinstein [2]). Then price c as a function of σ is monotone increasing on $(-\infty, 0)$ and on $(0, \infty)$, which guarantees the existence and uniqueness of the solution to equation (1). If $S < K\exp(-r(T-t))$, then from (2), $A < 0$, and

$$\lim_{\sigma \to +0} c = S\Phi(-\infty) - K\exp(-r(T-t))\Phi(-\infty) = 0,$$

$$\lim_{\sigma \to -0} c = S\Phi(\infty) - K\exp(-r(T-t))\Phi(\infty) = S - K\exp(-r(T-t)) < 0.$$

Figure 1. Smoothed histogram of TNT96 implied volatilities.

As c increases with σ, for any negative σ, c is also negative. If $S \geq K\exp(-r(T-t))$, then constant $A \geq 0$, and

$$\lim_{\sigma \to +0} c = S\Phi(\infty) - K\exp(-r(T-t))\Phi(\infty) = S - K\exp(-r(T-t)) \geq 0,$$

$$\lim_{\sigma \to -0} c = S\Phi(-\infty) - K \exp(-r(T-t))\Phi(-\infty) = 0. \qquad (3)$$

From (3), and the fact that c is monotone increasing with σ, it follows that $c < 0$ when $\sigma < 0$.

Distribution of implied volatility

We analyze annual records of options, from 1996 to 1998, including 1097 option prices of TNT in 1996 (TNT96), 5298 option prices of ANZ in 1996 (ANZ96), 8402 option prices of NAB in 1998 (NAB98), among others. The primary source database are option prices obtained from ASX of Australian banks (ANZ, NAB) and the TNT company.

In Figures 1-3, we give the plots of empirical densities (smoothed by an S-plus procedure) of implied volatilities computed from these data. The data exhibits heavier than Normal tails and excess kurtosis. These features prompt using the *generalized Student-t family* (GSTF) discussed in Bian and Tiku [1], Landsman and Makov [5] (see also McDonald and Newey [6]).

Figure 2. Smoothed histogram of ANZ96 implied volatilities.

Figure 3. Smoothed histogram of NAB98 implied volatilities.

The density functions of GSTF, have three parameters, μ, δ, p, as follows:

$$f(x \mid \mu, \delta, p) = c_p\left(1 + \frac{(x-\mu)^2}{(2p-3)\delta^2}\right)^{-p}, \quad x \in (-\infty, \infty).$$

Here μ is the location parameter, equal to the expectation of the random value, distributed GSTF, δ^2 is the variance, $p \geq 2$ is the shape parameter, characterizing a departure from the Normal distribution, and

$$c_p = \delta\sqrt{2p-3}\,\beta(0.5, p-0.2),$$

where $\beta(\alpha, \beta)$ is the beta function with parameters α and β. When $p = \infty$, GSTF is distributed Normal with mean μ and variance δ^2. The kurtosis of members of GSTF is given by

$$\gamma_2(p) = \frac{6}{2p-5}.$$

Varying the parameter p, one can tune the kurtosis of modeling the distribution in accordance with the data.

As σ cannot take negative values, we slightly modify the GSTF by truncating it at the point $\sigma = 0$. In other words, if X is distributed GSTF, then $X' = \max(0, X)$ is distributed GSTFT. We use the latter to fit our data. The fitting is done by choosing values for μ, σ and p that minimize the Cramer-von Mise's distance,

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(\sigma_{(i)}) - \frac{2i-1}{2n} \right)^2,$$

where $\sigma_{(1)} \le \sigma_{(2)} \le \cdots \le \sigma_{(n)}$ are the order statistics constructed by the corresponding data of implied volatility. In Table 1, the appropriate parameters for data sets considered in Figures 1-3 are given. Let us notice that for TNT96 and ANZ96, the fitted W^2 does not exceed the 95% confidence level, equal to 0.461, and for NAB98, the fitted W^2 does not exceed the 99% confidence level, equal to 0.743. In Figure 4, q-q plot is given, suggesting a reasonable fit.

Table 1. Parameter settings in fitting generalized Student-t

Data set	μ	δ	p	$W^2_{\min}$
TNT96	0.343	0.161	2.00	0.448
ANZ96	0.220	0.088	2.00	0.348
NAB98	0.249	0.103	2.05	0.648

Figure 4. Generalized Student-t q-q plot for TNT96 implied volatilities.

Modeling Prices and Smile

In this section we calculate the prices for given t, T, S and K by averaging Black-Scholes prices $c(t, T, S, K, \sigma)$ (see equation (1)) with respect to the model GSTFT density,

$$\bar{c}(t, T, S, K) = \int_0^\infty c(t, T, S, K, \sigma) f(\sigma | \mu, \delta, p) d\sigma,$$

with appropriate parameters μ, δ, p. Now, having the model distribution, we calculate $\bar{c}$ by simulating generalized t distributed random variables. In Figures 5-7, we give the graph of points having x-coordinate the market value prices, and y-coordinate the model value of price $\bar{c}$, for the same values of t, T, S, K. From the graph we can see that the model value of prices are close to market value of prices.

