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POSITIVE SOLUTIONS OF SINGULAR *n*th ORDER BOUNDARY VALUE PROBLEMS WITH NONLOCAL CONDITIONS

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Abstract

We discuss the existence of positive solutions of the following singular nth order three point boundary value problem

$$\begin{cases} u^{(n)}(t) + g(t)f(u) = 0, \quad t \in (0, 1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ \alpha u(\eta) = u(1), \end{cases}$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$ and g is allowed to have finitely many singularities. The existence of positive solutions of the problem is established by applying the fixed point index theorem under suitable conditions.

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1. Introduction

Investigation of positive solutions of nonlocal boundary value problems, initiated by II'in and Moiseev [7, 8], has recently been addressed by various authors, for instance, [2, 3, 4, 6, 9, 10, 11]. Nonlocal boundary value problems are also referred as multi-point nonlinear boundary value problems in the study of disconjugacy theory [1]. This work is motivated by Eloe and Ahmad [2] who addressed a nonlinear nth order BVP with nonlocal conditions. In fact, we extend the results of [2] by allowing g to have finitely many singularities.

In this paper, we shall establish some existence results for the following nth order singular differential equation

$$u^{(n)}(t) + g(t)f(u) = 0, \quad t \in (0, 1), \tag{1.1}$$

subject to the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad \alpha u(\eta) = u(1), \tag{1.2}$$

where $0 < \eta < 1,\ 0 < \alpha\eta^{n-1} < 1$.

Throughout this paper, we assume that

(A_1) $f : [0, \infty) \to [0, \infty)$ is continuous;

(A_2) $g \in L^1(0, 1),\ g(s) \geq 0$, a.e., and there exist $a, b \in [\eta, 1]$ with $a < b$ such that $\int_a^b g(s)ds > 0$.

It is clear that the following condition is a special case of the condition (A_2):

(A_2') For given points $t_1, \ldots, t_m$, $g : E = [0, 1] \setminus \{t_i : i = 1, \ldots, m\} \to [0, \infty)$ is continuous and the integral $\int_0^1 g(s)ds$ exists, and there exist $a, b \in [\eta, 1]$ with $a < b$ such that $\int_a^b g(s)ds > 0$.

We emphasize that the condition (A_2') allows g to have finitely many singularities at $t_1, \ldots, t_m$.

(A_3) $0 \leq f^0 < M_1$, and $m_1 < f_\infty \leq \infty$.

(A_4) $0 \leq f^\infty < M_1$ and $m_1 < f_0 \leq \infty$, where

$$M_1 = \left(\max_{0\le t\le 1}\int_0^1 -G(t, s)g(s)ds\right)^{-1}, \quad m_1 = \left(\min_{a\le t\le b}\int_a^b -G(t, s)g(s)ds\right)^{-1},$$

$$G(\eta; t, s) = \begin{cases} \dfrac{a(\eta; s)t^{n-1}}{(n-1)!}, & 0 \leq t \leq s \leq 1, \\ \dfrac{a(\eta; s)t^{n-1} + (t-s)^{n-1}}{(n-1)!}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$a(\eta(s); s) = \begin{cases} -\dfrac{(1-s)^{n-1}}{1-\alpha\eta^{n-1}}, & \eta \leq s, \\ -\dfrac{(1-s)^{n-1} - (\eta - s)^{n-1}}{1-\alpha\eta^{n-1}}, & s \leq \eta. \end{cases} \tag{1.3}$$

Here $G(\eta; t, s)$ is the Green's function for the BVP (1.1)-(1.2).

Notation. $f^\beta = \lim_{x\to\beta}\sup \dfrac{f(x)}{x}$, $f_\beta = \lim_{x\to\beta}\inf \dfrac{f(x)}{x}$, where β denotes either 0 or ∞.

