JP Jour. Fixed Point Theory & Appl. **1**(1) (2006), 1-13
This paper is available online at http://www.pphmj.com

FIXED POINT THEOREMS FOR MULTIMAPS HAVING KKM PROPERTY ON *G*-CONVEX UNIFORM SPACES

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Abstract

We establish a new fixed point theorem for multimaps having KKM property on G-convex uniform spaces, not necessarily on locally G-convex uniform spaces. This result is parallel to our previous one for topological vector spaces in [6].

1. Introduction

Many problems in nonlinear analysis can be formulated as the fixed point problems for multimaps on topological spaces without linear structure. In this direction, there have appeared a few kinds of abstract

2000 Mathematics Subject Classification: 47H10, 54H25.

Keywords and phrases: fixed point, KKM property.

Received May 9, 2006

convexity notions. For example, the hyperconvex metric spaces due to Aronszajn and Panitchpakdi [1], the convex spaces due to Lassonde [9], the c-spaces due to Horvath [5] which are called the *H-spaces* by Bardaro and Ceppitelli [2]. These different kinds of concepts can be unified as G-convex spaces introduced by Park and Kim [13]. Since then, many results about fixed point theory and KKM theory on topological vector spaces were extended to G-convex spaces, cf. [10, 12, 15, 16, 17]. However, the underlying spaces for their fixed point results are locally G-convex uniform spaces. This paper intends to establish a fixed point theorem for multimaps having KKM property on G-convex uniform spaces, not necessarily on locally G-convex uniform spaces. This result is parallel to that for topological vector spaces in [6] which in turn revises and extends the Kim's fixed point theorem for lower semicontinuous multimaps in Hausdorff topological vector spaces as well as that of Park in [11].

We now recall some basic definitions and facts. Throughout this paper, $\langle Y \rangle$ denotes the class of all nonempty finite subsets of a nonempty set Y. The notation $T : X \multimap Y$ stands for a multimap from a set X into $2^Y \setminus \{\varnothing\}$. For a multimap $T : X \to 2^Y$, the following notations are used:

(a) $T(A) = \bigcup_{x \in A} T(x)$ for $A \subseteq X$;

(b) $T^-(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$;

(c) $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$ for $B \subseteq Y$;

(d) $T^c(x) = Y \setminus T(x)$ for $x \in X$.

All topological spaces are supposed to be Hausdorff. Let X and Y be two topological spaces. Then a multimap $T : X \to 2^Y$ is said to be

(a) *upper semicontinuous* (u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y;

(b) *lower semicontinuous* (l.s.c.) if $T^-(B)$ is open in X for each open subset B of Y;

(c) *compact* if $T(X)$ is contained in a compact subset of Y;

(d) *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

It is well known that if Y is a regular space and T is a closed-valued u.s.c. multimap, then T is closed. The converse is true if Y is compact, cf. [9, Lemma 1].

$\overline{T} : X \to 2^Y$ is defined to be $\overline{T}(x) = \overline{T(x)}$ for each $x \in X$.

The unit coordinate vectors in $\mathbb{R}^{n+1}$ are denoted by $\mathbf{e_0}, \ldots, \mathbf{e_n}$, and Δ_n stands for the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{(\lambda_0, \ldots, \lambda_n) : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \lambda_i = 1\right\}.$$

Later on, if $J \in \langle\{0, 1, \ldots, n\}\rangle$, then $\mathrm{co}\{\mathbf{e}_i : i \in J\}$ will be denoted by Δ_J.

Park and Kim [13] unified many different kinds of abstract convexity in the following way.

Definition 1.1. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X and a map $\Gamma : \langle D\rangle \multimap X$ such that for each $A = \{a_0, \ldots, a_n\} \in \langle D\rangle$ with $|A| = n + 1$, there exists a continuous function $\varphi_A : \Delta_n \to \Gamma(A)$ such that if $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, then $\varphi_A(\mathrm{co}\{\mathbf{e}_{i_0}, \ldots, \mathbf{e}_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, \ldots, a_{i_k}\})$. When $D = X$, $(X, D; \Gamma)$ is abbreviated to $(X; \Gamma)$. In this paper, we assume that a G-convex space $(X; \Gamma)$ always satisfies the extra condition: $x \in \Gamma(\{x\})$ for any $x \in X$.

A subset K of a G-convex space (X, Γ) is said to be Γ-*convex* if for any $A \in \langle K\rangle$, $\Gamma(A) \subseteq K$.