Now using the simulated prices $\bar{c}$ we can find the modeling implied volatility $\bar{\sigma}(T - t, S, K)$ by solving the equation

$$\bar{c} = S\Phi(d_1) - K \exp(-r(T - t))\Phi(d_2),$$

with respect to σ. Fixing t and T we can find a curve $\bar{\sigma} = \bar{\sigma}(K/S)$, which represents the modeling smile curve. At the same time, the solution $\sigma = \sigma(K/S)$ of equation (1) gives us the market smile curve. In Figure 8, we give plots of market and modeling smile curves for ANZ96 and NAB98.

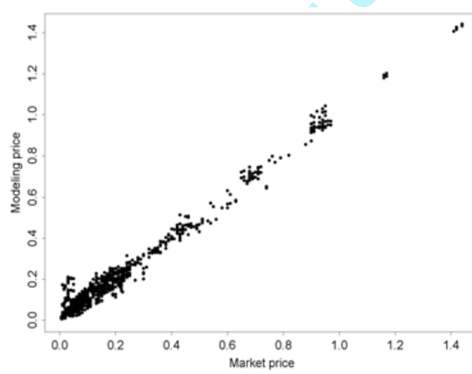

Figure 5. Modeling Black-Scholes price versus market price for TNT96 options.

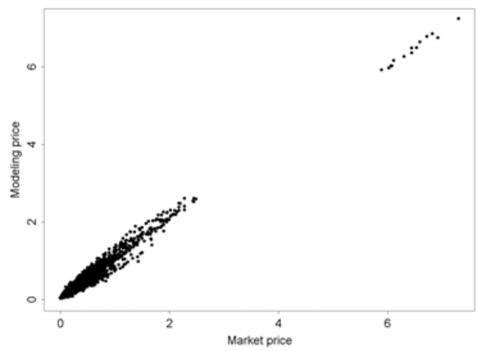

Figure 6. Modeling Black-Scholes price versus market price for ANZ96 options.

Conclusion

The paper presents an empirical study designed to find a possible distribution of stock's volatility when the latter is taken to be a random variable. To this end various distributions were fitted into the values of volatilities implied by options prices, and found that the generalized t-distribution is a possible model for use in practice. Checked on three companies on the Australian Stock Exchange, it was found that this model returns prices similar to the market prices of options, and explains to a degree the smile effect. Therefore, this work suggests using generalized t-distribution of volatility for options pricing.

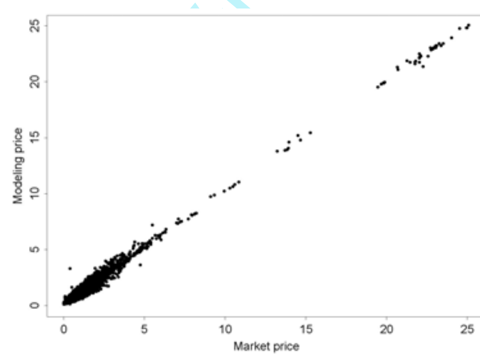

Figure 7. Modeling Black-Scholes price versus market price for NAB98 options.

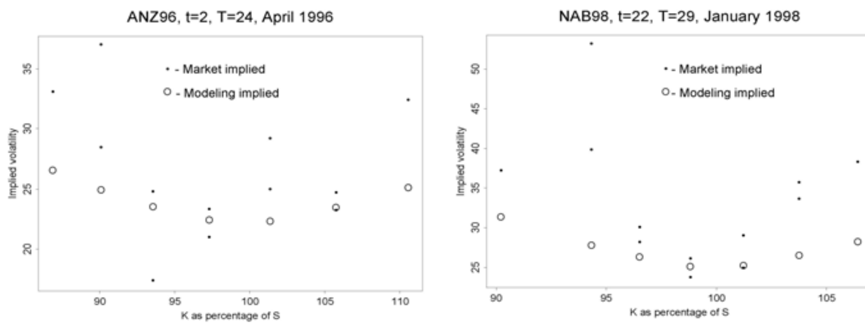

Figure 8. Modeling and market smile curves for ANZ96 and NAB98 options.

References

[1] G. Bian and M. L. Tiku, Bayesian inference based on robust priors and MML estimators: Part I, symmetric location-scale distributions, Statistics 29(4) (1997), 317-345.

[2] J. Cox and M. Rubinstein, Options Markets, Prentice-Hall, New York, 1987.

[3] J. C. Hull, Options, Futures and Other Derivative Securities, Prentice-Hall, New York, 1993.

[4] J. C. Hull and A. White, The pricing of options on assets with stochastic volatilities, J. Finance 42 (1987), 281-300.

[5] Z. Landsman and U. Makov, Sequential credibility evaluation for symmetric location claim distributions, Insurance Math. Econom. 24(3) (1999), 291-300.

[6] L. B. McDonald and W. K. Newey, Partially adaptive estimation of regression model via the generalized T distribution, Econometric Rev. 12 (1988), 103-124.

[7] S. J. Taylor and X. Xu, The magnitude of implied volatility smiles: theory and empirical evidence for exchange rates, Rev. Futures Markets 13 (1994), 355-380.