By a solution to (1.1)-(1.2) we mean a function $u \in C^{n-2}[0, 1]$, $u^{(n-1)}(t) \in AC[0, 1]$ and u satisfies (1.1)-(1.2), where $AC[0, 1]$ denotes the space of absolutely continuous functions defined on $[0, 1]$.

Evidently $u \in C[0, 1]$ is a positive solution of (1.1)-(1.2) if and only if $u \in C[0, 1]$ is a positive solution of the following integral equation:

$$u(t) = -\int_0^1 G(t, s)g(s)f(u(s))ds. \tag{1.4}$$

Remark 1.1. We allow f to satisfy either $0 \leq \lim_{x\to 0}\sup \dfrac{f(x)}{x} < a$ and $b < \lim_{x\to\infty}\inf \dfrac{f(x)}{x} \leq \infty$ or $0 \leq \lim_{x\to\infty}\sup \dfrac{f(x)}{x} \leq a$ and $b < \lim_{x\to 0}\inf \dfrac{f(x)}{x} \leq \infty$ for suitable a and b, which implies that f is not necessary superlinear or sublinear. This relaxation on f improves the results of [2] even if we require g to be a continuous function.

Let K be a cone in a Banach space X and let $K_r = \{x \in K : \| x \| < r\}$, $\partial K_r = \{x \in K : \| x \| = r\}$ and $\overline{K}_{\rho, r} = \{x \in K : \rho \leq \| x \| \leq r\}$, where $0 < \rho < r < \infty$.

Lemma 1.1 [5]. *Let K be a cone in a Banach space X and $A : K_r \to K$ be a compact map. Assume that the following conditions hold.*

(i) $\| Ax \| \leq \| x \|$ *for* $x \in \partial K_r$.

(ii) *There exists an* $e \in \partial K_1$ *such that* $x \neq Ax + \lambda e$ *for* $x \in \partial K_\rho$ *and* $\lambda > 0$.

Then A has a fixed point in $\overline{K}_{\rho, r}$. *The same condition remains valid if* (i) *holds on* ∂K_ρ *and* (ii) *holds on* ∂K_r.

We shall need the following well-known results.

Lemma 1.2 [2]. *Let* $u \in C[0, 1]$ *satisfy the differential inequality* $u^{(n)}(t) < 0$, *together with the boundary conditions* (1.2) *and* $0 < \alpha\eta^{n-1} < 1$. *Then* $u \geq 0$ *on* $[0, 1]$.

Lemma 1.3 [2]. *Let* $0 < \alpha\eta^{n-1} < 1$. *Let u satisfy* $u^{(n)}(t) \leq 0$, $0 < t < 1$ *with the nonlocal conditions* (1.2). *Then* $\inf_{t \in [\eta, 1]} u(t) \geq \gamma \| u \|$, *where* $\gamma = \min\{\alpha\eta^{n-1}, \alpha(1-\eta)(1-\alpha\eta)^{-1}, \eta^{n-1}\}$.

Lemma 1.4 [2]. *For each* $s \in (0, 1)$, *set* $| G(\tau(s), s) | = \max_{t \in [0, 1]} | G(t, s) |$. *Then* $| G(t, s) | \geq \gamma | G(\tau(s), s) |$, $\forall 0 \leq s \leq 1$.

2. Main Results

Theorem 2.1. *Assume that conditions* (A_1), (A_2) *and* (A_3) *hold. Then BVP* (1.1)-(1.2) *has at least one positive solution* $u(t)$ *with* $u(t) \not\equiv 0$ *for* $t \in [0, 1]$.