For convenience, we also express $\Gamma(A)$ by Γ_A.

A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that

(a) each member of $\mathcal{U}$ contains the diagonal Δ;

(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in $\mathcal{U}$;

(d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

(e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

Every member V in $\mathcal{U}$ is called an *entourage*. An entourage V is said to be *symmetric* if $(x, y) \in V$ whenever $(y, x) \in V$.

If $(X, \mathcal{U})$ is a uniform space, then the topology $\mathcal{T}$ induced by $\mathcal{U}$ is the family of all subsets W of X such that for each x in W, there is a U in $\mathcal{U}$ such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. For details on uniform spaces we refer to [7].

Definition 1.2. A *G-convex uniform space* $(E; \Gamma, \mathcal{U})$ is a G-convex space so that its topology is induced by a uniformity $\mathcal{U}$.

A G-convex uniform space $(E; \Gamma, \mathcal{U})$ is said to be a *locally G-convex uniform space* if it satisfies the following conditions:

(a) $\Gamma_{\{x\}} = \{x\}$ for any $x \in E$;

(b) The uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of open symmetric entourages such that for each $V \in \mathcal{B}$, $V[K] := \{y \in X : (x, y) \in V$ for some $x \in K\}$ is Γ-convex whenever K is Γ-convex.

By the definition of G-convex uniform space $(E; \Gamma, \mathcal{U})$, it is easy to check that $A \subseteq \Gamma_A$ for any $A \in \langle E \rangle$.

For a G-convex space $(E; \Gamma)$, a multimap $F : E \multimap E$ is called a *KKM map* if $\Gamma_A \subseteq F(A)$ for each $A \in \langle E \rangle$. The following result is well known.

Theorem 1.3 (The KKM Principle)**.** *Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a KKM map (that is,* $\mathrm{co}(A) \subseteq F(A)$ *for each $A \in \langle D \rangle$). Then* $\bigcap_{a \in D} \overline{F(a)} \neq \emptyset$.

The following known results in the literature will be quoted in the sequel.

Lemma 1.4 [14]. *Suppose X is a compact topological space and* $(Y; \Gamma)$ *is a G-convex space. If* $T : X \multimap Y$ *satisfies*

(a) $T(x)$ *is* Γ*-convex for each* $x \in X$,

(b) $T^{-}(y)$ *is open in X for each* $y \in Y$,

then T has a continuous selection.

Lemma 1.5 [16]. *Let* $(X; \Gamma)$ *be a compact locally G-convex uniform space. If* $p : X \to \Delta_n$ *is continuous and* $Q : \Delta_n \multimap X$ *is u.s.c. with compact and* Γ*-convex values, then* $p \circ Q : \Delta_n \multimap \Delta_n$ *has a fixed point.*

Lemma 1.6 [5]. *Let X and Y be two topological spaces. If* $T : X \multimap Y$ *is l.s.c. and* $g : X \to Y$ *is continuous and* $V : Y \multimap Y$ *has open graph, then* $x \to T(x) \cap V(g(x))$ *is l.s.c.*

2. The Main Results

In this section, we shall establish a new fixed point theorem for multimaps having KKM property in G-convex uniform spaces. To begin with, recall that

Definition 2.1. Suppose $(X; \Gamma)$ is a G-convex space and Y is a topological space. If $F, T : X \multimap Y$ are two multimaps satisfying that

$$T(\Gamma_A) \subseteq F(A)$$

for each $A \in \langle X \rangle$, then F is called a *KKM map* with respect to T. If the multimap T satisfies the requirement that for any KKM map F with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The collection of all multimaps from X to Y that have the KKM property is denoted by $\mathrm{KKM}(X, Y)$.

In what follows, $\Gamma(V[y])$ is defined as $\Gamma(V[y]) = \bigcup_{A \in \langle V[y] \rangle} \Gamma_A$.

Lemma 2.2. *Let X be a nonempty* Γ*-convex subset of a G-convex uniform space* $(E; \Gamma, \mathcal{U})$ *which has a uniformity* $\mathcal{U}$ *with a base* $\mathcal{B}$ *of open* *symmetric entourages. Suppose that for any* $V \in \mathcal{B}$, $x \in \Gamma(V[y])$ *if and only if* $y \in \Gamma(V[x])$. *If* $T : X \multimap X$ *is compact and has the KKM property, then for any* $V \in \mathcal{B}$, *there is an* $x_V \in X$ *such that* $\Gamma(V[x_V]) \cap T(x_V) \neq \varnothing$.