Proof. Let K be a cone in $C[0, 1]$ given by

$$K = \{u : u \in C[0, 1], u \geq 0, \min_{t \in [\eta, 1]} u(t) \geq \gamma \| u \|\}.$$

Define an operator as follows:

$$(Au)(t) = -\int_0^1 G(t, s) g(s) f(u(s)) ds.$$

Then from Lemma 1.3 we can get $A : K \to K$. Now, we prove that A is compact. We first prove that A is continuous. Let $M = \max\{-G(t, s) : t, s \in [0, 1]\}$. Assume that $u_n, u_0 \in K$ and $u_n \to u_0$. Then $\| u_n \| \leq L$ for every $n \geq 0$. Since f is continuous on $[0, L]$, it is uniformly continuous. Therefore, for $\varepsilon > 0$ there exists a $\delta > 0$ such that $|z' - z''| < \delta$ implies that $|f(z') - f(z'')| < \varepsilon\left(M\int_0^1 g(s)ds\right)^{-1}$. Since $u_n \to u_0$, there exists an $n_0 \in \mathbf{N}$ such that $\| u_n - u_0 \| < \delta$ for $n \geq n_0$. Thus we have

$$|f(u_n(t)) - f(u_0(t))| < \varepsilon\left(M\int_0^1 g(s)ds\right)^{-1}, \quad \text{for } n \geq n_0 \text{ and } t \in [0, 1].$$

This implies that

$$| Au_n(t) - Au_0(t) | \leq \int_0^1 | G(t, s) | g(s) | f(u_n(t)) - f(u_0(t)) | ds < \varepsilon,$$

for $t \in [0, 1]$ and $n \geq n_0$, and therefore $\| Au_n(t) - Au_0(t) \| \leq \varepsilon$ for $n \geq n_0$.

Next, let $B \in K$ be bounded, i.e., $\| x \| \leq m$ for all $x \in B$ and some $m > 0$, and let $b = \max\{f(x) : 0 \leq x \leq m\}\int_0^1 g(s)ds$. Then $A(B)$ is uniformly bounded. Thus we have $\| Ax \| \leq bM$ for $x \in B$. To see $A(B)$ is compact, it is sufficient to prove that $A(B)$ is equicontinuous. In fact, since $G(t, s)$ is uniformly continuous for any $(t, s) \in [0,1] \times [0,1]$, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $| G(t_1, s) - G(t_2, s) | < \varepsilon b^{-1}$ for $|t_1 - t_2| < \delta$ and $s \in [0, 1]$. This implies that $| Au(t_1) - Au(t_2) | \leq \int_0^1 | G(t_1, s) - G(t_2, s) | g(s) f(u(s)) ds < \varepsilon$.

Assume that (A_3) holds, by the first part of (A_3), there exists a $\rho > 0$ such that $f(x) \le M_1\rho$ for $0 \le x \le \rho$. So for every $u \in \partial K_\rho$, we have

$$Au(t) = -\int_0^1 G(t, s)g(s)f(u(s))ds \le M_1\rho\int_0^1 -G(t, s)g(s)ds \le \rho = \| u \|.$$

This implies $\| Au \| \le \| u \|$ for every $u \in \partial K_\rho$.

By the second part of (A_3), there exists an $\eta > \gamma\rho$ such that $f(x) \ge m_1 x$ for $x \ge \eta$. Let $r = \gamma^{-1}\eta$. Then we have

$$\min\{u(t) : a \le t \le b\} \ge \gamma\| u \| = \eta, \text{ for } u \in \partial K_r.$$

Let $\phi(t) \equiv 1$, for $t \in [0, 1]$. Then $\phi \in \partial K_1$. We prove that

$$u \ne Au + \lambda\phi, \text{ for } u \in \partial K_r \text{ and } \lambda > 0.$$

In fact, if not, there exist $u_0 \in \partial K_r$ and $\lambda_0 > 0$ such that $u_0 = Au_0 + \lambda_0\phi$. Let $\mu = \min\{u_0(t) : a \le t \le b\} \ge \eta$. Then we have, for $a \le t \le b$,

$$\begin{aligned} u_0(t) &= -\int_0^1 G(t, s)g(s)f(u_0(s))ds + \lambda_0\phi(t) \\ &\ge m_1\int_a^b -G(t, s)g(s)u_0(s)ds + \lambda_0 \\ &\ge m_1\mu\int_a^b -G(t, s)g(s)ds + \lambda_0 \\ &\ge \mu + \lambda_0. \end{aligned}$$

This implies $\mu \ge \mu + \lambda_0 > \mu$, a contradiction. It follows from Lemma 1.1 that A has a fixed point $u \in \overline{K}_{\rho, r}$.