Proof. On the contrary, assume there is a $V \in \mathcal{B}$ such that for any $x \in X$, $\Gamma(V[x]) \cap T(x) = \varnothing$. Let $K = \overline{T(X)}$. Then K is compact in X. Define $F : X \to 2^X$ by $F(x) = K \setminus V[x]$. Clearly, for each $x \in X$, $F(x)$ is closed. Using the fact that $x \in \Gamma_{\{x\}}$ it is easy to check that $F(x)$ is nonempty. We now show that F is a KKM map with respect to T. If not, there is an $A = \{x_1, ..., x_n\} \in \langle X \rangle$ such that $T(\Gamma_A) \not\subseteq F(A)$. Choose $z \in T(\Gamma_A)$ such that $z \notin F(A)$. Then there is a $y \in \Gamma_A$ such that $z \in T(y)$ but $z \notin F(A)$. Since

$$\begin{aligned} z \notin F(A) &= \bigcup_{i=1}^{n} F(x_i) \\ &= \bigcup_{i=1}^{n} (K \setminus V[x_i]) \\ &= K \setminus \bigcap_{i=1}^{n} V[x_i], \end{aligned}$$

we conclude that $z \in V[x_i]$ for all $i = 1, ..., n$, that is, $(x_i, z) \in V$ for all $i = 1, ..., n$. By symmetry, $(z, x_i) \in V$ for all $i = 1, ..., n$. So, $x_i \in V[z]$ for all $i = 1, ..., n$. Consequently, $A \subseteq V[z]$, and hence $\Gamma_A \subseteq \Gamma(V[z])$. In particular, $y \in \Gamma(V[z])$. By hypothesis, $z \in \Gamma(V[y])$. Therefore, we reach the conclusion that $z \in T(y) \cap \Gamma(V[y])$, a contradiction. So, F is a KKM map with respect to T. Since T has the KKM property, $\{F(x) : x \in X\}$ has the finite intersection property. Now, noting that for each $x \in X$, $F(x) \subseteq K$ and K is compact, we infer that $\bigcap_{x \in X} F(x) \neq \varnothing$. Choose $\xi \in \bigcap_{x \in X} F(x)$. Then $\xi \in F(\xi) = K \setminus V[\xi]$, which implies that $\xi \notin V[\xi]$, a contradiction. This completes the proof.

Corollary 2.3. *Suppose* $(E; \Gamma, \mathcal{U})$ *is a locally G-convex uniform space with a base* $\mathcal{B}$ *of symmetric open entourages and X is a* Γ*-convex subset of*

E. If $T : X \multimap X$ *is compact and has the KKM property, then for any* $V \in \mathcal{B}$, *there is* $x_V \in X$ *such that* $V[x_V] \cap T(x_V) \neq \varnothing$.

Proof. Since $\Gamma(V[x]) = V[x]$ in a locally G-convex uniform space, the result follows immediately.

Theorem 2.4. *Suppose X is a nonempty compact* Γ*-convex subset of a G-convex uniform space* $(E; \Gamma, \mathcal{U})$ *with a base* $\mathcal{B}$ *of symmetric open entourages such that for any* $V \in \mathcal{B}$, $x \in \Gamma(V[y])$ *if and only if* $y \in \Gamma(V[x])$. *If* $T \in \mathrm{KKM}(X, X)$ *is closed-valued and satisfies the following condition* $(*)$:

$(*)$ *If* $y \in X$ *satisfies that* $y \notin U[T(y)]$ *for some open entourage U, then* $y \notin \mathrm{cl}\{x \in X : x \in \Gamma(V[T(x)])\}$ *for some* $V \in \mathcal{B}$,

then T has a fixed point.