Theorem 2.2. *Assume that conditions* (A_1), (A_2) *and* (A_4) *hold. Then BVP* (1.1)-(1.2) *has at least one positive solution* $u(t)$ *with* $u(t) \not\equiv 0$ *for* $t \in [0, 1]$.

Proof. Assume that (A_4) holds. Choose some $\beta \in (f^\infty, M_1)$. By the first part of (A_4) there exists an $r_1 > 0$ such that $f(x) \le \beta x$ for $x \ge r_1$. Since

f is continuous, we have $c = \max\{f(x) : 0 \le x \le r_1\} < \infty$. Hence $0 \le f(x) \le c + \beta x$ for $0 \le x < \infty$. Let $r = c(M_1 - \beta)^{-1}$ and $P_r = \{x \in K \mid \| x \| < r\}$. Then we have for $z \in \partial P_r$,

$$| Az(t) | \le (c + \beta \| z \|) \int_0^1 | G(t, s) | g(s) ds \le \frac{c + \beta \| z \|}{M_1} = r = \| z \|.$$

This implies $\| Az \| \le \| z \|$ for every $z \in \partial P_r$. On the other hand, by the last part of (A_4) there exists a $\rho \in (0, r)$ such that $f(x) \ge m_1 x$ for $0 \le x \le \rho$. By a similar argument to that used in Theorem 2.1 we have $z \ne Az + \lambda\phi$ for $z \in \partial P_\rho$ and $\lambda > 0$. The result follows from Lemma 1.1.

Remark 2.1. In the proof of Theorem 2.1 and also of Theorem 2.2 one of the key steps is to find the function ϕ, which is difficult to obtain by using norm-type cone expansion and compression theorem.

Remark 2.2. In Theorem 2.1 or Theorem 2.2 if g satisfies the condition (A_2'), then we have for any $u \in K$,

(R_1) $(Au)^{(n)}(t) = -g(t) f(u(t))$ for $t \in E$,

(R_2) $Au(t) \in C^n(E) \cap C^{n-2}[0, 1]$.

3. Application

As an application of Theorem 2.1 or Theorem 2.2, we consider the following eigenvalue problem

$$u^{(n)}(t) + \lambda g(t) f(u(t)) = 0, \tag{3.1}$$

with the boundary condition (1.2). We list the following conditions:

(P) $\int_\eta^1 g(s) ds > 0$;

(P_1) $f_\infty > 0$, $f^0 \ne \infty$ and $m_1/f_\infty < M_1/f^0$;

(P_2) $f^\infty \ne \infty$, $f_0 > 0$ and $m_1/f_0 < M_1/f^\infty$.

We write $m_1/f_\beta = 0$ if $f_\beta = \infty$ and $M_1/f^\beta = \infty$ if $f^\beta = 0$, where $\beta = 0, \infty$.

We have the following new result on existence for the eigenvalue problem (3.1).

Theorem 3.1. *Assume that* (P) *and* (P_1) *hold. Then for every* $\lambda \in (m_1/f_\infty, M_1/f^0)$, *equation* (3.1) *with* (1.2) *has a solution* u *with* $u \geq 0$ *for* $t \in [0, 1]$ *and* $u(t) \not\equiv 0$ *on* $[0, 1]$. *The same result remains valid for every* $\lambda \in (m_1/f_0, M_1/f^\infty)$, *if* (P) *and* (P_2) *hold.*

Proof. If (P_1) holds, $0 < \lambda f^0 < M_1$ and $\lambda f_\infty > m_1$, then similar to the proof of Theorem 2.1, it is easy to prove Theorem 3.1.

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