Proof. For each $U \in \mathcal{B}$, put $F_U = \{x \in X : x \in \Gamma(U[T(x)])\}$ and $G_U = \{x \in X : x \in U[T(x)]\}$. By Lemma 2.2, there is an $x_U \in X$ such that $\Gamma(U[x_U]) \cap T(x_U) \neq \varnothing$. Choose a $y \in \Gamma(U[x_U]) \cap T(x_U)$. It follows from the assumption that $x_U \in \Gamma(U[y])$, and hence $x_U \in \Gamma(U[T(x_U)])$. Consequently, for each $U \in \mathcal{B}$, $F_U \neq \varnothing$. Since $\{\overline{F_U} : U \in \mathcal{B}\}$ has the finite intersection property and since X is compact, we conclude that $\bigcap_{U \in \mathcal{B}} \overline{F_U} \neq \varnothing$. Also, it is obvious that $\bigcap_{U \in \mathcal{B}} G_U \subseteq \bigcap_{U \in \mathcal{B}} \overline{F_U}$ by noting that $G_U \subseteq \overline{F_U}$ for each $U \in \mathcal{B}$. For the reverse inclusion, if there is a $y \in \bigcap_{U \in \mathcal{B}} \overline{F_U}$ such that $y \notin \bigcap_{U \in \mathcal{B}} G_U$, then $y \notin U[T(y)]$ for some $U \in \mathcal{B}$, hence by $(*)$ we can choose a $V \in \mathcal{B}$ such that $y \notin \overline{F_V}$, a contradiction. Therefore, $\bigcap_{U \in \mathcal{B}} G_U = \bigcap_{U \in \mathcal{B}} \overline{F_U} \neq \varnothing$. Letting $\xi \in \bigcap_{U \in \mathcal{B}} G_U$, we see

$$\xi \in \bigcap_{U \in \mathcal{B}} U[T(\xi)] = T(\xi),$$

where the equality holds is due to the fact that $T(\xi)$ is closed. This completes the proof.

When X is a topological vector space, this is our previous result [6, Theorem 2.2].

In a locally G-convex uniform space, we have $\Gamma(V[x]) = V[x]$, and so $y \in V[x]$ if and only if $x \in V[y]$ for any symmetric open entourage V.

Proposition 2.5. *Suppose X is a nonempty compact Γ-convex subset of a locally G-convex uniform space $(E; \Gamma, \mathcal{U})$ and $T : X \multimap X$ is closed and has Γ-convex values, then the condition $(*)$ holds*:

$(*)$ *If $y \in X$ satisfies that $y \notin U[T(y)]$ for some open entourage U, then $y \notin \mathrm{cl}\{x \in X : x \in \Gamma(V[T(x)])\}$ for some $V \in \mathcal{B}$.*

Proof. Choose a $V \in \mathcal{B}$ such that $V \subseteq \overline{V} \subseteq U$. Then the set $A := \{x \in X : x \in \overline{V}[T(x)]\}$ is closed. Indeed, for any $x \in \overline{A}$, choose a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$. Since $x_\alpha \in A$, we can choose $z_\alpha \in T(x_\alpha)$ such that $(x_\alpha, z_\alpha) \in \overline{V}$. By the compactness of X, we may assume that $z_\alpha \to z$ for some $z \in X$. So $(x_\alpha, z_\alpha) \to (x, z)$, and hence $(x, z) \in \overline{V}$ by noting that $\{(x_\alpha, z_\alpha)\} \subset \overline{V}$ and $\overline{V}$ is closed. Meanwhile, since T is closed, we have that $z \in T(x)$, so $x \in \overline{V}[T(x)] \cap X$, which shows that A is closed. Now, if $y \notin U[T(y)] \cap X$, then $y \notin \overline{V}[T(y)] \cap X$, and so $y \notin A = \overline{A}$. Hence $y \notin \mathrm{cl}\{x \in X : x \in V[T(x)]\}$. Moreover, since $V[T(x)]$ is Γ-convex, $\Gamma(V[T(x)]) = V[T(x)]$. Therefore,

$$y \notin \mathrm{cl}\{x \in X : x \in V[T(x)]\} = \mathrm{cl}\{x \in X : x \in \Gamma(V[T(x)])\}.$$

Corollary 2.6. *Suppose X is a nonempty compact Γ-convex subset of a locally G-convex uniform space $(E; \Gamma, \mathcal{U})$. If $T \in \mathrm{KKM}(X, X)$ is closed and has Γ-convex values, then T has a fixed point.*

Proof. This follows from Theorem 2.4 and Proposition 2.5.

Here we like to mention that the condition that T has Γ-convex values can be dropped by means of Corollary 2.3 and the technique in [4, Theorem 3.2].

3. The KKM Class

In this section, we further study the KKM class.

Proposition 3.1. *If $f : X \to Y$ is a continuous function from a G-convex space (X, Γ) to a topological space Y, then $f \in \mathrm{KKM}(X, Y)$.*

Proof. Let $F : X \multimap Y$ be any closed-valued KKM map with respect to f and $A = \{a_0, \ldots, a_n\}$ be any member in $\langle X \rangle$. Since X is a G-convex space, there is a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \le i_0 < \cdots < i_k \le n$,

$$\phi_A(\mathrm{co}(e_{i_0}, \ldots, e_{i_k})) \subseteq \Gamma_{\{a_{i_0}, \ldots, a_{i_k}\}} \cap \phi_A(\Delta_n),$$

so

$$\mathrm{co}(e_{i_0}, \ldots, e_{i_k}) \subseteq \phi_A^{-}(\Gamma_{\{a_{i_0}, \ldots, a_{i_k}\}} \cap \phi_A(\Delta_n)). \tag{1}$$

Also, since F is a KKM map with respect to f, we have

$$f(\Gamma_{\{a_{i_0}, \ldots, a_{i_k}\}}) \subseteq F(\{a_{i_0}, \ldots, a_{i_k}\}),$$

that is,

$$\Gamma_{\{a_{i_0}, \ldots, a_{i_k}\}} \subseteq f^{-}(F(\{a_{i_0}, \ldots, a_{i_k}\})). \tag{2}$$

It follows from (1) and (2) that

$$\mathrm{co}(e_{i_0}, \ldots, e_{i_k}) \subseteq \phi_A^{-}(f^{-}(F(\{a_{i_0}, \ldots, a_{i_k}\})) \cap \phi_A(\Delta_n)).$$

Noting that $e_i \multimap \phi_A^{-}(f^{-}(F(a_i)) \cap \phi_A(\Delta_n))$ is a KKM map, we infer from the KKM principle that

$$\bigcap_{i=0}^{n} \phi_A^{-}(f^{-}(F(a_i)) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which implies that $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. Hence $f \in \mathrm{KKM}(X, Y)$.

Lemma 3.2. *Let X be a compact topological space and $(Y; \Gamma, \mathcal{U})$ be a locally G-convex uniform space. If $T : X \multimap Y$ is a l.s.c. multimap with Γ-convex values, then for any symmetric entourage V of the uniformity of Y, there exists a continuous function $f : X \to Y$ such that, for any $x \in X$, $T(x) \cap V[f(x)] \neq \emptyset$.*

Proof. We may assume that V is open in $Y \times Y$. For each $x \in X$, taking $y \in T(x)$, we see that $y \in T(x) \cap V[y]$. Define $F : X \multimap Y$ by $F(x) = \{y \in Y : T(x) \cap V[y] \neq \varnothing\}$. Since $V[T(x)] = \{y \in Y : (z, y) \in V$ for some $z \in T(x)\}$, we have $F(x) = V[T(x)]$ for each $x \in X$. Due to $T(x)$ is Γ-convex, we infer that $V[T(x)]$ is Γ-convex, so $F(x)$ is Γ-convex. Also, for any $y \in Y$, since

$$\begin{aligned} F^{-}(y) &= \{x \in X : y \in F(x)\} \\ &= \{x \in X : T(x) \cap V[y] \neq \varnothing\} \\ &= T^{-}(V[y]) \end{aligned}$$

and since $V[y]$ is open in Y, it follows from the lower semi-continuity of T that $F^{-}(y)$ is open in X. By Lemma 1.4, F has a continuous selection $f : X \to Y$, so $f(x) \in F(x)$, which implies that $T(x) \cap V[f(x)] \neq \varnothing$.

Definition 3.3. Let $(Y; \Gamma)$ be a G-convex space. We call it a *locally G-convex metric space* if (Y, d) is a metric space such that for any $\varepsilon > 0$,

$$\{y \in Y : d(y, E) < \varepsilon\}$$

is a Γ-convex set whenever $E \subseteq Y$ is a Γ-convex set, and all open balls are Γ-convex.

Proposition 3.4. *Let X be a compact topological space and $(Y; \Gamma)$ be a complete locally G-convex metric space. If $T : X \multimap Y$ is a l.s.c. multimap with closed Γ-convex values, then T has a continuous selection.*

Proof. For each $n \in \mathbb{N} \cup \{0\}$, let $V_n[y] = \left\{z \in Y : d(y, z) < \frac{1}{2^n}\right\}$. By Lemma 3.2, there exists a continuous function $f_0 : X \to Y$ such that $T(x) \cap V_0[f_0(x)] \neq \varnothing$. Define $T_1 : X \multimap Y$ by $T_1(x) = T(x) \cap V_0[f_0(x)]$. By Lemma 1.6, T_1 is l.s.c. Obviously, for each $x \in X$, $T_1(x)$ is Γ-convex. So, it follows from Lemma 3.2 that there exists a continuous function $f_1 : X \to Y$ such that $T_1(x) \cap V_1[f_1(x)] \neq \varnothing$. Continuing in this manner, we obtain a sequence of multimaps $T_n : X \multimap Y$ and a sequence of continuous functions $f_n : X \to Y$ such that for each $n \in \mathbb{N} \cup \{0\}$ and for each $x \in X$, $T_n(x) \cap V_n[f_n(x)] \neq \emptyset$. Since $d(f_{n+1}(x), f_n(x)) \leq \frac{1}{2^{n+1}} + \frac{1}{2^n}$ for all $x \in X$, the sequence $\{f_n\}$ is uniformly Cauchy, its limit function f is continuous and it is clear that $f(x) \in T(x)$ for each $x \in X$.

Proposition 3.5. *Let* $(X; \Gamma)$ *be a compact locally G-convex metric space. If* $T : X \multimap X$ *is a l.s.c. multimap with closed* Γ*-convex values, then* $T \in \mathrm{KKM}(X, X)$.

Proof. By Proposition 3.4, T has a continuous selection $f : X \to X$. Since every KKM map F with respect to T is also a KKM map with respect to f and since $f \in \mathrm{KKM}(X, X)$, it follows immediately that $T \in \mathrm{KKM}(X, X)$.

Corollary 3.6. *Let X be a nonempty compact* Γ*-convex subset of a locally G-convex uniform space* $(E; \Gamma, \mathcal{U})$. *If* $T : X \multimap X$ *is a l.s.c. multimap with closed* Γ*-convex values, then T has a fixed point.*

Proof. By Proposition 3.4, T has a continuous selection t which has a fixed point by Park [12, Theorem 3]. Therefore T has a fixed point.

Proposition 3.7. *Let* $(X; \Gamma)$ *be a compact locally G-convex uniform space. If* $T : X \multimap X$ *is a u.s.c. closed-valued multimap with* Γ*-convex values, then* $T \in \mathrm{KKM}(X, X)$.

Proof. If $T \notin \mathrm{KKM}(X, X)$, then there exist a closed-valued KKM map $F : X \multimap X$ with respect to T and a nonempty finite subset $A = \{x_0, \ldots, x_n\}$ of X such that $\bigcap_{i=0}^{n} F(x_i) = \emptyset$. Since (X, Γ) is a G-convex space, there exists a continuous function $\phi_A : \Delta_n \to X$ satisfying that

(i) $\phi_A(\Delta_n) \subseteq \Gamma_A$, and

(ii) $\phi_A(\Delta_J) \subseteq \Gamma_{\{x_i : i \in J\}}$, for any $J \in \langle\{0, 1, \ldots, n\}\rangle$.

Moreover, since $X = \bigcup_{i=0}^{n} F^{c}(x_i)$, it has a partition of unity $\{\alpha_i\}_0^n$ subordinated to the open covering $\{F^c(x_i)\}_0^n$ of X. Define $f : X \to \Delta_n$ by $f(x) = \sum_{i=0}^{n} \alpha_i(x)e_i$. Obviously, f is continuous, and so $T \circ \phi_A : \Delta_n \multimap X$ is u.s.c. with compact and Γ-convex values. By Watson [16] $f \circ T \circ \phi_A : \Delta_n \multimap \Delta_n$ has a fixed point $\hat{x}$. Choose $\hat{y} \in T \circ \phi_A(\hat{x})$ such that $\hat{x} = f(\hat{y})$. Denote by J the set of indices i such that $\alpha_i(\hat{y}) > 0$. Then $\hat{y} \in F^c(x_i)$ for all $i \in J$, $\hat{x} = f(\hat{y}) \in \Delta_J$ and $\phi_A(\hat{x}) \in \phi_A(\Delta_J) \subseteq \Gamma_{\{x_i : i \in J\}}$. So $\hat{y} \in T(\Gamma_{\{x_i : i \in J\}})$, and hence $T(\Gamma_{\{x_i : i \in J\}}) \not\subseteq \bigcup_{i \in J} F(x_i)$. This is a contradiction. Therefore, $T \in \mathrm{KKM}(X, X)$.

Corollary 3.8. *Suppose X is a nonempty compact Γ-convex subset of a locally G-convex uniform space $(E; \Gamma, \mathcal{U})$. If $T : X \multimap X$ is u.s.c. with closed and Γ-convex values, then T has a fixed point.*

Proof. Since T is u.s.c. and closed-valued, it is closed, and so the result follows from Proposition 3.7 and Corollary 2.6.

This corollary is due to Watson [16, Theorem 4.1].

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